Assignment 12

Oral questions

1. Consider the fractional linear transformation $z \mapsto \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc \neq 0$. Introduce $z = z_1 + z_2 i$ and calculate explicitly the imaginary part of $\frac{az+b}{cz+d}$. Prove that the imaginary part of the image is positive for all $z_2 > 0$ if and only if $ad - bc > 0$.

   Now show that a conjugate fractional linear map $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$ takes the upper half plane into itself if and only if $ad - bc < 0$.

2. Explain why a dilation, centered at the origin, represents a congruence in the Poincaré half plane model. Show that each such dilation may be written as a composition of two inversions, where both circles are centered at the origin. Keeping in mind that these inversions correspond to reflections, help visualize the congruence represented by a dilation by comparing it to the composition of two reflections about two parallel lines in the Euclidean plane.

Question to be answered in writing

1. Prove that a fractional linear transformation that takes the Poincaré upper half plane onto itself may be written as $f(z) = \frac{az+b}{cz+d}$ where a, b, c, d are real numbers. (Hints: walk through the cases in the proof of Theorem 1 in the handout on fractional linear transformations. When c is not zero, you may assume it is a real number. You know that any fractional linear transformation may be written as a composition transformations that preserve half planes, except for a single inversion. That inversion should not take your half plane into the interior or the exterior of a circle.)