Sample Final Exam Questions (Mandatory Part)

The actual final exam will have a mandatory and an optional section. The optional questions will be similar to the ones on the previous (sample) tests, and need to be answered only if you do not want me to re-use your average (unadjusted) test score. The list of questions below is supposed to help you prepare for the mandatory part of the final.

1. Find the number of ways to form a circle of 12 people.

2. How many ways are there to seat 5 couples around a circular table such that no wife sits next to her husband?

3. Describe a divide and conquer algorithm to alphabetically sort a list of names, and find the exact number of comparisons needed to perform the most efficient algorithm known on a list with 2^m entries.

4. Describe a divide and conquer algorithm to find the largest and second largest element of a set of numbers, and find the exact number of comparisons needed to perform the most efficient algorithm on a set with 2^m elements.

5. In a tennis tournament each player receives k hundreds of dollars where k is the number of players "proven to be worse" than the player in question. (This includes everybody the player defeated, the victims of the player's victims, and so on, but not the player itself. In particular, losers of the first round receive no money.) Find the total prize money for a tournament that has 2^m contestants by solving the appropriate recurrence relation.

6. Describe a divide and conquer algorithm to efficiently multiply two large numbers, and estimate the number of single-digit multiplications needed to multiply two numbers with n digits. Compare your answer to the number of single-digit multiplications needed when using the "normal way" to multiply numbers.

7. Using the table on page 292 of our textbook (I will provide a copy), solve the recurrence relation

$$a_n = 5a_{n/2} + n$$

   assuming n is a power of 2 (leave the constant A undetermined).

8. Using the same table as in the previous question, find the exact solution to

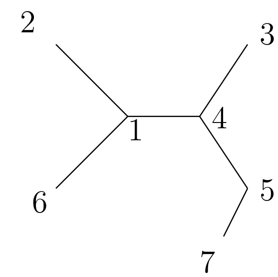
$$a_n = 2a_{n/2} + 3$$

   for all values of n of the form $n = 2^m$, subject to the initial condition $a_1 = 7$. Once you found the exact formula, prove its correctness by induction on m.

9. Draw the labeled tree whose Prüfer code is $(3, 2, 3, 1)$.

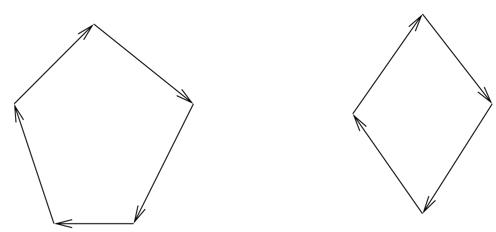
10. State the formula for the number of labeled trees on n vertices, and justify it by explaining why your answer is the number of valid Prüfer codes.

11. Find the Prüfer code of the labeled tree shown in the picture.



12. Two players take turns to remove straws from a pile. At each turn a player is allowed to remove at most 11 straws. At the beginning the pile contains 144 straws. The player taking the last straw wins. Which player has a winning strategy? Describe the winning strategy.

13. Which of the following two graphs has a kernel? Justify your answer!



14. Two players take turns to remove straws from a pile. At each turn a player is allowed to remove 2 or 5 straws. At the beginning the pile contains 15 straws. Which player has a winning strategy? What is the Grundy number of the game?

15. There are three piles of straws in front of two players, containing 20, 30, and 50 straws respectively. The players take alternate turns. At each turn a player is allowed to take any number of straws from exactly one pile. The player taking the last straw wins. Which player has a winning strategy? What is the Grundy number of this game?

16. Write the permutation
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 7 & 2 & 6 & 4 \end{pmatrix}$$
as a product of disjoint cycles.

17. Find the number of ways to color the vertices of a floating rectangle with 10 colors.

18. Find the cycle index for the group of symmetries of the regular hexagon. Use your answer to calculate the number of ways to color the vertices of a floating regular hexagon with 5 colors.

Good luck. Gábor Hetyei