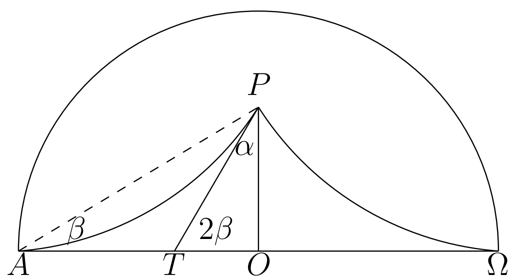
Lobachevsky's formula

Theorem 1 *In the Poincaré disk model, the angle of parallelism* $\Pi(x)$ *satisfies the equation*

$$e^{-x} = \tan\left(\frac{\Pi(x)}{2}\right).$$

Proof: Since the angle of parallelism depends only on the distance x between a line ℓ and a point P, we may assume that the line ℓ is represented by the horizontal diameter $A\Omega$ and that the point P is on the vertical diameter, as shown in the picture below:



The limiting parallel lines to ℓ through P are represented by circular arcs through P which are perpendicular to the base circle, thus $A\Omega$ is a tangent to these arcs. The angle of parallelism $\alpha = \Pi(x)$ is the angle between the vertical line OP and the tangent at P to either of these arcs. Let us denote the intersection of the left tangent with the line $A\Omega$ with T. The triangle $APT_{\triangle}$ is isosceles, as the line segments AT and TP are parts of tangent lines from T to the same circle. Hence $\angle TAP = \angle TPA = \beta$ and the exterior angle $\angle PTO$ has measure 2β. This angle, and $\alpha = \angle TPO$ are the acute agles of the right triangle $TPO_{\triangle}$. Thus we have

$$2\beta + \alpha = \frac{\pi}{2} \quad \text{implying} \quad \beta = \frac{\pi}{4} - \frac{\alpha}{2}.$$

By inspecting the right triangle $APO_{\triangle}$ we obtain that the Euclidean length of OP is $\tan(\beta)$. Using the formula

$$\tan(u - v) = \frac{\tan(u) - \tan(v)}{1 + \tan(u)\tan(v)}$$

we obtain

$$\tan(\beta) = \frac{1 - \tan\left(\frac{\alpha}{2}\right)}{1 + \tan\left(\frac{\alpha}{2}\right)}.$$

Hence the hyperbolic length x of OP satisfies

$$e^x = \frac{1 + \tan(\beta)}{1 - \tan(\beta)} = \frac{1 + \frac{1-\tan\left(\frac{\alpha}{2}\right)}{1+\tan\left(\frac{\alpha}{2}\right)}}{1 - \frac{1-\tan\left(\frac{\alpha}{2}\right)}{1+\tan\left(\frac{\alpha}{2}\right)}} = \frac{1 + \tan\left(\frac{\alpha}{2}\right) + 1 - \tan\left(\frac{\alpha}{2}\right)}{1 + \tan\left(\frac{\alpha}{2}\right) - 1 + \tan\left(\frac{\alpha}{2}\right)} = \frac{1}{\tan\left(\frac{\alpha}{2}\right)}, \quad \text{implying}$$

$$e^{-x} = \tan\left(\frac{\alpha}{2}\right).$$

$\diamondsuit$