Fractional linear transformations

Definition 1 *A* fractional linear transformation *is a function of the form*

$$z \mapsto \frac{az+b}{cz+d}$$

from the set $\mathbb{C} \cup \{\infty\}$ *of extended complex numbers into itself. Here* a, b, c, d *are complex numbers satisfying* $ad - bc \neq 0$, ∞ *is sent into* a/c *and* $-d/c$ *is sent into* ∞.

Proposition 1 *A fractional linear transformation has an inverse which is also a fractional linear transformation.*

Proposition 2 *The composition of two fractional linear transformations is a fractional linear transformation.*

Theorem 1 *Every fractional linear transformation may be written as a composition of dilations, rotations, reflections, inversions, and translations.*

Proof: Assume the fractional linear is given by

$$z \mapsto \frac{az+b}{cz+d},$$

and consider first the case when $c = 0$. Then we must have $d \neq 0$ since $ad - bc \neq 0$. Thus

$$\frac{az+b}{cz+d} = \frac{a}{d}z + \frac{b}{d}.$$

which may be written a composition of the map $z \mapsto a/d \cdot z$ and the translation $z \mapsto b/d$. The map $z \mapsto a/d \cdot z$ is a rotation, composed with a dilation.

From now on we may assume $c \neq 0$. Then

$$\frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d)}{cz+d} + \frac{b - \frac{ad}{c}}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c} \cdot \frac{1}{cz+d}.$$

The statement now follows from the following sequence of observations: $z \mapsto cz$ is a dilation composed with a rotation, $z \mapsto z + d$ is a translation, $z \mapsto 1/z$ is an inversion composed with a reflection, $z \mapsto (bc - ad)/c \cdot z$ is a dilation composed with a rotation, and $z \mapsto a/c + z$ is a translation. $\diamondsuit$

Corollary 1 *Fractional linear transformations take lines and circles into lines and circles. They preserve angles and the cross-ratio.*

References

[1] D. Royster, "Non-Euclidean Geometry and a Little on How We Got There," Lecture notes, December 11, 2011.