Assignment 9

We consider the Newton sums $S_m(n) = 1^m + 2^m + \ldots + n^m$, and their generating function

$$F_m(t) = \sum_{n \geq 0} S_m(n) \cdot t^n.$$

1. Explain why $G_m(t) = \sum_{n \geq 0} n^m t^n$ is given by $G_m(t) = (1-t)F_m(t)$, and prove the recursion formula

$$G_{m+1}(t) = t \cdot \frac{d}{dt} G_m(t).$$

2. Prove that the generating functions $F_m(t)$ satisfy the recursion formula

$$F_{m+1}(t) = \frac{t}{1-t} \cdot \frac{d}{dt} \left((1-t) \cdot F_m(t) \right).$$

   Use this and $F_0(t) = 1/(1-t)^2$ to calculate $F_4(t)$.

3. Using the result of the previous exercise, prove that the generating functions $F_m(t)$ are of the form

$$F_m(t) = \frac{t \cdot p_m(t)}{(1-t)^{m+2}},$$

   where the polynomials $p_m(t)$ are given by $p_0(t) = 1$ and the recursion formula

$$p_{m+1}(t) = (p_m(t) + t \cdot p'_m(t)) \cdot (1-t) + (m+1) \cdot t \cdot p_m(t).$$

m	$p_m(t)$
0	1
1	1
2	$1 + t$
3	$1 + 4t + t^2$
4	$1 + 11t + 11t^2 + t^3$
5	$1 + 26t + 66t^2 + 26t^3 + t^4$

4. Introduce $a_{m,k}$ for the coefficients of $p_m(t)$, i.e., assume $p_m(t) = \sum_{k=0}^{m-1} a_{m,k} t^k$. Using the result of the previous exercise, find a recursion formula for the numbers $a_{m,k}$.

5. Using the formulas $F_m(t) = \dfrac{t \cdot p_m(t)}{(1-t)^{m+2}}$ and $p_m(t) = \sum_{k=0}^{m-1} a_{m,k} t^k$, express the Newton sums $S_m(n)$ in terms of the numbers $a_{m,k}$. Use the formula

$$\frac{1}{(1-t)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} t^k.$$