Abstract Combinatorial Games

Arthur Holshouser
3600 Bullard St.
Charlotte, NC, USA

Harold Reiter
Department of Mathematics,
University of North Carolina Charlotte,
Charlotte, NC 28223, USA
hbreiter@email.uncc.edu

Received: ; Revised: ; Accepted:
MR Subject Classifications: 11B37,11B39, 05A10

Introduction.

The purpose of this paper is to define an abstract model for the move properties of a 2-player combinatorial game. Our model serves as a definition of a two-player combinatorial game. The two players are called Art and Beth. The model uses an abstract position set P of arbitrary cardinality. A position $x_0 \in P$ is designated the initial position, and as the game progresses, there is a sequence of positions $x_0 \to x_1 \to x_2 \to \cdots$ where $\forall i, x_i \in P$ and $x_{i+1} = x_i$ is a possibility. The moves are not specified. Instead we only specify the properties that a move $x_i \to x_{i+1}$ must have. The terms "perfect play" and "encounter" are undefined terms.

The paper deals only with the move properties. The outcome of the game is not defined. The entire theory of combinatorial games can then be developed abstractly the same way that point set topology can be derived abstractly using abstract topological spaces. This leads to new problems and challenges that do not exist with concrete games. In section 1, we define the idea of an abstract game. In section 2, we define a restricted abstract game. In section 3, we show that a concrete game is also an abstract game. In section 4, we defend the model, and in the appendix, we give an example. In later papers, we will develop the standard properties of these abstract games, including Nim values and the balanced and unbalanced positions.

Abstract Combinatorial Games 1

Let P be a set. For all x in P, we are given two non-empty sets $A(x), B(x)$ that satisfy the following conditions.

1. $A(x), B(x)$ is a partition of 2^P, the collection of all subsets of P including the empty set $\emptyset$. Thus, $\forall x \in P$, each member of $A(x), B(x)$ is a subset of P and each subset of P is a member of exactly one of $A(x), B(x)$.

2. $\forall \theta \subseteq P, \forall \psi \subseteq P$, if $\theta \subseteq \psi$ then (a), $\theta \in A(x) \Rightarrow \psi \in A(x)$. Since $A(x), B(x)$ is a partition of 2^P, and $A(x), B(x)$ are non-empty, we see that 2-b,3 are also true. (b) $\psi \in B(x) \Rightarrow \theta \in B(x)$.

3. $P \in A(x), \emptyset \in B(x)$.

   Two players, Art and Beth, play this game. The positions of the game are $x \in P$. At the beginning of the game, a position $x_0 \in P$ is designated the initial position. During the course of the game, the positions change, and this leads to a sequence of positions $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$ where $x_i = x_{i+1}$ is a possibility. The moves of the game are undefined, and we only list the properties that a move must have. Also, we will not even specify whose move it is.

   Suppose during the course of the game the position is $x_k \in P$. The move $x_k \rightarrow x_{k+1}$ has the following properties.

   The move $x_k \rightarrow x_{k+1}$ is determined by some undefined encounter (whatever that is) between Art and Beth that satisfies the following.

   (1) $x_{k+1} \in P$.

   (2) $\forall \theta \in A(x_k)$, if Art uses perfect play (whatever that is), he can force $x_{k+1} \in \theta$.

   (3) $\forall \theta \in B(x_k)$, if Beth uses perfect play, she can force $x_{k+1} \in P \backslash \theta$. That is, she can force x_{k+1} to lie outside of θ.
   In general both players interact in determining the outcome of $x_k \rightarrow x_{k+1}$. (See the appendix for an example.)

Notation 1. *We denote this game by* $(P, A(x), B(x))$.

Note that for all $\theta \in A(x_k)$, even though Art can force $x_{k+1} \in \theta$, in general he will not be able to determine precisely where inside of θ the position x_{k+1} ends up. The same is true for Beth when $\theta \in B(x_k)$ and she forces x_{k+1} to belong to $P \setminus \theta$.

