Étiquetage des régions dans les arrangements d'hyperplans

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12 février, 2024
GT "Combinatoire et interactions"
LaBRI, Université de Bordeaux I

Hyperplane arrangements

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A hyperplane arrangement $\mathcal{A}$ is a finite collection of hyperplanes in a d-dimensional real vector space, which partition the space into regions.

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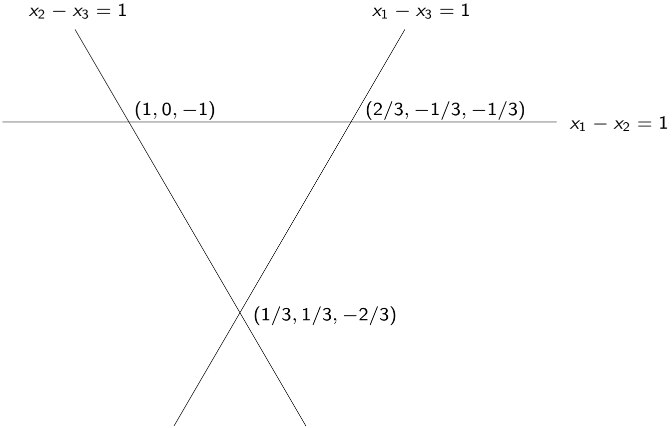

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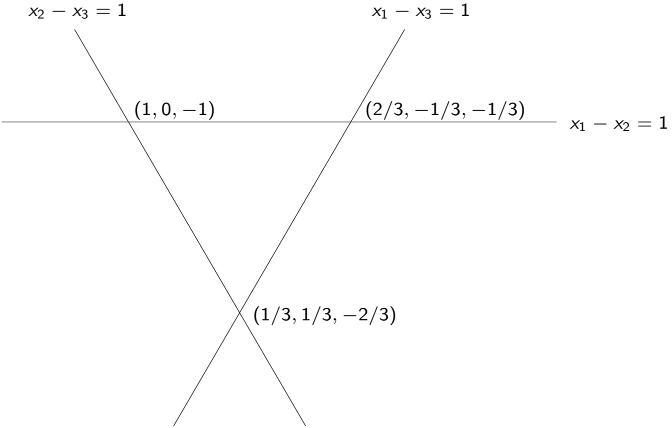

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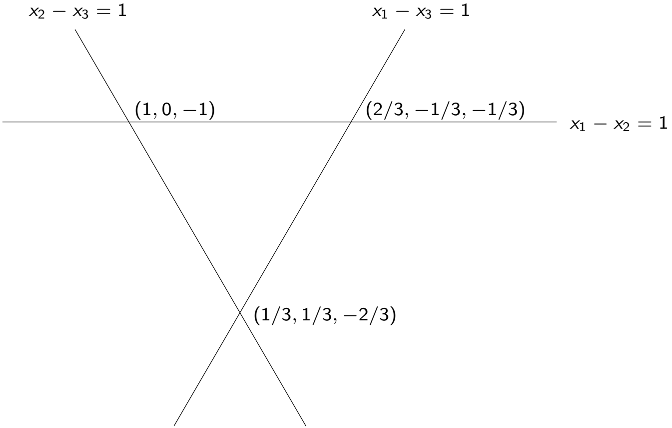

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The *truncated affine arrangements* $\mathcal{A}_{n-1}^{a,b}$ (where $a + b \geq 2$) contain the hyperplanes are $x_i - x_j = 1 - a, 2 - a, \ldots, b - 1$ for $1 \leq i < j \leq n$.

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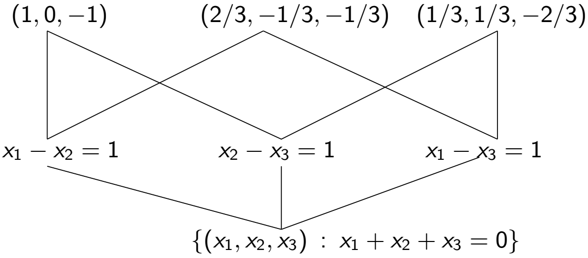

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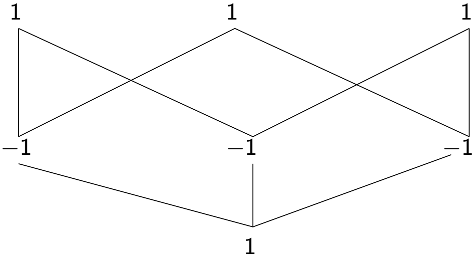

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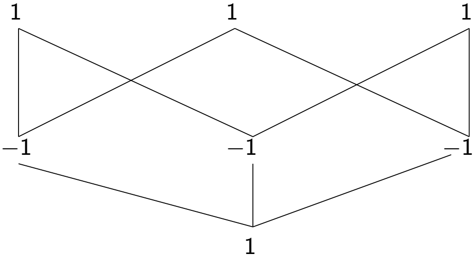

and compute the *characteristic polynomial* $\chi(\mathcal{A}, q) = \sum_{x \in L_{\mathcal{A}}} \mu(\widehat{0}, x) q^{\dim(x)} = 1 - 3q + 3q^2$.

