Sample Final Exam questions.

This study guide is subject to updates until the last day of clases.

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The actual final exam will have a mandatory and an optional section. The optional questions will be similar to the ones on the previous (sample) tests, and need to be answered only if you do not want me to re-use your average (adjusted) test score. The list of questions below is supposed to help you prepare for the mandatory part of the final. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. Below I am only listing questions related to the definitions, theorems, and proofs I expect you to know. There will be also application exercises, similar to the already discussed homework questions.

1. Recall that the Motzkin number M_n is the number of lattice paths from $(0,0)$ to $(n,0)$ using up steps $(1,1)$, down steps $(1,-1)$ and horizontal steps $(1,0)$ that never go below the x axis. Express the Motzkin number M_n in terms of the Catalan numbers and binomial coefficients.

2. Write down and explain the quadratic equation satisfied by the ordinary generating function of the Motzkin numbers.

3. Write down the generating functions counting all nonnegative integer solutions of the equation $3z_1 + 12z_2 = n$.

4. Write down the generating functions counting all nonnegative integer solutions of the equation $3z_1 + 12z_2 = n$, subject to $0 \leq z_1 \leq 100$ and $1 \leq z_2 \leq 10$.

5. Give a closed-form formula for the Fibonacci number F_n and prove it.

6. Prove that the Lucas number L_n is given by $L_n = F_{n-2} + F_n$. Explain why this formula shows that L_n counts the tilings of the circular n-board with 1- and 2-tiles.

7. Given the set $[n] = \{1, 2, \ldots, n\}$, for some $k \in [n]$ you put an circular permutation on the first k elements and you select a (possibly empty) subset of the remaining $n - k$ elements. Describe the ordinary generating function of the number of pairs of structures you may obtain this way. Which rule will you be using?

8. Write down a closed form formula for the ordinary generating function for the number of (integer) partitions of n with parts size at most k and justify your formula.

9. Write down a closed form formula for the ordinary generating function for the number of (integer) partitions of n with exactly k parts and justify your formula.

10. Using generating functions prove that the number of (integer) partitions of n into odd parts is the same as the number of partitions of n into distinct parts.

11. Express $(-1)^n\binom{1/2}{n}$ as a multiple of the Catalan number C_{n-1}.

12. The Catalan number C_n is defined as the number of sequences $a_1, \ldots, a_{2n}$ such that exactly n of the a_is is 1, the remaining a_is are -1, and we have $a_1 + a_2 + \cdots + a_m \geq 0$ for all $m \leq n$. Express C_n using binomial coefficients. Prove your formula, using the reflection principle or by considering rotational equivalence classes.

13. Find a recurrence for the Catalan numbers and use it to express their ordinary generating function.

14. Find a closed form formula for $\sum_{n\geq 0}^{\infty} n^2 x^n$.

15. Explain how the formula
$$f(n) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(0) \quad \text{for } n \geq 0,$$
may be used to find a closed form formula for a higher order arithmetic sequence.

16. Using difference tables, find a closed-form formula for $f(n) = 1^2 + 3^2 + \cdots + (2n-1)^2$.

17. Prove the formula
$$\Delta^m f(n) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(n+m-k).$$

18. Given a function $f : \mathbb{N} \to \mathbb{R}$, write $\Delta^4 f(n)$ as a linear combination of $f(n)$, $f(n+1)$, $f(n+2)$, $f(n+3)$ and $f(n+4)$.

19. Find the ordinary generating function $f_k(x) = \sum_{n=0}^{\infty} S(n,k)x^n$ of the Stirling numbers of the second kind $S(n,k)$. Prove your formula.

20. Find the exponential generating function $g_k(x) = \sum_{n=0}^{\infty} S(n,k)x^n/n!$ of the Stirling numbers of the second kind $S(n,k)$. Prove your formula.

21. A football coach splits the team into to groups. Each group has to form a line and each member of the second group must put on an orange, a yellow or a white shirt. Write the exponential generating function for the number of ways all this can happen.

22. n soldiers are lined up. We create a certain number of teams by breaking the line of soldiers into parts, and we select a leader from each team. Write the ordinary generating function for the number of ways to perform the selection. *Do not simplify!*

23. We form a certain number of teams from n soldiers (who were not lined up). We arrange the teams in a cyclic order and we select a (possibly empty) subset of each team. Find the exponential generating function for the number of ways to perform this operation. *Do not simplify!*

Good luck. Gábor Hetyei