Apollonius Problems

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Several years ago one of the authors (SM) posed in ΠME journal the following problem (#1136, Vol. 12 & 6, Spring 2006): Four planar circles are pair-wise externally tangent. Three of the circles are also tangent to a line L. If the radius of the fourth circle is one unit, what is the distance of its center from L?

The problem belongs to the class of the problems on tangent circles, which we will call

Apollonius type problems, named after the great Greek geometer Apollonius. As an example using a compass and straight-edge, construct the circle, tangent to each of three given circles. Among other questions, we would like to know how many solutions are there.

The Apollonius constructions (including the inversion transform) is the basis for the study of special class of the fractals sets, so called, *Apollonius carpets*. See [1] which contains rich information about such carpets. The following from [1] is a good example:

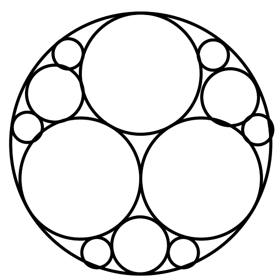

Fig. 1

We use repeatedly here a class of mappings of $R^2 \cup \{\infty\}$ to $R^2 \cup \{\infty\}$ called *circular inversions* $R^2 \cup \{\infty\} \to R^2 \cup \{\infty\}$ (with respect to the unit circle):

$$\begin{aligned}
\vec{x} &= (x, y) \to I(\vec{x}) = (x_1, y_1) \qquad \text{(when } (x,y) \neq (0,0)\text{)} \\
x_1 &= \frac{x}{x^2+y^2},\ y_1 = \frac{y}{x^2+y^2}.\ \text{Also, } I(0,0) = \infty, I(\infty) = (0,0) \\
x_1^2 + y_1^2 &= \frac{1}{x^2+y^2},\ x = \frac{x_1}{x_1^2+y_1^2},\ y = \frac{y_1}{x_1^2+y_1^2}
\end{aligned}$$

Thus, $I\left(I\left(\vec{x}\right)\right) = I^2\left(\vec{x}\right) = \vec{x}$.

Alternate version: $I : R^2 \cup \{\infty\} \to R^2 \cup \{\infty\}$ defined by

$$I(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right) \text{(when } (x, y) \neq (0, 0))$$

and

$$I(0, 0) = \infty, I(\infty) = (0, 0).$$

Theorem 1 *Inversion transforms circles into circles (we consider the straight line as a circle of the infinite radius).*

Let's prove this theorem using analytic geometry. Put $C\left(\vec{a}, r\right) = \left\{\vec{x} \in R^2 : |\vec{x} - \vec{a}| = r\right\}, \vec{a} = (a, b)$ is the center of the circle, $r > 0$ is its radius, $\frac{1}{r} = c$ is the curvature. Then $(x - a)^2 + (y - b)^2 = r^2$ is the equation of $C\left(\vec{a}, \vec{r}\right) \Rightarrow x^2 + y^2 - (2ax + 2by) = r^2 - (a^2 + b^2)$. Assume that $|\vec{a}| = \sqrt{a^2 + b^2} > r$. The analysis when $\sqrt{a^2 + b^2} < r$ and $\sqrt{a^2 + b^2} = r$ is very similar.

The transformed equation has a form

$$\begin{aligned}
\frac{1}{x_1^2 + y_1^2} - \frac{2ax_1 + 2by_1}{x_1^2 + y_1^2} &= -\rho^2 = r^2 - (a^2 + b^2) \\
\rho^2\left(x_1^2 + y_1^2\right) - (2ax_1 + 2by_1) + 1 &= 0 \\
x_1^2 + y_1^2 - 2\left(\frac{a}{\rho^2}x_1 + \frac{b}{\rho^2}y_1\right) + \frac{1}{\rho^2} &= 0 \\
\left(x_1 - \frac{a}{\rho^2}\right)^2 + \left(y_1 - \frac{b}{\rho^2}\right)^2 &= \frac{a^2 + b^2 - \rho^2}{\rho^4} = \frac{r^2}{\rho^4}.
\end{aligned}$$

This is a new circle.

$$\overline{C}\left(\overrightarrow{\overline{a}}, \overline{r}\right), \overline{a} = \frac{(a, b)}{\rho^2}, \overline{r} = \frac{r}{\rho^2}$$

If $|\overrightarrow{a}| = \sqrt{a^2 + b^2} < r$, $\rho^2 = r^2 - (a^2 + b^2)$, the new circle is $\overline{C}(\overrightarrow{\overline{a}}, \overline{r}), \overline{a} = -\frac{(a,b)}{\rho^2}, \overline{r} = \frac{r}{\rho^2}$.