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The numbers $r(\mathcal{A})$ and $b(\mathcal{A})$ of all, respectively bounded regions are given by

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In our example

$$r(\mathcal{A}) = (-1)^2 (1 - 3 \cdot (-1) + 3 \cdot (-1)^2) = 7$$

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Related approaches: finite field method (case of integer coefficients), Whitney's formula and the gain graph method (deformations of graphical arrangements).

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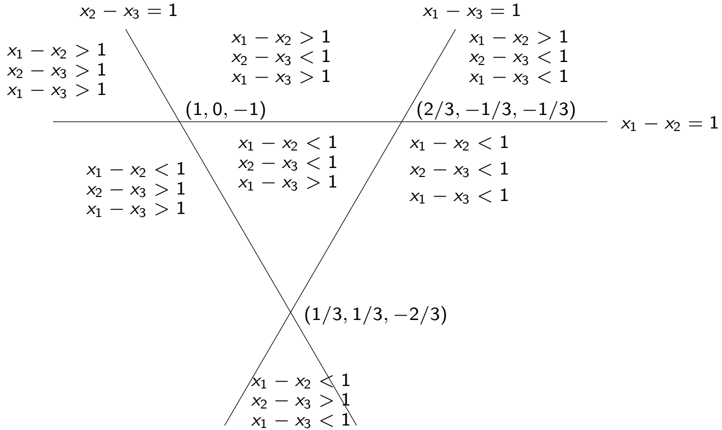

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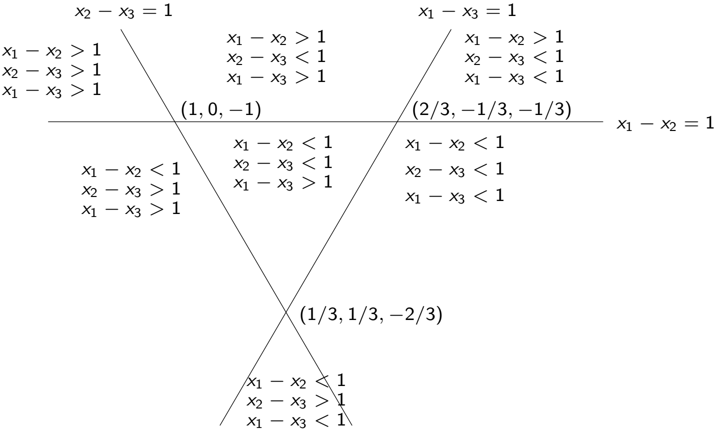
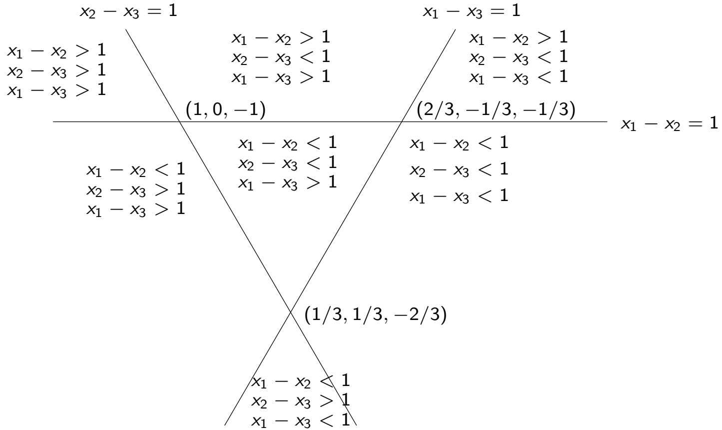


One possibility is missing:

Regions defined by sets of inequalities



$x_1 - x_2 > 1$ and $x_2 - x_3 > 1$ imply $x_1 - x_3 > 1$.

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The hyperplanes $x_i - x_j = 1 - a, 2 - a, \ldots, a$ (where $1 \leq i < j \leq n$) define the *extended Shi arrangement* in V_{n-1}, These have a *Stanley-Pak labeling* and an *Athanasiadis-Linusson labeling*.

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For a graph G on $\{1, 2, \ldots, n\}$ and a set of parameters $\{a_{i,j} \, : \, \{i,j\} \in E(G)\}$, the set of hyperplanes $\{x_i - x_j = a_{i,j} \, : \, \{i,j\} \in E(G)\}$ define a *bigraphical arrangement*.
They have a *Hopkins-Perkinson labeling*.

Two key lemmas

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The following variant of the Farkas Lemma was also used by Hopkins and Perkinson:

Lemma (Carver)

The system of inequalities $Ax < b$ *has no solution if and only if there is a nonzero real* $m \times 1$ *row vector* y *satisfying* $y \geq 0$, $yA = 0$ *and* $yb \leq 0$.

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We will apply the flow decomposition theorem to circulations:

Theorem (Gallai)

Every not identically zero circulation f can be written as a positive linear combination of directed cycles. Moreover, a directed edge e appears in at least one of these cycles if and only if $f(e) > 0$.

