Assignment 8

Oral questions

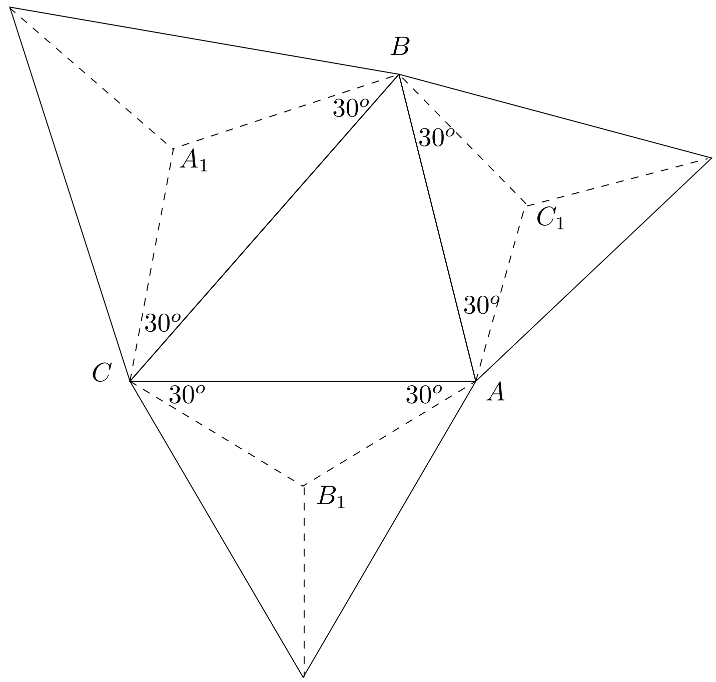
1. Let O be the center of the circle of inversion, P' the inverse of P and Q' the inverse of Q. Assume that O, P, and Q form a triangle. Show that $OPQ_{\triangle}$ is similar to $OQ'P'_{\triangle}$. Use this result to show that inversion preserves the cross-ratio: if A, B, P, and Q are four points distinct from the center O of the circle of inversion and A', B', P', and Q' are their inverses then $(AB, PQ) = (A'B', P'Q')$.

2. Prove that inversion preserves the angle of two circles or lines, in the special case when you have two lines that intersect in a point that is different from the center of the base circle. (Recall that the inverses of these lines are circles that contain the center of the base circle.)

Questions to be answered in writing

1. Prove that inversion about the circle given by $x^2 + y^2 = 1$ takes the point $(x, y) \neq 0$ into the point $(x/(x^2 + y^2), y/(x^2 + y^2))$.

2. Prove Napoleon's theorem: Given an arbitrary triangle $ABC_{\triangle}$, the centers of the equilateral triangles exterior to $ABC_{\triangle}$ form an equilateral triangle. (Illustration and hints on next page.)



Hints: Represent the points A, B, C, A_1, B_1, C_1 with complex numbers a, b, c, a_1, b_1, c_1. Observe that multiplying with

$$\rho := \frac{1}{\sqrt{3}} (\cos(30^o) + i \cdot \sin(30^o))$$

rotates the vector $\overrightarrow{BA} = a - b$ into $\overrightarrow{BC_1} = c_1 - b$. Use this observation to express c_1 in terms of a, b and ρ. Express then a_1 and c_1 similarly in terms of a, b, c and ρ. Show that $c_1 - a_1$ is obtained by multiplying $b_1 - a_1$ with

$$\frac{\rho}{1-\rho} = \frac{2\rho - 1}{\rho} = \frac{\rho - 1}{2\rho - 1}.$$

It is probably easier to do so if you find the quadratic equation whose roots are ρ and its conjugate. Finally show that

$$\frac{\rho}{1-\rho} = \cos(60^o) + i \cdot \sin(60^o)$$

meaning that $\overrightarrow{A_1C_1}$ is obtained from $\overrightarrow{A_1B_1}$ by a 60^o rotation.