In case $|\overrightarrow{a}| = \sqrt{a^2+b^2} = r$, the new circle is the line $2ax_1 + 2by_1 = 1$. Of course, $|\overrightarrow{a}| = \sqrt{a^2+b^2} = r$ means that the circle $C(\overrightarrow{a}, r)$ passes through the origin $(0,0)$.

Note that the center $(\frac{a}{\rho^2}, \frac{b}{\rho^2})$ of the new circle is not the image of the center of the old circle under inversion.

Before stating and proving Theorem 2, we restate Theorem 1. In Theorem 1, suppose circle (O_1, R_1) inverts into circle $(\overline{O_1}, \overline{R_1})$, where $O_1, R_1, \overline{O_1}, \overline{R_1}$ are the centers and radii of the two circles. Of course, circle $(\overline{O_1}, \overline{R_1})$ inverts into circle (O_1, R_1). Let D_1 and $\overline{D_1}$ be the distances from the origin to O_1 and to $\overline{O_1}$ respectively. From Theorem 1, we know that $D_1 > R_1$ if and only if $\overline{D_1} > \overline{R_1}$. Also, if $D_1 > R_1, \overline{D_1} > \overline{R_1}$, then

$$\overline{D_1} = \frac{D_1}{D_1^2 - R_1^2}, \qquad D_1 = \frac{\overline{D_1}}{\overline{D_1}^2 - \overline{R_1}^2}$$

$$\overline{R_1} = \frac{R_1}{D_1^2 - R_1^2}, \qquad R_1 = \frac{\overline{R_1}}{\overline{D_1}^2 - \overline{R_1}^2}$$

Theorem 2 *The five circles $(O_1, R_1), (O_2, R_2), (O_3, R_3), (O_4, R_4), (O_5, R_5)$ are each tangent to one another as shown in Fig. 2. Also, $C_1 = \frac{1}{R_1}, C_2 = \frac{1}{R_2}, C_3 = \frac{1}{R_3}, C_4 = \frac{1}{R_4}, C_5 = \frac{1}{R_5}$ are the curvatures of the five circles. Then*