Note $\forall x \in P$, the two sets $A(x), B(x)$ form a structure.

Observation $\forall x \in P$, define

$$\begin{aligned} \overline{B}(x) &= A^c(x) = \{P \backslash \theta : \theta \in A(x)\}, \\ \overline{A}(x) &= B^c(x) = \{P \backslash \theta : \theta \in B(x)\}. \end{aligned}$$

Then, $\forall x \in P, \overline{A}(x), \overline{B}(x)$ satisfies the following conditions,

1. $\overline{A}(x), \overline{B}(x)$ is a partition of 2^P.

2. $\forall\theta \subseteq P, \forall\psi \subseteq P$, if $\theta \subseteq \psi$ then

   (a) $\theta \in \overline{A}(x) \Rightarrow \psi \in \overline{A}(x)$. Therefore,

   (b) $\psi \in \overline{B}(x) \Rightarrow \theta \in \overline{B}(x)$.

3. $P \in \overline{A}(x), \emptyset \in \overline{B}(x)$.

   Also, a move $x_k \rightarrow x_{k+1}$ has the following properties.

   (1) $x_{k+1} \in P$.

   (2) $\forall\theta \in \overline{A}(x_k)$, if Beth uses perfect play, she can force $x_{k+1} \in \theta$.

   (3) $\forall\theta \in \overline{B}(x_k)$, if Art uses perfect play, he can force $x_{k+1} \in P \backslash \theta$.

Conjecture 1. *Any 2-player combinatorial game can be represented as an abstract combinatorial game* $(P, A(x), B(x))$.

However, since we cannot possibly prove this, we prefer to use this as a definition.

Definition 1. A 2-player game G is a combinatorial game if and only if there exists an abstract combinatorial game $(P, A(x), B(x))$ such that the positions and move properties of G can be represented by $(P, A(x), B(x))$.

As we illustrate later, in order to represent a 2-player combinatorial game as $(P, A(x), B(x))$, it may be necessary to extend the position set to a bigger set. The entire theory of combinatorial games can be developed abstractly using this abstract model similar to the way that point set topology can be derived abstractly using abstract topological spaces. However, it is usually more expedient to use other equivalent models which we do not discuss here.

Restricted Abstract Combinatorial Games 2.

Suppose $\overline{P}$ is an abstract set of arbitrary cardinality. For all $x \in \overline{P}$, we are given three sets $\overline{S}(x), \overline{A}(x), \overline{B}(x)$ that satisfy the following conditions.

1. $\overline{S}(x) \subseteq \overline{P}, \overline{S}(x) \neq \emptyset$, and $\overline{A}(x), \overline{B}(x)$ is a partition of $2^{\overline{S}(x)}$.

2. $\forall\theta \subseteq \overline{S}(x), \forall\psi \subseteq \overline{S}(x)$ if $\theta \subseteq \psi$ then

   (a) $\theta \in \overline{A}(x) \Rightarrow \psi \in \overline{A}(x)$. Therefore,

   (b) $\psi \in \overline{B}(x) \Rightarrow \theta \in \overline{B}(x)$.

3. $\overline{S}(x) \in \overline{A}(x), \emptyset \in \overline{B}(x)$.

As before, Art and Beth play the games. The positions of the game are $x \in \overline{P}$. $x_0 \in \overline{P}$ is designated the initial position, and during the course of the game the positions change and this leads to a sequence of positions $x_0 \to x_x \to x_2 \to \cdots$. The moves are undefined, but a move has the following properties. Also, we do not specify whose move it is. Suppose during the course of the game the position is $x_k \in \overline{P}$. The move $x_k \to x_{k+1}$ has the following properties.

(1) $x_{k+1} \in \overline{S}(x_k)$.

(2) $\forall \theta \in \overline{A}(x_k)$, if Art uses perfect play, he can force $x_{k+1} \in \theta$.

(3) $\forall \theta \in \overline{B}(x_k)$, if Beth uses perfect play, she can force $x_{k+1} \in \overline{S}(x_k) \backslash \theta$.

