Elementary product expansion of the determinant

1 Permutations and inversions

A *permutation* of the set $\{1, 2, \ldots, n\}$ is a bijection $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. The number n is the *order* of the permutation. To write permutations we use sometimes the *two-row notation*, other times the *cycle decomposition*. For example, for $n = 4$, the permutation π given by $\pi(1) = 1$, $\pi(2) = 3$, $\pi(3) = 4$, $\pi(4) = 2$ may be written as

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

in the two-row notation, and $\pi = (1)(234)$ or $\pi = (234)$ is the cycle decomposition of π. (Cycles of length 1, also known as *fixed points* may be omitted when we write the cycle decomposition.) There are $n!$ permutations of order n, they form a group, the *symmetric group* S_n of order n.

An *inversion* of a permutation π is a pair (i, j) such that $i < j$ and $\pi(i) > \pi(j)$. A permutation is even if it has an even number of inversions, otherwise it is odd. Even permutations of order n form a normal subgroup of S_n, the *alternating group* A_n.

Permutations of order n are in bijection with maximal rook placements on an $n \times n$ chess-board, as follows. We may associate to $\pi \in S_n$ the rook placement which places a rook in row i and column $\pi(i)$ for each i. Thus we place exactly one rook in each row and each column. Inversions correspond then to the pairs of rooks in the placement of π which are in "anti-diagonal" position.

A cycle of odd length is an even permutation, a cycle of even length is an odd permutation. Thus a permutation is even, if and only if the number of even cycles in its cycle decomposition is odd.

2 The elementary product expansion

Given an $n \times n$ matrix $A = (a_{i,j})$, and *elementary product* of A is a product $a_{1,\pi(1)} \cdot a_{2,\pi(2)} \cdots a_{n,\pi(n)}$, where π is any permutation of order n. In other words, we select exactly one entry in each row and each column of A and we multiply them. Our main result is the following

Theorem 1 *The determinant* $\det(A)$ *of an* $n \times n$ *matrix* A *is given by*

$$\det(A) = \sum_{\pi \in S_n} (-1)^{\mathrm{inv}(\pi)} \cdot a_{1,\pi(1)} \cdot a_{2,\pi(2)} \cdots a_{n,\pi(n)}.$$

Here $\mathrm{inv}(\pi)$ *is the number of inversions of the permutation* π.

Proof: We proceed by induction on n, assuming the definition given in [1]. For $n = 1$ the statement is obvious, $\det(A)$ equals the only entry in it, either way. Assume the statement is true for all $(n-1) \times (n-1)$ matrices, and consider an $n \times n$ matrix A. By definition,

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1,j} \cdot \det(\widetilde{A}_{1,j}) \tag{1}$$

where $\widetilde{A}_{1,j}$ is the matrix obtained by removing the first row and the j-th column from A. By our induction hypothesis, $\det(\widetilde{A}_{1,j})$ may be obtained by summing over all elementary products of $\widetilde{A}_{1,j}$ and multiplying each elementary product by (-1) raised to the number of anti-diagonal pairs in the rook placement associated to the elementary product. Each elementary product of A contains exactly one entry $a_{1,j}$ in the first row, and the remaining terms form an elementary product of $\widetilde{A}_{1,j}$. Conversely each elementary product of $\widetilde{A}_{1,j}$, multiplied by $a_{1,j}$ yields an elementary product of A. Thus, replacing each $\det(\widetilde{A}_{1,j})$ with its elementary product expansion in (1) gives a sum in which each elementary product of A appears exactly once, with coefficient 1 or -1. We only need to check that this coefficient is 1 exactly when the underlying permutation even.

When we decompose an elementary product of A as $a_{1,j}$ times an elementary product of $\widetilde{A}_{1,j}$, we may distinguish between two types of anti-diagonal pairs in the underlying rook placement: those involving $a_{1,j}$, and those forming an anti-diagonal pair in the underlying rook placement of the corresponding elementary product of $\widetilde{A}_{1,j}$. Thus the sign of the elementary product in $\det(A)$ may be obtained by multiplying the sign of the corresponding elementary product of $\det(\widetilde{A}_{1,j})$ with (-1) raised to the number of anti-diagonal pairs involving $a_{1,j}$. This number is $(-1)^{j-1}$ since $a_{1,j}$ forms an anti-diagonal pair with the terms in the first $j-1$ columns, and only with these. There are exactly $(j-1)$ entries selected in the first $(j-1)$ columns. Note finally that $(-1)^{j-1} = (-1)^{j+1}$. $\diamond$

3 Consequences of the elementary product expansion

Corollary 1 *For a* 3×3 *matrix* A *we have*

$$\det(A) = a_{1,1}a_{2,2}a_{3,3} - a_{1,1}a_{2,3}a_{3,2} - a_{2,1}a_{1,2}a_{3,3} - a_{1,3}a_{2,2}a_{3,1} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2}.$$

In fact, S_3 has 6 elements, of which the identity, (123) and (132) are even permutations, and the transpositions (12), (23) and (31) are odd permutations.

Proposition 1 *Let A be a square matrix, and let B be the matrix obtained from A by exchanging two adjacent rows in A. Then* $\det(B) = -\det(A)$.

Proof: Assume B is obtained from A by exchanging the i-th and $(i+1)$-st rows. Compare the elementary row expansions of $\det(A)$ and $\det(B)$. The same terms appear in both, and the inversions are almost the same. The only difference between A and B is that, for the same elementary product, the entry selected in the i-th row of A is in inversion with the entry selected in the $(i+1)$-st row if and only if the same two entries are not in inversion in B. ◇

Corollary 2

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \cdot \det(\widetilde{A}_{i,j}).$$

In fact, using the previous proposition we may transform the cofactor expansion by the first row into the cofactor expansion by any row. It takes $(i-1)$ exchanges of adjacent rows to arrive at the cofactor expansion by the i-th row.

Proposition 2 *For any square matrix A we have* $\det(A) = \det(A^T)$.

Proof: Reflecting a maximal rook placement about the main diagonal transforms the underlying permutation into its inverse, and leaves the number of inversions unchanged. Thus we get

$$\begin{aligned} \det(A^T) &= \sum_{\pi \in S_n} (-1)^{\mathrm{inv}(\pi)} \cdot a_{1,\pi^{-1}(1)} \cdot a_{2,\pi^{-1}(2)} \cdots a_{n,\pi^{-1}(n)} \\ &= \sum_{\pi \in S_n} (-1)^{\mathrm{inv}(\pi^{-1})} \cdot a_{1,\pi^{-1}(1)} \cdot a_{2,\pi^{-1}(2)} \cdots a_{n,\pi^{-1}(n)} = \det(A). \end{aligned}$$

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References

[1] Stephen H. Friedberg, Arnold J. Insel and Lawrence E. Spence, *"Linear Algebra, 4th Edition,"* Prentice Hall, 2003.