Weighted digraphical polytopes

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A weighted digraphical polytope is the solution set of a system of inequalities

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We create an *associated weighted digraph*: For each $i < j$, if $m_{ij} > -\infty$, we create directed edge $i \to j$ with weight m_{ij} and if $M_{ij} < \infty$ we also create a directed edge $i \leftarrow j$ with weight $-M_{ij}$.

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An *m-ascending cycle* in the associated weighted digraph is a directed cycle, along which the sum of the labels is nonnegative.
We call the associated weighted digraph *m-acyclic*, if it contains no *m-ascending cycle*.

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Proof.

(Sketch) By Carver's variant of the Farkas Lemma the polytope is empty if and only if there is an "m-ascending circulation". By the Flow Decomposition Theorem every m-ascending circulation contains an m-ascending cycle. □

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Corollary

If we think of the weight $w(e)$ as money we gain when we walk along e then the system of inequalities has a nonempty solution set if and only if we lose money along any closed walk.

Semiacyclic tournaments

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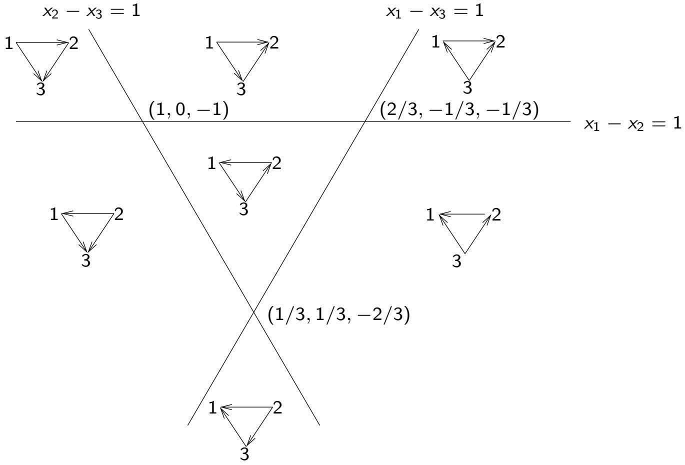

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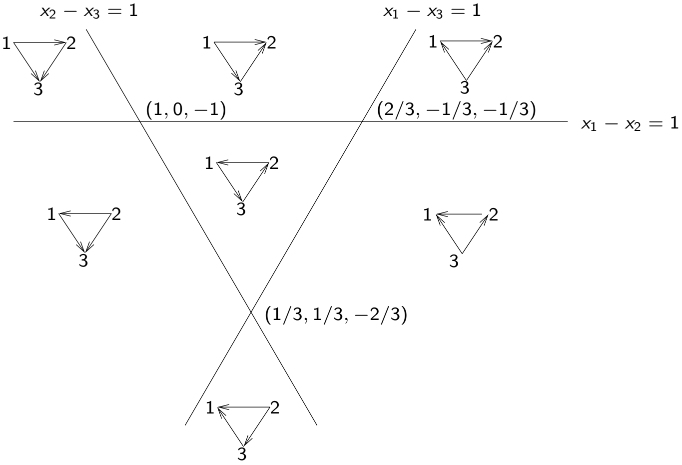

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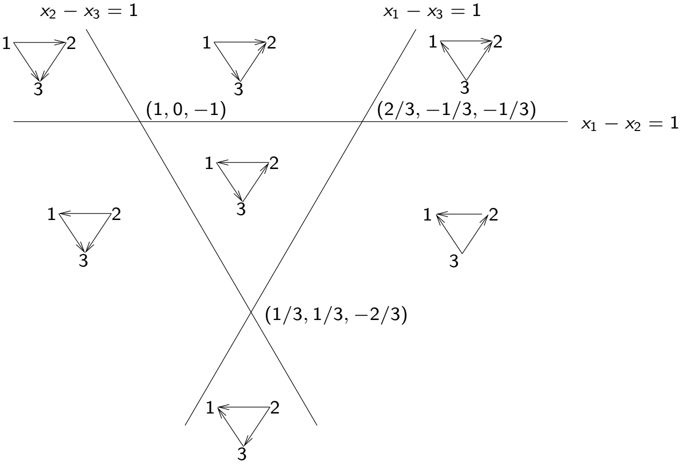

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If all arrows go from V_2 to V_1 then $(x_1, \ldots, x_n)$ may be replaced with $(x'_1, \ldots, x'_n)$ where

$$x'_v = \begin{cases} x_v + \frac{t}{|V_1|} & \text{if } v \in V_1 \\ x_v - \frac{t}{|V_2|} & \text{if } v \in V_2 \end{cases}$$

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Example

Each region of the Linial arrangement is described by a set of inequalities $\{m_{ij} < x_i - x_j < M_{ij} \ : \ 1 \leq i < j \leq n\}$*, each inequality is either* $-\infty < x_i - x_j < 1$ *or* $1 < x_i - x_j < \infty$*. The associated weighted digraph is a* tournament*, it contains no m-ascending cycle if and only if it is* semiacyclic*. Bounded regions correspond to strongly connected semiacyclic tournaments.*

Exponential arrangements

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Let $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots)$ be a sequence of deformations of the braid arrangement, such that each $\mathcal{A}_n$ is a hyperplane arrangement in $\mathbb{R}^n$. For each $S \subseteq \{1, 2, \ldots\}$ we define $\mathcal{A}_n^S$ as the subcollection of hyperplanes $x_i - x_j = c$ of $\mathcal{A}_n$ satisfying $\{i, j\} \subseteq S$. $\mathcal{A}$ is *exponential* if $r(\mathcal{A}_n^S)$ depends only on $k = |S|$ and it is the number $r(\mathcal{A}_k)$ of regions of $\mathcal{A}_k$.