(a) $C_4 = C_1 + C_2 + C_3 + 2\sqrt{C_1C_2 + C_1C_3 + C_2C_3}$ *and*

(b) $C_5 = C_1 + C_2 + C_3 - 2\sqrt{C_1C_2 + C_1C_3 + C_2C_3}$

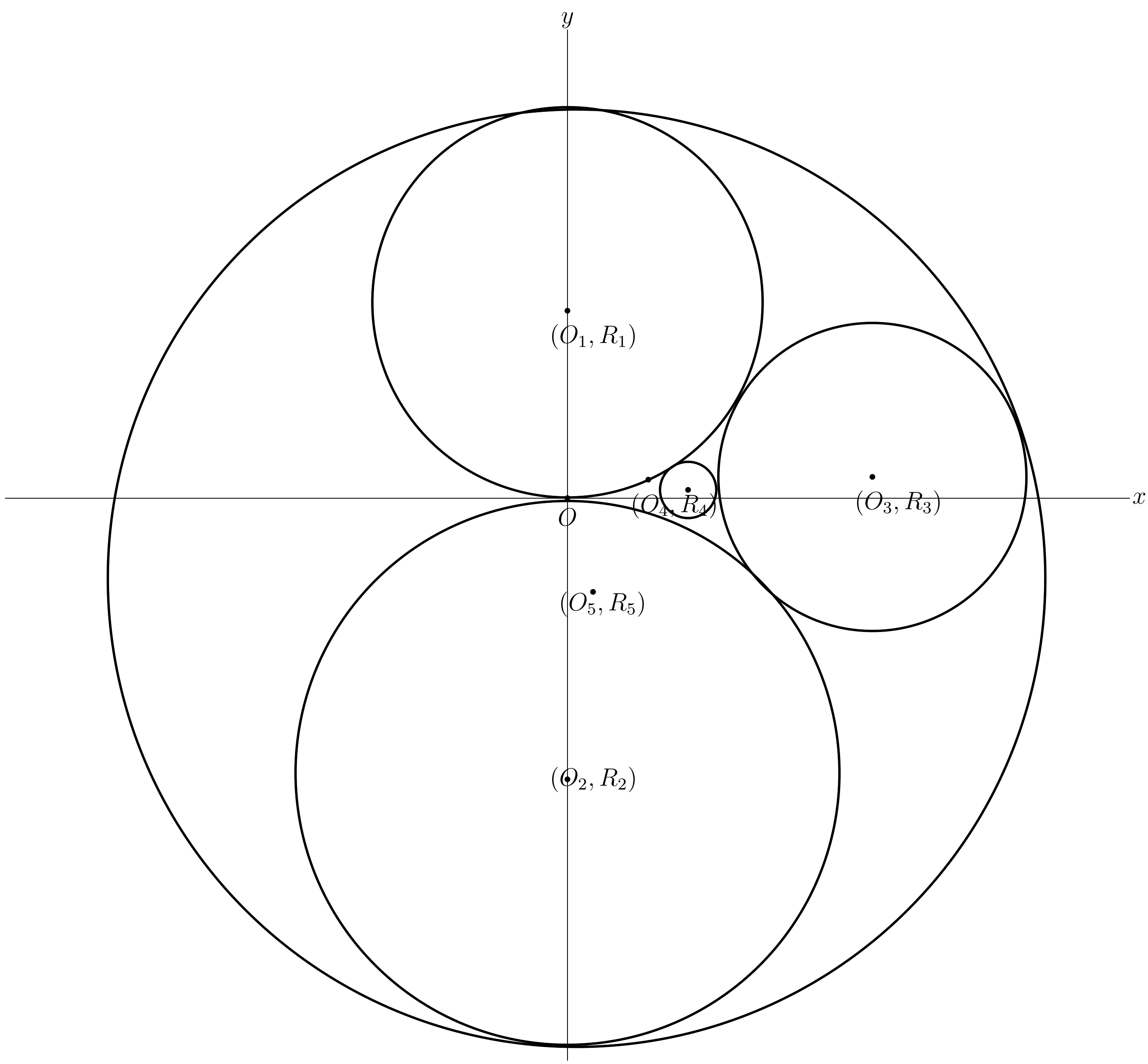

Fig.2

Note 1. From (a) and (b) it is easy to show that C_4, C_5 are the roots of the equation

$$(x + C_1 + C_2 + C_3)^2 = 2(C_1^2 + C_2^2 + C_3^2 + x^2),$$

which is called Descartes equation. If $C_5 < 0$, then the circles $(O_1, R_1), (O_2, R_2), (O_3, R_3), (O_4, R_4)$ lie inside the circle (O_5, R_5). If $C_5 > 0$, then the circles $(O_1, R_1), (O_2, R_2), (O_3, R_3), (O_4, R_4)$ lie outside the circle (O_5, R_5). If $C_5 = 0$, then the circle (O_5, R_5) is a straight line that is tangent to the three circles $(O_1, R_1), (O_2, R_2), (O_3, R_3)$.

Proof. We prove formula (a). The proof of formula (b) is nearly identical. The proof uses nothing more that Theorem 1 and the Pythagorean Theorem. Inverting the circles $(O_1, R_1), (O_2, R_2), (O_3, R_3)$ and (O_4, R_4) of Figure 2 with respect to the unit circle $x^2+y^2 = 1$, we have Figure 3, where the circles (O_1, R_1) and (O_2, R_2) invert into straight lines $y = \frac{C_1}{2}$ and $y = -\frac{C_2}{2}$ respectively, circle (O_3, R_3) inverts into circle $(\overline{O_3}, \overline{R_3})$ and the circle (O_4, R_4) inverts into $(\overline{O_4}, \overline{R_4})$. Tangency is preserved.

