Sample Test II.

The real test will have less questions and you will have about 80 minutes to answer them. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work.

1. Using generating functions, evaluate the sum $1^3 + 2^3 + \cdots + n^3$.

   Solution: We have $1/(1-x) = \sum_{n\geq 0} x^n$. The derivative is

$$\frac{1}{(1-x)^2} = \sum_{n\geq 0} nx^{n-1}.$$

Multiplying both sides with x gives

$$\frac{x}{(1-x)^2} = \sum_{n\geq 1} nx^n.$$

Taking the derivatve again gives

$$\frac{1+x}{(1-x)^3} = \sum_{n\geq 1} n^2x^{n-1}.$$

Multiply both sides again by x:

$$\frac{x+x^2}{(1-x)^3} = \sum_{n\geq 1} n^2x^n.$$

Take the derivative of both sides again:

$$\frac{1+4x+x^2}{(1-x)^4} = \sum_{n\geq 1} n^3x^{n-1}.$$

Multiply both sides by $x/(1-x)$:

$$\frac{x+4x^2+x^3}{(1-x)^5} = \sum_{n\geq 1}(1^3+\cdots+n^3)x^n.$$

Thus $1^3 + \cdots + n^3$ equals to the coefficient of x^n in $(x + 4x^2 + x^3)/(1-x)^5$:

$$\begin{aligned}
1^3+\cdots+n^3 &= [x^n]\frac{x+4x^2+x^3}{(1-x)^5} = [x^{n-1}]\frac{1}{(1-x)^5} + 4[x^{n-2}]\frac{1}{(1-x)^5} + [x^{n-3}]\frac{1}{(1-x)^5} \\
&= \left(\!\!\binom{5}{n-1}\!\!\right) + 4\left(\!\!\binom{5}{n-2}\!\!\right) + \left(\!\!\binom{5}{n-3}\!\!\right) \\
&= \binom{n+3}{n-1} + 4\binom{n+2}{n-2} + \binom{n+1}{n-3} = \binom{n+3}{4} + 4\binom{n+2}{4} + \binom{n+1}{4} \\
&= \frac{n^2(n+1)^2}{4}.
\end{aligned}$$

2. Find a recurrence relation for the number of steps a_n to solve the towers of Hanoi puzzle with n rings.

3. Find a recurrence relation for the number a_n of regions into which the plane is divided by n lines in general position. (No two lines are parallel, no three lines meet in the same point.) Use the recurrence relation to prove a closed formula for the number of these regions.

4. Find the recursion formula for the number F_n of ways an elf can climb a staircase of n stairs if the elf is able to leap one or two stairs at once. Solve the recurrence relation and give a closed formula.

5. Give a closed formula for the sequence a_n given by $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = 6a_n - 9a_{n-1}$.

   Solution: The characteristic equation $q^2 = 6q - 9$ has a double root 3. In such situations, besides 3^n, $n3^n$ is also a solution to the recurrence. (See p. 297 in our book). We look for a solution in the form
   $$a_n = \alpha 3^n + \beta n 3^n.$$
   Substituting $n = 0$ and $n = 1$ we get
   $$\begin{aligned} \alpha \cdot 3^0 + \beta \cdot 0 &= 0 \\ \alpha \cdot 3^1 + \beta \cdot 3^1 &= 1 \end{aligned}$$
   Thus $\alpha = 0$, $\beta = 1/3$, and $a_n = n \cdot 3^{n-1}$.

6. Write a recursion formula for the number a_n of ways to make a pile of red and blue chips of height n in such a way that there are no consecutive red chips in the pile.

   Solution: Consider a pile of height n. If the first chip is blue then we can continue the pile in a_{n-1} ways. If the first chip is red then the next one most be blue. After that we can continue the pile in a_{n-2} ways. Thus $a_n = a_{n-1} + a_{n-2}$.

7. Let a_n be the number of ways to place parentheses in a product of n numbers. Write a recurrence relation for these numbers.

8. Find $\binom{1/3}{5}$.

9. Let a_n be given by $a_0 = 0$, $a_1 = 1$, and the recursion formula $a_n = a_1 a_{n-1} + a_2 a_{n-2} + \cdots + a_{n-1} a_1$. Write an equation satisfied by the generating function $F(x) = \sum_{n=0}^{\infty} a_n x^n$. Solve the equation and use it to get a closed formula for a_n.

10. Write a recursion formula for the Stirling numbers $s_{n,k}$ of the first kind, and provide a table for them for $n \leq 6$.

11. A movie ticket costs 5 dollars. There are $2n$ people standing in a line, n of them have 5 dollar bills, n of them have 10 dollar bills. The box office has no money at the time it opens. What is the probability that the cashier will be able to provide change during the entire ticket sale

process, without asking any person with a 5 dollar bill to step out of the line and come forward. Justify your answer.

12. State and prove the inclusion-exclusion formula.

13. n persons attend a party. A fire breaks out in the building, while outside there is a heavy rain. Everybody rushes to the wardrobe, picks up an umbrella, and leaves. What is the probability that no one picked their own umbrella. Give an exact formula as a function of n and an approximate number for large n. Justify your answer.

14.* The Stirling number $S(n, m)$ of the second kind is defined as the number of ways to partition an n element set into an unordered collection $A_1, \ldots, A_m$ of pairwise disjoint nonempty subsets. Prove that these numbers may be computed using the formula

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n.$$

15.* Find the number of ways to color the vertices of a square with n colors in such a way that adjacent vertices have different colors.

16.* How many n-digit quaternary $(0, 1, 2, 3)$ sequences are there that contain at least one copy of each of the four digits?

17.** The Euler ϕ function $\phi(n)$ gives the number of positive integers $\leq n$ relative prime to n. Assuming that n is of the form $p_1^{k_1}p_2^{k_2}p_3^{k_3}$ where p_1, p_2 and p_3 are primes, find a formula for $\phi(n)$. Generalize this to $n = p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$.

18.* Explain how $\phi(n) = \sum_{d|n} d\mu(n/d)$ follows from $\sum_{d|n}\phi(d) = n$. Here $\phi(n)$ is the Euler ϕ function, and $\mu(m)$ is the number theoretic Möbius function.

Good luck. Gábor Hetyei