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$$B_{\mathcal{A}}(t) = 1 - \frac{1}{R_{\mathcal{A}}(t)}.$$

Exponential arrangements (cont'd)

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$$r(\mathcal{A}_n) = \sum_{k=1}^{n} \sum_{\substack{n_1+\cdots+n_k=n \\ n_1,\ldots,n_k>0}} \binom{n}{n_1, n_2, \ldots, n_k} \prod_{i=1}^{k} b(\mathcal{A}_{n_i}) \quad \text{for all } n \geq 1.$$

Posets of gains

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Definition

Given a valid m-acyclic weighted digraph D on $\{1, 2, \ldots, n\}$, we define $i <_D j$ if there is a directed path $i = i_0 \to i_1 \to \cdots \to i_k = j$ such that the weight of each directed edge $i_s \to i_{s+1}$ is nonnegative. We call the set $\{1, 2, \ldots, n\}$, ordered by $<_D$ the *poset of gains induced by D*.

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Example

The posets of gains of the Linial arrangement are the *sleek posets*.

Sparse deformations

Sparse deformations

Definition

a deformation of the braid arrangement, is *sparse* if $1 \leq n_{i,j} \leq 2$ holds for all $i < j$, and the signs of the numbers $a_{i,j}^{(k)}$ satisfy the following for all $i < j$:

1. $a_{i,j}^{(1)} > 0$ holds, whenever $n_{i,j} = 1$,
2. $a_{i,j}^{(1)} < 0 < a_{i,j}^{(2)}$ holds, whenever $n_{i,j} = 2$.

We call $\mathcal{A}$ an *interval order arrangement* if $n_{i,j} = 2$ holds for all $i < j$.

Sparse deformations

Proposition

Consider a sparse deformation of the braid arrangement and any valid m-acyclic weighted digraph D associated to it. In the induced poset of gains, $i <_D j$ holds exactly when there is a single directed edge $i \to j$ of positive weight. For any pair $\{i, j\}$ of incomparable vertices satisfying $i < j$, the edge $j \to i$ is always present, and any edge between i and j has negative weight.

Sparse deformations

Theorem

Let D be a valid m-acyclic weighted digraph associated to a sparse deformation of the braid arrangement in V_{n-1}. If D is strongly connected then the incomparability graph of the induced poset of gains is connected. The converse is also true when $n_{i,j} = 2$ holds for all $1 \leq i < j \leq n$.

Sparse deformations

Example

Consider the Linial arrangement and the semiacyclic tournament D containing a directed edge $i \leftarrow j$ of weight -1 for each $i < j$. This is a valid m-acyclic weighted digraph, it is in fact acyclic. The induced poset of gains is an antichain, the incomparability graph is the complete graph, it is connected. However, D is not strongly connected.

$\underline{a}$-generalized Linial arrangements

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Definition

Let $\underline{a} = (a_1, a_2, \ldots, a_n) \in {\mathbb{R}_{\geq 0}}^n$. The $\underline{a}$-generalized Linial arrangement is

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Alternation acyclic tournaments label the regions of the *homogenized Linial arrangement* $\{x_i - x_j = y_j \ : \ 1 \leq i < j \leq n\}$.

Separated deformations

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We call a deformation of the braid arrangement $\mathcal{A}$ *separated* if 0 belongs to the set $\{a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(n_{ij})}\}$ for each $1 \leq i < j \leq n$.

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Corollary

For a separated deformation of the braid arrangement, the induced poset of gains associated to any valid m-acyclic weighted digraph is a totally ordered set.

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For a separated deformation of the braid arrangement, the induced poset of gains associated to any valid m-acyclic weighted digraph is a totally ordered set.

Equivalently, each region is included in a region $x_{\sigma(1)} > x_{\sigma(2)} > \cdots > x_{\sigma(n)}$ of the braid arrangement.

A structure theorem

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Theorem

Let $\mathcal{R}$ be a region of a separated deformation of the braid arrangement and let $\sigma(1)\sigma(2)\cdots\sigma(n)$ be its total order of gains. Then there is a unique decomposition
$\sigma = (\sigma(i_0)\cdots\sigma(i_1))\cdot(\sigma(i_1+1)\cdots\sigma(i_2))\cdots(\sigma(i_{k-1}+1)\cdots\sigma(i_k))$
satisfying

1. *For each* $j=-1,0,\ldots,k-1$, $\mathcal{R}\cap \operatorname{span}(e_{\sigma(i_j+1)},e_{\sigma(i_j+2)},\ldots,e_{\sigma(i_{j+1})})$ *is bounded.*
2. *If* $S\subseteq\{1,2,\ldots,n\}$ *contains indices* j_1 *and* j_2 *such that* $\sigma(j_1)$ *and* $\sigma(j_2)$ *belong to different subwords in the above decomposition then* $\mathcal{R}\cap \operatorname{span}((e_{\sigma(j)}\ :\ j\in S)$ *is unbounded.*

Gain functions

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For each $i \in \{1, 2, \ldots, n\}$ we define the *gain function* $g(\sigma(i))$ as the maximum weight of a directed path beginning at $\sigma(1)$ and ending at $\sigma(i)$. In particular, we set $g(\sigma(1)) = 0$. Here σ is the total order of gains.