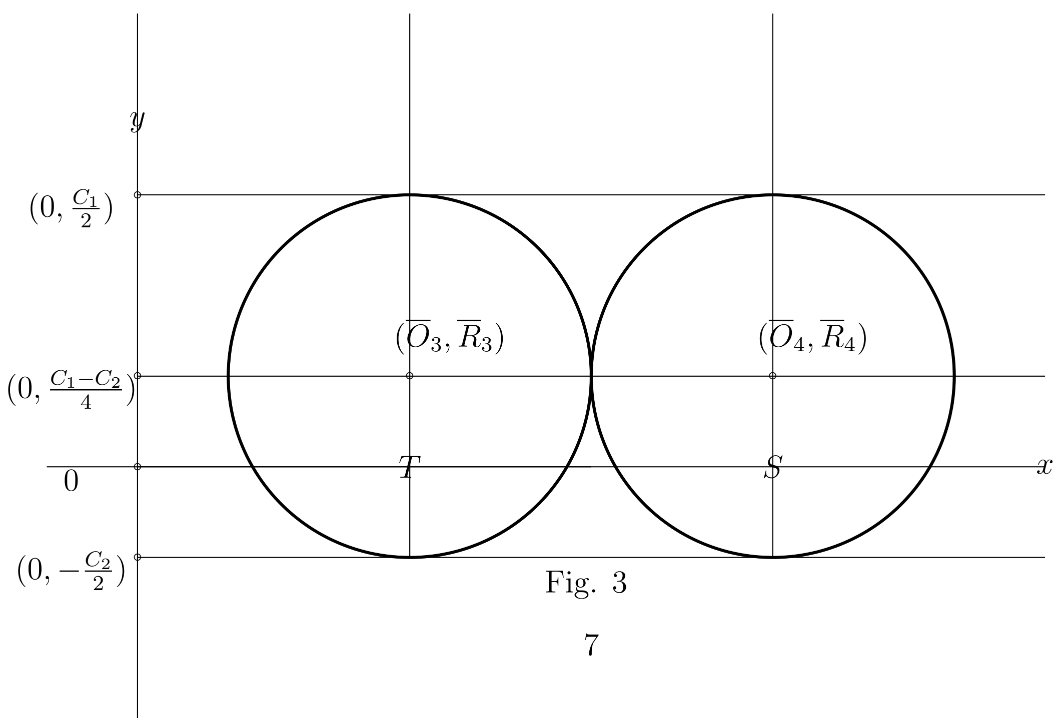
y
$(0, \frac{C_1}{2})$
$(\overline{O}_3, \overline{R}_3)$
$(\overline{O}_4, \overline{R}_4)$
$(0, \frac{C_1 - C_2}{4})$
0
T
S
x
$(0, -\frac{C_2}{2})$

Fig. 3

In Fig. 2 and Fig. 3, the points $O, O_3, \overline{O}_3$ are colinear and $O, O_4, \overline{O}_4$ are colinear. In Fig.3, note that $\overline{R}_3 = \overline{R}_4 = \frac{C_1+C_2}{4}, TS = \overline{R}_3 + \overline{R}_4 = \frac{C_1+C_2}{2}$, $T\overline{O}_3 = S\overline{O}_4 = \frac{C_1-C_2}{4}$. In Fig.2 we denote $OO_3 = D$ and in Fig. 3 we denote $O\overline{O}_3 = \overline{D}, O\overline{O}_4 = \overline{d}$. From the inversion $(O_3, R_3) \leftrightarrow (\overline{O}_3, \overline{R}_3)$, we know from Theorem 1 that

$$\overline{R}_3 = \frac{C_1 + C_2}{4} = \frac{R_3}{D^2 - R_3^2}.$$

Therefore,

$$(**) \qquad D^2 = R_3^2 + \frac{4R_3}{C_1 + C_2}.$$

Also from $(O_3, R_3) \leftrightarrow (\overline{O}_3, \overline{R}_3)$, using Theorem 1 and by $(**)$, we know that

