The Fermat point

Let ABC be a triangle whose internal angles are all less than 120^o. We define its Fermat point as the point P satisfying $\angle APB = \angle BPC = \angle CPA = 120^o$. Using this definition, the Fermat point clearly exist since it is the intersection of the (open) arc $\{Q : \angle AQB = 120^o\}$ with the (open) arc $\{Q : \angle BQC = 120^o\}$.

Construct the regular triangles $A'BC_\triangle$, $AB'C_\triangle$, and $ABC'_\triangle$ as shown in Fig. **??**.

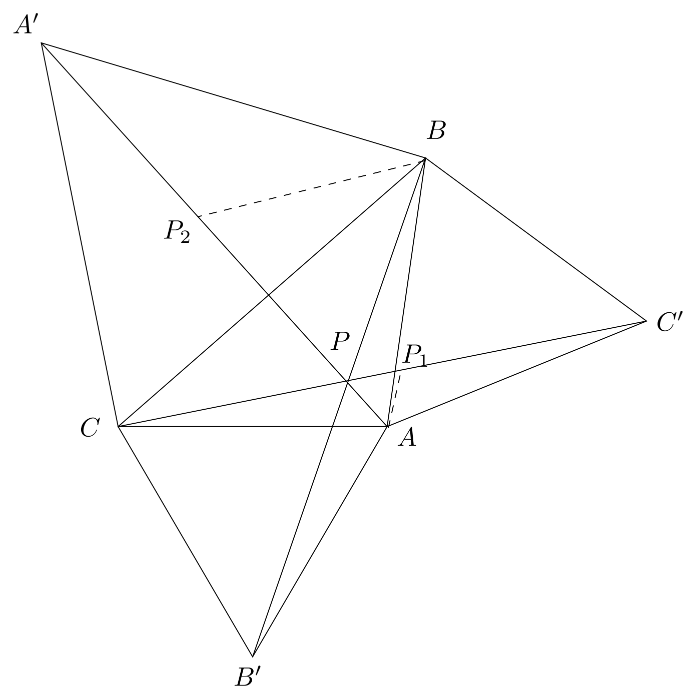

Figure 1: Constructing the Fermat point using regular triangles

Proposition 1 *The lines AA', BB' and CC' meet in the Fermat point of the triangle.*

Proof: Rotation by 60^o around A sends $AB'B_\triangle$ into $ACC'_\triangle$. Introducing P for the intersection of $B'B$ and $C'C$ we see that $\angle CPB' = 60^o$ and so $\angle CPB = 120^o$. Let us introduce P_1 for the image of P under this rotation. Then $|P_1C'| = |PB|$ by the congruence induced by the rotation, and $|PP_1| = |PA|$ since $PP_1A_\triangle$ is a regular triangle. Therefore

$$|CC'| = |CP| + |PP_1| + |P_1C'| = |PA| + |PB| + |PC|.$$

Consider now the rotation by 60^o around B. This takes $BC'C_\triangle$ into $BAA'_\triangle$ and let us denote the image of P under this rotation by P_2. (Note that we know only that P is on CC', we don't know yet whether it is also on AA'.) By the congruence $BC'C_\triangle \cong BAA'_\triangle$ we have $|AA'| = |CC'|$ and so $|AA'| = |PA| + |PB| + |PC|$. Observe also that $|P_2A'| = |PC|$ by the congruence induced by the rotation and that $|PP_2| = |PB|$ since $PP_2B_\triangle$ is a regular triangle. Therefore we get that

$$|A'P_2| + |P_2P| + |PA| = |PC| + |PB| + |PA| = |A'A|.$$

If P or P_2 is not on the line AA' then (by the triangle inequality), the sum $|A'P_2| + |P_2P| + |PA|$ is strictly greater than $|A'A|$, in contradiction with the above equality. Thus P is also on the line AA'. Now $\angle CPA = 120^o$ follows in analogy with $\angle CPB = 120^o$. $\diamond$

The first half of the proof of Proposition **??** is almost repeated in the proof of the following characterization of the Fermat point.

Proposition 2 *The Fermat point minimizes the sum $|QA| + |QB| + |QC|$ among all points Q in the plane.*

Proof: Consider any point Q in the plane that is different from the Fermat point. W.l.o.g. we may

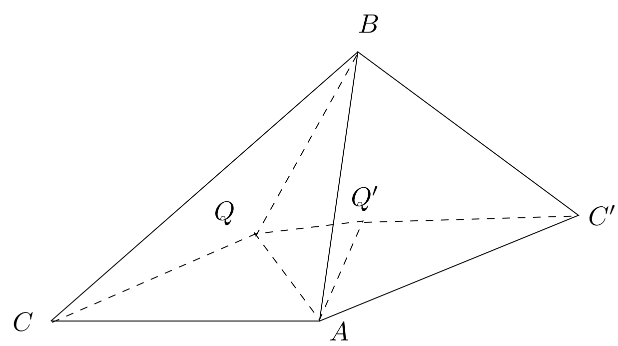

Figure 2: Extremal property of the Fermat point

assume that $\angle AQC \neq 120^0$. Let us rotate $QAB_\triangle$ around A by 60^o, as shown in Fig. **??**. Denote the image of Q and B respectively by Q' and C' respectively. By rotational congruence we have $|QB| = |Q'C'|$ and we also have $|QQ'| = |AQ'|$ since $AQQ'_\triangle$ is regular. Thus

$$|QA| + |QB| + |QC| = |CQ| + |QQ'| + |Q'C|.$$

Since $\angle AQC \neq 120^0$, the angle $\angle AQQ' \neq 180^o$ and, by the triangle inequality, $|CQ| + |QQ'| + |Q'C|$ is strictly more than $|CC'$. In the proof of the previous proposition we have seen that the Fermat point satisfies $|PA| + |PB| + |PC| = |CC'|$. $\diamond$