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Lemma

Every gain function has the weakly increasing property

$$g(\sigma(1)) \leq g(\sigma(2)) \leq \cdots \leq g(\sigma(n)).$$

Gain functions

Definition

We call a deformation $\mathcal{A}$ of the braid arrangement *integral* if all the numbers $a_{i,j}^{k}$ appearing in in its definition are integers. We say that $\mathcal{A}$ satisfies the *weak triangle inequality* if for all triplets (i, j, k), the inequalities $w(i, j) \geq 0$ and $w(j, k) \geq 0$ imply

$$w(i, k) \leq w(i, j) + w(j, k) + 1$$

in any valid m-acyclic associated weighted digraph.

Gain functions

Theorem

Let $\mathcal{A}$ be a separated integral deformation of the braid arrangement satisfying the weak triangle inequality, and let D be an associated m-acyclic weighted digraph. Let σ be the total order of gains associated to D and let g be the gain function. Then, for each $i > 1$ there is a directed path from $\sigma(1)$ to $\sigma(i)$ such that all weights in the path are nonnegative and the total weight of the edges in the path is $g(\sigma(i)) - g(\sigma(1))$.

Contiguous integral deformations

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An integral deformation of the braid arrangement in V_{n-1} *contiguous* if, for every $i < j$, the set $\{a_{i,j}^{(1)}, a_{i,j}^{(2)}, \ldots, a_{i,j}^{(n_{i,j})}\}$ is a contiguous set $[\alpha(i,j), \beta(i,j)] = \{\alpha(i,j), \alpha(i,j)+1, \ldots, \beta(i,j)\}$ of integers.

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Since $x_i - x_j = c \Leftrightarrow x_j - x_i = -c$, we may set

$$\alpha(j,i) = -\beta(i,j) \quad \text{and} \quad \beta(j,i) = -\alpha(i,j) \quad \text{for } 1 \leq i < j \leq n.$$

Minimal obstructions

Minimal obstructions

Theorem

If $\beta(i,k) \leq \beta(i,j) + \beta(j,k) + 1$ holds for all $\{i,j,k\}$. then any valid associated weighted digraph is m-acyclic if and only if it contains no m-ascending cycle of length at most four.

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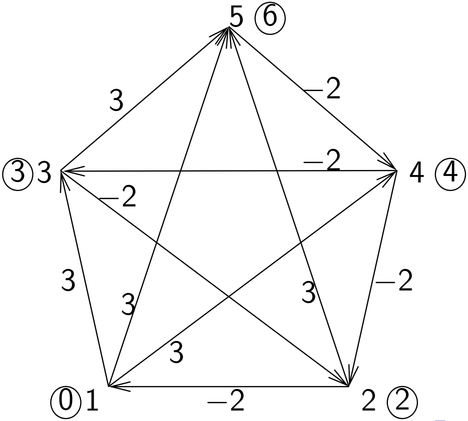
Theorem

If the truncated affine arrangement $\mathcal{A}_{n-1}^{a,b}$ satisfies $a, b \geq 0$, then a valid associated weighted digraph is m-acyclic if and only if it contains no m-ascending cycle of length at most four.

Minimal obstructions

There is a minimal m-ascending cycle of length 5 in $\mathcal{A}_{n-1}^{-1,3}$ for $n \geq 5$.



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We define the *Pak-Stanley label* $(f(1), \ldots, f(n))$ of a region as

$$f(i) = \sum_{i <_{\sigma^{-1}} j} w(i,j) + |\{(i,j) \ : \ i <_{\sigma^{-1}} j \text{ and } i > j\}|.$$

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The sum $\sum_{i <_{\sigma^{-1}} j} w(i,j)$ is the number of *separations*, and $|\{(i,j) \ : \ i <_{\sigma^{-1}} j \text{ and } i > j\}|$ is the number of *inversions*.

The Pak-Stanley labeling

Lemma (Stanley)

Given $i <_{\sigma^{-1}} j$, if $i > j$ or $w(i,j) > 0$ holds then we have $f(i) > f(j)$.

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Theorem (Stanley)

The labels of the regions of the extended Shi arrangement are the a-parking functions of length n, each occurring exactly once.

Given an a-parking function $(f(1), \ldots, f(n))$, we insert the labels i into σ one by one and show the uniqueness of the place and of the function values $w(i,j)$ one step at a time. (Still "tedious", but fits on a single page.)