Notation 2. *We denote this game by* $\left(\overline{P}, \overline{S}(x), \overline{A}(x), \overline{B}(x)\right)$.

Lemma 1. *A restricted abstract combinatorial game* $\left(\overline{P}, \overline{S}(x), \overline{A}(x), \overline{B}(x)\right)$ *is an abstract combinatorial game* $\left(\overline{P}, \overline{A}(x), \overline{B}(x)\right)$.

Proof. For all $x \in \overline{P}$, let us define

$$\begin{aligned} A(x) &= \left\{\theta : \theta \subseteq \overline{P} \text{ and } \theta \cap \overline{S}(x) \in \overline{A}(x)\right\}, \\ B(x) &= \left\{\theta : \theta \subseteq \overline{P} \text{ and } \theta \cap \overline{S}(x) \in \overline{B}(x)\right\}. \end{aligned}$$

It is easy to see that $\left(\overline{P}, \overline{S}(x), \overline{A}(x), \overline{B}(x)\right)$ is also the abstract combinatorial game $\left(\overline{P}, \overline{A}(x), \overline{B}(x)\right)$ □

Applications 3.

A Concrete Combinatorial Game Suppose $\overline{P}$ is an abstract set of arbitrary cardinality. For all $x \in \overline{P}$, non-empty sets $S_a(x) \subseteq \overline{P}, S_b(x) \subseteq \overline{P}$ are given. Two players called Art and Beth alternate moving. Suppose $x_0 \subseteq \overline{P}$ is designated the initial position. As always, the sequence of positions is $x_0 \to x_1 \to x_2 \to \cdots$.

If Art moves first, then Art will make the moves $x_0 \to x_1, x_2 \to x_3, \cdots$ and Beth will make the moves $x_1 \to x_2, x_3 \to x_4, \cdots$ and vice-versa if Beth moves first.

If $x_k \in \overline{P}$ and Art makes the move $x_k \to x_{k+1}$, then $x_{k+1} \in S_a(x_k)$ and x_{k+1} can be anything that Art chooses.

If $x_k \in \overline{P}$ and Beth makes the move $x_k \to x_{k+1}$, then $x_{k+1} \in S_b(x_k)$ and x_{k+1} can be anything that Beth chooses.

Abstract Model of Game Define $P_a = \overline{P} \times \{a\} = \left\{(x,a) : x \in \overline{P}\right\}, P_b = \overline{P} \times \{b\} = \left\{(x,b) : x \in \overline{P}\right\}$, and $P = P_a \cup P_b$. P is the position set.

If Art moves first, the move sequence is $(x_0, a) \to (x_1, b) \to (x_2, a) \to (x_3, b) \to \cdots$. If Beth moves first, the move sequence is $(x_0, b) \to (x_1, a) \to (x_2, b) \to (x_3, a) \to \cdots$. Also, if $(x_k, a) \in P_a$, Art can move $(x_k, a) \to (x_{k+1}, b)$ where $x_{k+1} \in S_a(x_k)$ is any element that he chooses. Also, if $(x_k, b) \in P_b$, Beth can move $(x_k, b) \to (x_{k+1}, a)$ where $x_{k+1} \in S_b(x_k)$ is any element that she chooses.

For all x in P, we need to define $A(x), B(x)$ so that the abstract combinatorial game $(P, A(x), B(x))$ is a model of this game. The easy proof is left to the reader. If $x = (\overline{x}, a), \overline{x} \in \overline{P}$, define

$$\begin{aligned} A(x) &= A(\overline{x}, a) = \{\theta : \theta \subseteq P \text{ and } \theta \cap [S_a(\overline{x}) \times \{b\}] \neq \emptyset\} \text{ and} \\ B(x) &= B(\overline{x}, b) = \{\theta : \theta \subseteq P \text{ and } \theta \cap [S_a(\overline{x}) \times \{b\}] = \emptyset\}. \end{aligned}$$

If $x = (\overline{x}, b), \overline{x} \in \overline{P}$, define

$$\begin{aligned} A(x) &= A(\overline{x}, b) = \{\theta : \theta \subseteq P \text{ and } [S_b(\overline{x}) \times \{a\}] \subseteq \emptyset\}, \\ B(x) &= B(\overline{x}, b) = \{\theta : \theta \subseteq P \text{ and } [S_b(\overline{x}) \times \{a\}] \nsubseteq \emptyset\}. \end{aligned}$$

Defending the model 4.

