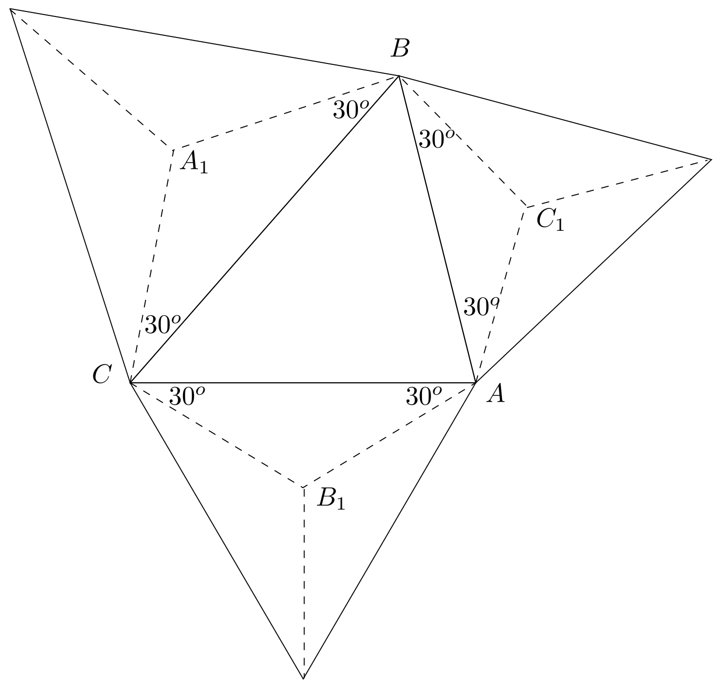
Prove Napoleon's theorem: Given an arbitrary triangle $ABC_{\triangle}$, the centers of the equilateral triangles exterior to $ABC_{\triangle}$ form an equilateral triangle. (Illustration and hints below.)



Hints: Represent the points A, B, C, A_1, B_1, C_1 with complex numbers a, b, c, a_1, b_1, c_1. Observe that multiplying with

$$\rho := \frac{1}{\sqrt{3}} \left(\cos(30^o) + i \cdot \sin(30^o) \right)$$

rotates the vector $\overrightarrow{BA} = a - b$ into $\overrightarrow{BC_1} = c_1 - b$. Use this observation to express c_1 in terms of a, b and ρ. Express then a_1 and c_1 similarly in terms of a, b, c and ρ. Show that $c_1 - a_1$ is obtained by multiplying $b_1 - a_1$ with

$$\frac{\rho}{1-\rho} = \frac{2\rho - 1}{\rho} = \frac{\rho - 1}{2\rho - 1}.$$

It is probably easier to do so if you find the quadratic equation whose roots are ρ and its conjugate. Finally show that

$$\frac{\rho}{1-\rho} = \cos(60^o) + i \cdot \sin(60^o)$$

meaning that $\overrightarrow{A_1C_1}$ is obtained from $\overrightarrow{A_1B_1}$ by a 60^o rotation.