The Pak-Stanley labeling

Remark

Mazin has shown that the Pak-Stanley labeling of the regions of the extended Shi arrangement is surjective. Together with Stanley's above result we have a self-contained proof of the fact that the Pak-Stanley labeling is a bijection between the regions of the regions of the extended Shi arrangement and the a-parking functions.

Athanasiadis-Linusson diagrams

Athanasiadis-Linusson diagrams

Definition

The regions of a contiguous, separated and integral deformation of the braid arrangement $\{x_i - x_j = m \,:\, 1 \leq i < j < n, m \in [-\beta(j,i), \beta(i,j)]\}$ *have Athanasiadis-Linusson diagrams* if $\{\beta(i,j) \,:\, i \neq j\}$ contains at most two consecutive nonnegative integers for each $j \in \{1, 2, \ldots, n\}$. We set $\beta(j) = \min_{i \neq j} \beta(i,j)$ for all j.

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2 1 4 3 5

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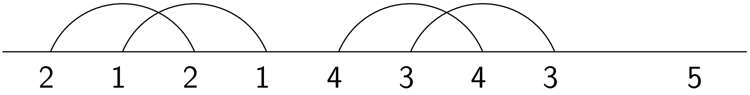

1. Fix a representative $\underline{x}$ of the region. This satisfies $x_{\sigma(1)} > x_{\sigma(2)} > \cdots > x_{\sigma(n)}$.
2. For each j satisfying $\beta(j) > 0$ we also mark $x_j + \beta(j), x_j + \beta(j) - 1, \ldots, x_j + 1$ on the reversed number line and we draw an arc connecting $x_j + k + 1$ with $x_j + k$ for $k = 0, 1, \ldots, \beta(j) - 1$. We label all of these points with j.

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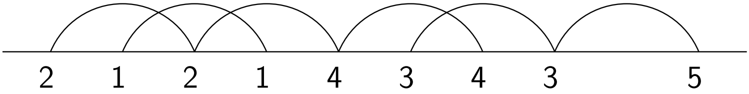

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3. For each $\{i, j\} \subseteq \{1, 2, \ldots, n\}$ we also draw an arc between x_i and $x_j + \beta(j)$ if $\beta(i, j) = \beta(j) + 1$ $x_i - x_j > \beta(i, j)$ holds.

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4. We remove all nested arcs, that is, all arcs that contain another arc.

Athanasiadis-Linusson diagrams

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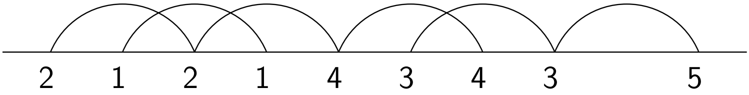

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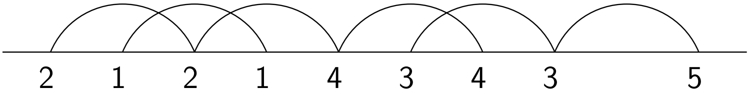
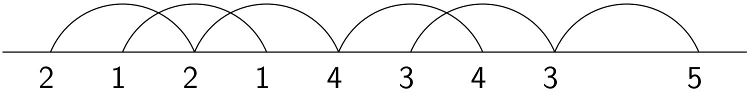


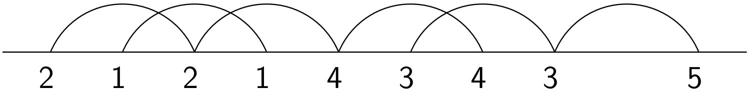
Without 5 this is an example of Athanasiadis and Linusson in $\mathcal{A}_3^{1,2}$. For all $\{i, j\} \subset \{1, 2, 3, 4\}$ we have $\beta(i, j) = 2$ if $i < j$ and $\beta(i, j) = 1$ if $i > j$. We add $\beta(i, 5) = \beta(5, i) = 0$ for $i = 1, 2, 4$, and we add $\beta(3, 5) = 1$ and $\beta(3, 5) = 0$.

Athanasiadis-Linusson diagrams



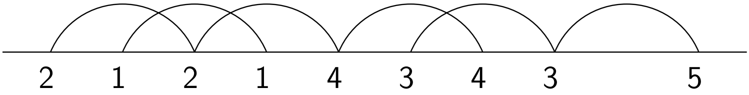
For each $i \in \{1, 2, \ldots, n\}$ we define $f(i)$ as the position of the leftmost element of the continuous component of i. We call the resulting $(f(1), f(2), \ldots, f(n))$ the β-*parking function* of the region.

Athanasiadis-Linusson diagrams



For each $i \in \{1, 2, \ldots, n\}$ we define $f(i)$ as the position of the leftmost element of the continuous component of i. We call the resulting $(f(1), f(2), \ldots, f(n))$ the *β-parking function* of the region. Here we have $f(1) = 2$, $f(2) = f(4) = 1$ and $f(3) = f(5) = 6$.

Athanasiadis-Linusson diagrams



For each $i \in \{1, 2, \ldots, n\}$ we define $f(i)$ as the position of the leftmost element of the continuous component of i. We call the resulting $(f(1), f(2), \ldots, f(n))$ the β-*parking function* of the region. Here we have $f(1) = 2$, $f(2) = f(4) = 1$ and $f(3) = f(5) = 6$. As before, we may reconstruct the diagram from its β-parking function.