$$\overline{D} = \frac{D}{D^2 - R_3^2} = \frac{D(C_1 + C_2)}{4R_3}.$$

Now from Fig. 3, $\overline{D}^2 = OT^2 + T\overline{O}_3^2 = OT^2 + \left(\frac{C_1-C_2}{4}\right)^2$. Therefore,

$$\begin{aligned} OT^2 &= \overline{D}^2 - \left(\frac{C_1 - C_2}{4}\right)^2 \\ &= \frac{D^2(C_1 + C_2)^2}{16R_3^2} - \left(\frac{C_1 - C_2}{4}\right)^2, \text{using } (**), \\ &= \left[R_3^2 + \frac{4R_3}{C_1 + C_2}\right]\left(\frac{(C_1 + C_2)^2}{16R_3^2}\right) - \left(\frac{C_1 - C_2}{4}\right)^2 \\ &= \frac{C_1 + C_2}{4R_3} + \frac{(C_1 + C_2)^2}{16} - \left(\frac{C_1 - C_2}{4}\right)^2 \\ &= \frac{C_1C_2 + C_1C_3 + C_2C_3}{4} \end{aligned}$$

since $1/R_3 = C_3$. Therefore

$$OT = \frac{\sqrt{C_1C_2 + C_1C_3 + C_2C_3}}{2}.$$

From Fig 3,

$$
\begin{aligned}
O\overline{O}_4^2 &= \overline{d}^2 = (OT+TS)^2 + (S\overline{O}_4)^2 \\
&= \left(OT + \frac{C_1+C_2}{2}\right)^2 + \left(\frac{C_1-C_2}{4}\right)^2 \\
&= OT^2 + (C_1+C_2)OT + \left(\frac{C_1+C_2}{2}\right)^2 + \left(\frac{C_1-C_2}{4}\right)^2 .
\end{aligned}
$$

Using $OT = \frac{\sqrt{C_1C_2+C_1C_3+C_2C_3}}{2}$, we have

$$
\overline{d}^2 = \frac{5}{16}(C_1+C_2)^2 + \frac{C_3}{4}(C_1+C_2) + \frac{(C_1+C_2)\sqrt{C_1C_2+C_1C_3+C_2C_3}}{2}
$$

after simplifying. From the inversion $(O_4, R_4) \leftrightarrow (\overline{O}_4, \overline{R}_4)$, we have

$$
R_4 = \frac{\overline{R}_4}{\overline{d}^2 - \overline{R}_4^2} = \frac{\left(\frac{C_1+C_2}{4}\right)}{\overline{d}^2 - \left(\frac{C_1+C_2}{4}\right)^2}.
$$

Therefore,

$$
\begin{aligned}
C_4 &= \frac{1}{R_4} = \frac{\overline{d}^2 - \left(\frac{C_1+C_2}{4}\right)^2}{\frac{C_1+C_2}{4}} \\
&= \frac{4\overline{d}^2}{C_1+C_2} - \frac{C_1+C_2}{4} \qquad \text{using the formula for } \overline{d}^2 \\
&= \frac{5}{4}(C_1+C_2) + C_3 + 2\sqrt{C_1C_2+C_1C_3+C_2C_3} - \frac{C_1+C_2}{4} \\
&= C_1 + C_2 + C_3 + 2\sqrt{C_1C_2+C_1C_3+C_2C_3},
\end{aligned}
$$

which is what we needed to prove. To prove formula (b), we simply note that circle (O_5, R_5) in Fig 2 inverts into circle $(\overline{O}_5, \overline{R}_5)$, where $(\overline{O}_5, \overline{R}_5)$ lies to the left of circle $(\overline{O}_3, \overline{R}_3)$ in Fig 3.