Abstract Combinatorial Games with a Memory. P is an abstract set of arbitrary cardinality. Define $F = \{(x_0, x_1, x_2, \cdots, x_k) : \forall i = 0, 1, 2, \cdots, k, x_i \in P\}$. Thus F is the collection of all finite sequences in P.

For all $(x_0, x_1, \cdots, x_k) \in F$, we are given two non-empty sets $A(x_0, x_1, \cdots, x_k), B(x_0, x_1, \cdots, x_k)$ that satisfy the following conditions.

1. $A(x_0, x_1, \cdots, x_k), B(x_0, x_1, \cdots, x_k)$ is a partition of 2^P.
2. $\forall \theta \subseteq P, \forall \psi \subseteq P$ if $\theta \subseteq \psi$ then
   (a) $\theta \in A(x_0, x_1, \cdots, x_k) \Rightarrow \psi \in A(x_0, x_1, \cdots, x_k)$,
   (b) Therefore, $\psi \in B(x_0, x_1, \cdots, x_k) \Rightarrow \theta \in B(x_0, x_1, \cdots, x_k)$.
3. $P \in A(x_0, x_1, \cdots, x_k), \emptyset \in B(x_0, x_1, \cdots, x_k)$.

Two players, Art and Beth, play this game. The positions of the game are $x \in P$. At the beginning of the game, a position $x_0 \in P$ is designated the initial position.

During the course of the game, the positions change and this leads to a sequence of positions $x_0 \to x_1 \to x_2 \to \cdots$. The moves are undefined, and we only list the properties that a move must have, and again we will not specify whose move it is.

Suppose during the course of the game, the position is $x_k \in P$, and the move sequence is $x_0 \to x_1 \to x_2 \to \cdots \to x_k$.

The move $x_k \to x_{k+1}$ is determined by some undefined encounter between Art and Beth that has the following properties.

(1) $x_{k+1} \in P$.

(2) $\forall \theta \in A(x_0, x_1, \cdots, x_k)$, if Art uses perfect play, he can force $x_{k+1} \in \theta$.

(3) $\forall \theta \in B(x_0, x_1, \cdots, x_k)$, if Beth uses perfect play, he can force $x_{k+1} \in P \backslash \theta$.

Lemma 2. *An abstract combinatorial game with a memory can be represented as an abstract combinatorial game.*

Proof. We will first show that an abstract combinatorial game with a memory can be represented as a restricted abstract combinatorial game. Lemma 1 will then complete the proof.

Define $\overline{P} = F = \{(x_0, x_1, \cdots, x_k) : \forall i = 0, 1, \cdots, k, x_i \in P\}$. Thus, the position set $\overline{P}$ is the collection of all finite (move) sequences $(x_0, x_1, \cdots, x_k)$. $\forall (x_0, x_1, \cdots, x_k) \in \overline{P}$, we first define the three sets $\overline{S}(x_0, x_1, \cdots, x_k), \overline{A}(x_0, x_1, \cdots, x_k), \overline{B}(x_0, x_1, \cdots, x_k)$.

(a) $\overline{S}(x_0, x_1, \cdots, x_k) = \{(x_0, x_1, \cdots, x_k, x_{k+1}) : x_{k+1} \in P\} = \{(x_0, x_1, \cdots, x_k)\} \times P$.

(b) $\overline{A}(x_0, x_1, \cdots, x_k) = \{\{(x_0, x_1, \cdots, x_k)\} \times \theta : \theta \in A(x_0, x_1, \cdots, x_k)\}$.