Athanasiadis-Linusson trees

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4. Once we inserted the copies of all labels j satisfying $f(j) < i$, all copies of the labels j satisfying $f(j) = i$ will be the children of the node whose number is i.

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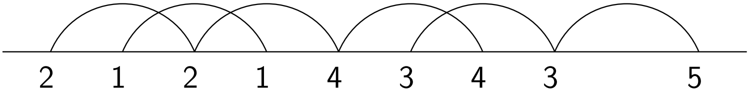

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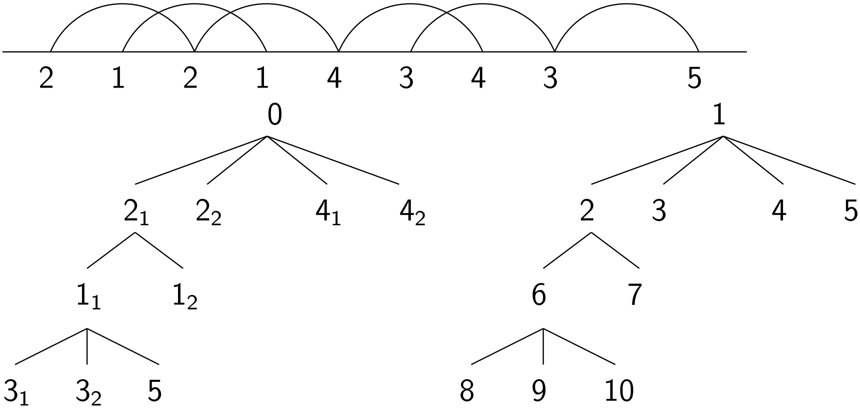

Athanasiadis-Linusson trees

Definition

For a sequence $\underline{\beta} \in \mathbb{N}^n$ we define the $\underline{\beta}$*-extended Shi arrangement* as the hyperplane arrangement

$$x_i - x_j = -\beta(j), -\beta(j) + 1, \ldots, \beta(j) + 1 \quad 1 \leq i < j \leq n \quad \text{in } V_{n-1}.$$

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The number of regions in a $\underline{\beta}$*-extended Shi arrangement* $\mathcal{A}$ *is*

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The proof uses a colored variant of the Prüfer code algorithm.

a-Catalan arrangements

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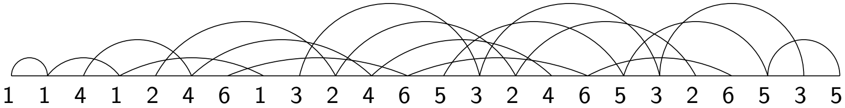

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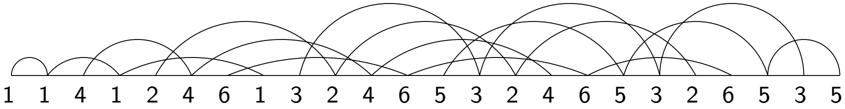

The Athanasiadis-Linusson diagrams are very simple: they connect points with the same label only.

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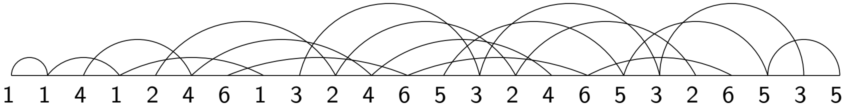

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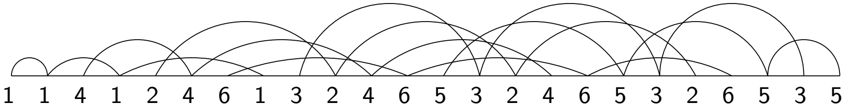

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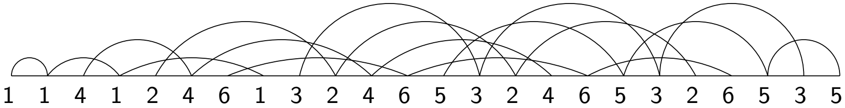

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$$r(\mathcal{A}^{a,a}_{n-1}) = an(an-1)\cdots((a-1)n+2)$$

first found by Postnikov and Stanley.

A mysterious labeling

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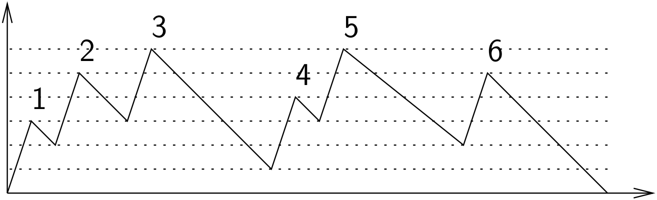