Applying theorem 1 to the inversion $O_3, R_3) \leftrightarrow (\overline{O_3}, \overline{R_3})$, we get

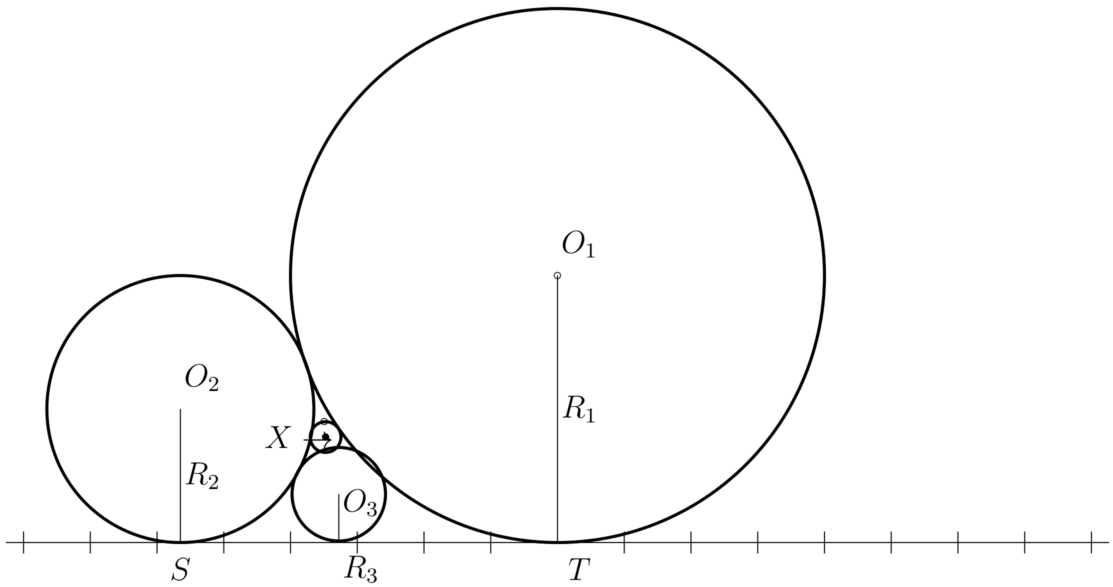

Fig. 4

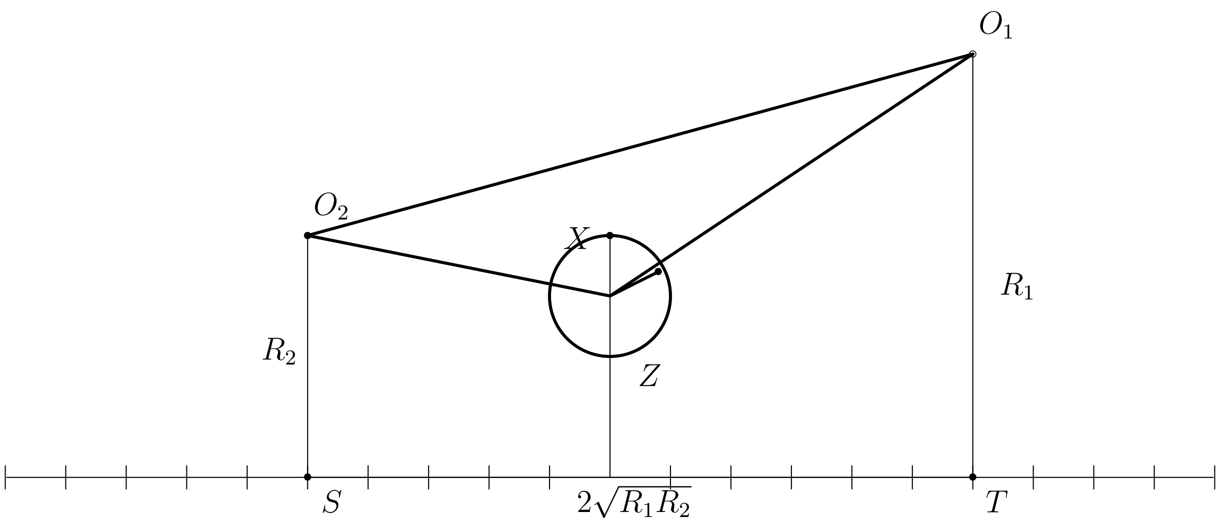

Fig. 5

Solution of 1136.

