Forward differences

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Let $f : \mathbb{N} \to \mathbb{R}$ be any function of the natural numbers. (In other words, the sequence $f(0)$, $f(1)$, $\ldots$, $f(n)$, $\ldots$ is a sequence of real numbers.) The *forward difference operator* Δ is defined by

$$\Delta f(n) := f(n+1) - f(n).$$

We may apply the forward difference operator as many times as we want, and get the functions Δf, $\Delta^2 f$, and so on. These functions may be calculated directly from f using the following result.

Theorem 1 *Let $f : \mathbb{N} \to \mathbb{R}$ be any function. Then we have*

$$\Delta^m f(n) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(n+m-k)$$

holds for $m \geq 1$. Introducing Δ^0 as the identity operator, the formula also holds for $m = 0$.

Proof: We proceed by induction on m. For $m = 0$ we have $\Delta^0 f(n) = f(n)$, for $m = 1$ we get

$$\Delta f(n) = \sum_{k=0}^{1} (-1)^k \binom{1}{k} f(n+1-k) = f(n+1) - f(n).$$

These are obviously true statements. Assume the formula holds for some $m \geq 1$. Applying Δ to both sides we get

$$\begin{aligned}\Delta^{m+1} f(n) &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} \Delta f(n+m-k) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (f(n+m+1-k) - f(n+m-k)) \\ &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(n+m+1-k) + \sum_{k=0}^{m} (-1)^{k+1} \binom{m}{k} f(n+m-k).\end{aligned}$$

Note that $n+m+k = (n+m+1) - (k+1)$ and summing from $k = 0$ to $k = m$ is the same as summing from $k+1 = 1$ to $k+1 = m+1$. Replacing $k+1$ with j in the second sum gives

$$\begin{aligned}\Delta^{m+1} f(n) &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(n+m+1-k) + \sum_{j=1}^{m+1} (-1)^j \binom{m}{j-1} f(n+m+1-j) \\ &= \sum_{k=0}^{m+1} (-1)^k f(n+m+1-k) \left(\binom{m}{k} + \binom{m}{k-1} \right).\end{aligned}$$

Note that, on the last line, we get the correct summand when $k = 0$, since $\binom{m}{-1} = 0$. We also get the correct summand when $k = m+1$, since $\binom{m}{m+1} = 0$. By Pascal's identity, we may replace $\binom{m}{k} + \binom{m}{k-1}$ with $\binom{m+1}{k}$, and we get exactly the statement we wanted to prove for $m + 1$. $\diamondsuit$

A shorter, albeit more abstract proof may be given by introducing the *forward shift operator* S given by $Sf(n) = f(n + 1)$. We may express the forward difference operator Δ as the difference of the identity operator I and the forward shift operator S:

$$\Delta f(n) = (S - I)f(n).$$

Here the operators I, S, and Δ belong to the ring of linear operators sending the vector space of all functions $f : \mathbb{N} \to \mathbb{R}$ into itself. We can think of these operators as infinite square matrices whose rows and columns are indexed with the natural numbers. This ring of operators is not commutative, but the identity operator does commute with the powers of S. Thus we may apply the binomial theorem to the difference $S - I$ and get

$$\Delta^m = (S - I)^m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} S^{m-k}.$$

Theorem 1 now follows from $S^{m-k} f(n) = f(n + m - k)$.

Conversely, we may use the values $\Delta^0 f(0)$. $\Delta^1 f(0)$, $\Delta^2 f(0)$, ... to express $f(n)$. To prove such a formula we will use the following lemma.

Lemma 2 *Any function $f : \mathbb{N} \to \mathbb{R}$ satisfies*

$$f(n) = \sum_{j=0}^{n-1} \Delta f(j) + f(0).$$

In fact, the right hand side is the "telescoping sum"

$$(f(n) - f(n-1)) + (f(n-1) - f(n-2)) + \cdots + (f(1) - f(0)) + f(0)$$

in which the numbers $f(n - 1)$, $f(n - 2)$, ..., $f(0)$ cancel. Now we are able to state our formula.

Theorem 3 *Let $f : \mathbb{N} \to \mathbb{R}$ be any function. Then we have*

$$f(n) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(0) \quad \textit{for } n \geq 0.$$

Proof: We proceed by induction on n. For $n = 0$ the statement is $f(0) = \Delta^0 f(0)$, which is true by the definition of Δ^0. Assume the statement is true up to some value of n. To show the statement for $n + 1$ we use Lemma 2 above to rewrite $f(n + 1)$ as

$$f(n+1) = \sum_{j=0}^{n} \Delta f(j) + f(0).$$

We apply the induction hypothesis for $j = 0, 1, \ldots, n$ to the right hand side. We get

$$\begin{aligned} f(n+1) &= \sum_{j=0}^{n}\sum_{k=0}^{j}\binom{j}{k}\Delta^k\Delta f(0) + f(0) = \sum_{j=0}^{n}\sum_{k=0}^{j}\binom{j}{k}\Delta^{k+1} f(0) + f(0) \\ &= \sum_{k=0}^{n}\left(\sum_{j=k}^{n}\binom{j}{k}\right)\Delta^{k+1} f(0) + f(0). \end{aligned}$$

Here we may replace $\sum_{j=k}^{n}\binom{j}{k}$ with $\binom{n+1}{k+1}$ and we get

$$f(n+1) = \sum_{k=0}^{n}\binom{n+1}{k+1}\Delta^{k+1} f(0) + f(0).$$

Introducing $j = k+1$ we may write

$$f(n+1) = \sum_{j=0}^{n+1}\binom{n+1}{j}\Delta^{j} f(0)$$

which is exactly what we wanted to prove. ◊

Using the operators I, S and Δ we also have a shorter (but more abstract) proof of Theorem 3. In fact, $\Delta = S - I$ may be rewritten as $S = I + \Delta$. Note that $f(n)$ is nothing else but $S^n f(0)$. Now apply the binomial theorem to $S^n = (I + \Delta)^n$ to get

$$f(n) = S^n f(0) = (I + \Delta)^n f(0) = \sum_{k=0}^{n}\binom{n}{k}\Delta^k f(0).$$

Theorem 3 has an important consequence for *higher order arithmetic sequences.*

Definition 4 *The function $f : \mathbb{N} \to \mathbb{R}$ is an* arithmetic sequence of order r *if $\Delta^{r+1}(f)$ is identically zero, but $\Delta^r f$ is not.*

For example, the arithmetic sequences of order zero are the nonzero constant sequences, the arithmetic sequences of order one are the non-constant ordinary arithmetic sequences. If a function $f : \mathbb{N} \to \mathbb{R}$ is an arithmetic sequence of order $r+1$ then we only need to sum up to $k = r$ on the right hand side of Theorem 3, no matter how large is n.

Corollary 5 *Any arithmetic sequence $f : \mathbb{N} \to \mathbb{R}$ of order r is given by*

$$f(n) = \sum_{k=0}^{r}\binom{n}{k}\Delta^k f(0).$$

In particular, $f(n)$ is a polynomial of n of degree r.