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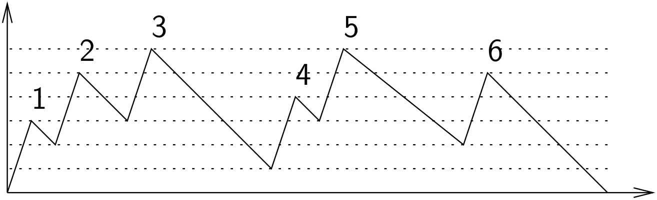

$$w(\pi(i),\pi(j)) = \begin{cases} \ell(\pi(j)) - \ell(\pi(i)) & \text{if } \ell(\pi(j)) - \ell(\pi(i)) \in [1-a, a-1] \\ -\infty & \text{if } \ell(\pi(j)) - \ell(\pi(i)) < 1-a \\ a-1 & \text{if } \ell(\pi(j)) - \ell(\pi(i)) > a-1 \end{cases}$$

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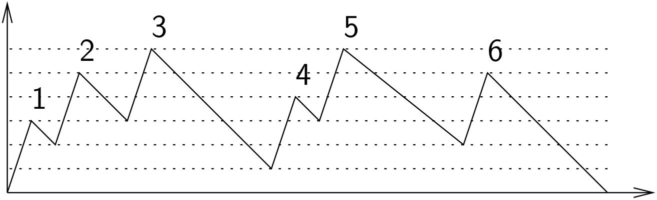

Lemma

The total order of gains $\sigma = \gamma \circ \pi$ is the order of the labels $\pi(1), \ldots, \pi(n)$ in increasing order of their levels, where $\pi(i)$ is listed before $\pi(j)$ if $\ell(\pi(i)) = \ell(\pi(j))$ and $i < j$ hold.

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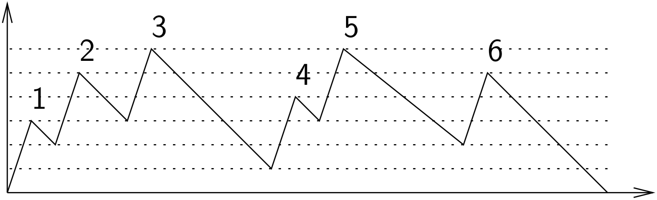

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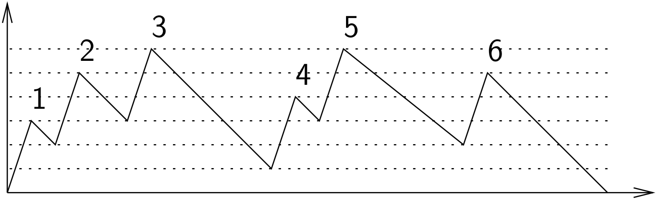

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Proposition

For the weighted digraph encoded by (π, Λ) *the gain function is the level function: we have* $g(\sigma(i)) = \ell(\sigma(i))$.

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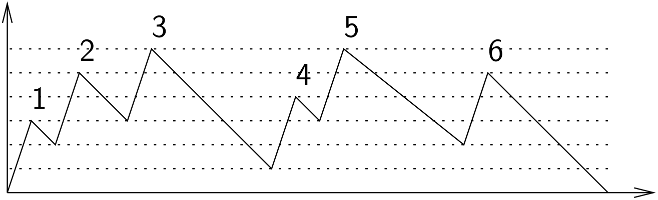

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Theorem

The correspondence between the pairs (π, Λ) and the valid weighted m-acyclic digraphs encoded by them is a bijection.

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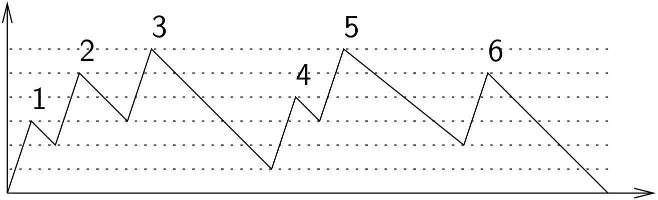

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The correspondence between the pairs (π, Λ) and the valid weighted m-acyclic digraphs encoded by them is a bijection.

We only prove injectivity and then we use the Postnikov-Stanley formula.

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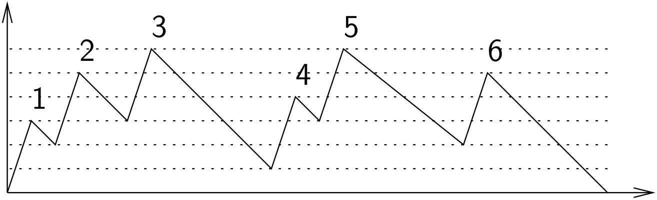

Here we get $\sigma = 142635$.

Proposition

A region of $\mathcal{A}^{a,a}_{n-1}$ is bounded if and only if the total order of gains σ satisfies $w(\sigma(i), \sigma(i+1)) < a-1$ for $1 \leq i \leq n-1$.

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For a fixed n and a fixed tree of gain functions, the number of regions of $\mathcal{A}_{n-1}^{a,a}$ associated to it is a polynomial of a.

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Conjecture

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This conjecture implies that the n-th a-Catalan number, considered as a polynomial of a, could be written as a sum of C_n polynomials, where C_n is the n-th Catalan number.

Thank you!

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Labeling regions in deformations of graphical arrangements

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`arXiv:2312.06513 [math.CO]`

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