(c) $\overline{B}(x_0, x_1, \cdots, x_k) = \{\{(x_0, x_1, \cdots, x_k)\} \times \theta : \theta \in B(x_0, x_1, \cdots, x_k)\}$.

It is very easy to show that $(\overline{P}, \overline{S}, \overline{A}, \overline{B})$ satisfies the 3 conditions 1, 2, 3 of the restricted abstract combinatorial games.

This defines the restricted abstract combinatorial game $\left(\overline{P}, \overline{S}(x), \overline{A}(x), \overline{B}(x)\right)$ where $x \in \overline{P}$.

Let us now consider the following game.

As always, Art and Beth play the games. Let the positions of the game be $(x_0, x_1, x_2, \cdots, x_k) \in F$ and let $(x_0) \in F$ be the initial position. Suppose during the course of the game the position is $(x_0, x_1, x_2, \cdots, x_k) \in F$. Let us agree that a move in this game is $(x_0, x_1, \cdots, x_k) \to (x_0, x_1, \cdots, x_k, x_{k+1})$ and that this is just an alternate way of expressing the move $x_k \to x_{k+1}$ in the original game. From this, it is obvious that the moves in this game, $(x_0, x_1, \cdots, x_k) \to (x_0, x_1, \cdots, x_k, x_{k+1})$, have the following properties.

(1) $(x_0, x_1, \cdots, x_k, x_{k+1}) \in \overline{S}(x_0, x_1, \cdots, x_k)$.

(2) $\forall \theta \in \overline{A}(x_0, x_1, \cdots, x_k)$, if Art uses perfect play, he can force $(x_0, x_1, \cdots, x_{k+1}) \in \theta$.

(3) $\forall \theta \in \overline{B}(x_0, x_1, \cdots, x_k)$, if Beth uses perfect play, he can force $(x_0, x_1, \cdots, x_{k+1}) \in \overline{S}(x_0, \cdots, x_k) \backslash \theta$.

Therefore, this game satisfies $\left(\overline{P}, \overline{S}(x), \overline{A}(x), \overline{B}(x)\right)$. Lemma 1 completes the proof. □

Appendix 5.

The general model used in the abstract combinatorial games is very useful in studying the following types of games. This also illustrates what an encounter between Art and Beth can mean.

Rules of Game Let $P = z$ be the set of all integers. Art and Beth alternate moving. The game is impartial (or symmetric) which means that both players have the same moves available. Let $x_0 \in z$ be the initial position. By symmetry suppose it is Art's move facing $x_k \in z$. Art's move $x_k \to x_{k+1}$ is defined as the following k steps.

1. If $x_k = y_0$ is even, Art subtracts $x \in \{1, 2, 3\}$ from x_k giving $x_k - x = y_1$. If x_k is odd, Art subtracts $x \in \{1, 2, 5\}$ giving $x_k - x = y_1$.

2. Beth has the option of subtracting x from y_1 giving $y_1 - x = y_2$, where x is the number Art chose in Step 1.

3. Beth has the option of blocking one of Art's options $\overline{x} \in \{1, 3, 7\}$.

4. Art subtracts $\overline{x}$ from y_2 giving $y_3 = y_2 - \overline{x}$ where $\overline{x} \in \{1, 3, 7\}$ and $\overline{x}$ is not blocked by Beth.

5. Art subtracts $x' \in \{x, 2x, 5x\}$ from y_3 giving $y_4 = y_3 - x'$, where x is the number that Art chose on step 1.

6. Beth subtracts $x^* \in \{x', 2x', 8x'\}$ from y_4 giving $y_5 = y_4 - x^*$, where x' is the number Art chose on Step 5,

$\vdots$

$(k-1)$.

(k) .

Art subtracts from the number in step $k-1$ the sum of the number $x, \overline{x}$ where $x, \overline{x}$ are the numbers chosen by Art in Steps 1, 4. At the end of these k steps, Art's move is over. If it is Beth's move and Beth is facing $x_k \in z$, the words Art and Beth are interchanged in the k steps defining Art's move.