In Fig. 4, we note that circle (O_3, R_3) lies inside of the triangular shaped region created by circles $(O_1, R_1), (O_2, R_2)$ and the line ST. In the case where circle (O_3, R_3) lies outside of this triangular shaped region, the answer to the problem is the same and the proof is nearly the same.

An important observation is that the picture is not definite. We know only that $x = 1$ and have to find z. It is clear physically that one can vary R_1 and R_2 in such a way that

$x = 1$ (we will derive the corresponding equation later). If in fact z depends only on x ($x = 1$) then to guess the answer we can consider the special case $R_1 = R_2 = 1 \Rightarrow R_3 = \frac{1}{4}; x = \frac{1}{12}$ (see Fig. 4)$\Rightarrow z = \frac{1}{12} + \frac{1}{4} + \frac{1}{4} = \frac{7}{12} \Rightarrow z = 7x$, i.e. the answer in 1136 is $z = 7$.

We will prove that this is true in general.

Let us find R_3 in terms of R_1 and R_2. Note that the curvature of a line ST in Fig. 4 is zero. If we call C_3 the curvature of the circle (O_3, R_3), then from formula (a), Theorem 2, we see that

$$C_3 = C_1 + C_2 + 0 + 2\sqrt{C_1C_2 + C_1 0 + C_2 0} = C_1 + C_2 + 2\sqrt{C_1C_2}.$$

And in different terms

$$R_3 = \frac{1}{C_3} = \frac{R_1R_2}{R_1 + R_2 + 2\sqrt{R_1R_2}}.$$

Since X is the radius of the small circle, $1/X$ is the curvature of the small circle. From formula (a), Theorem 2, we have

$$\frac{1}{X} = C_1 + C_2 + C_3 + 2\sqrt{C_1C_2 + C_1C_3 + C_2C_3}.$$

Also, since 0 is the curvatures of the line ST, from formula (b), Theorem 2, we have

$$0 = C_1 + C_2 + C_3 - 2\sqrt{C_1C_2 + C_1C_3 + C_2C_3}.$$

Therefore, $1/X = 2(C_1 + C_2 + C_3) = 4(C_1 + C_2) + 4\sqrt{C_1C_2}$, and therefore

$$X = \frac{R_1R_2}{4R_1 + 4R_2 + 4\sqrt{R_1R_2}}.$$

The equation

$$4\left(C_1 + C_2 + \sqrt{C_1C_2}\right) = 1$$

presents the relationship between $R_1 = \frac{1}{C_1}, R_2 = \frac{1}{C_2}$ which will give $x = 1$.

If we know x we can find the equation for z

In Fig. 4, it is easy to show that $ST = 2\sqrt{R_1 R_2}$ using the Pythagorean Theorem. Using Fig. 5, we see that

$$\begin{aligned} 2\sqrt{R_1 R_2} &= \sqrt{(R_1+x)^2 - (R_1 - z)^2} + \sqrt{(R_2+x)^2 - (R_2-z)^2} \\ &= \sqrt{2R_1(x+z) + x^2 - z^2} + \sqrt{2R_2(x+z) + x^2 - z^2} \\ x &= \frac{R_1 R_2}{4\left(R_1 + R_2 + \sqrt{R_1 R_2}\right)} \end{aligned}$$

This equation can be transformed easily into a quadratic.

Let's check directly that $z = 7x, x = \frac{R_1 R_2}{4\left(R_1+R_2+\sqrt{R_1 R_2}\right)}$

is the desirable solution by direct substitution into

$$\sqrt{16R_1 x - 48x^2} + \sqrt{16R_2 x - 48x^2} = 2\sqrt{R_1 R_2}.$$

Now

$$\begin{aligned} 16R_1 x - 48x^2 &= 16\left(R_1 x - 3x^2\right) \\ &= 16\left(\frac{R_1^2 R_2}{4\left(R_1 + R_2 + \sqrt{R_1 R_2}\right)} - \frac{3R_1^2 R_2^2}{16\left(R_1 + R_2 + \sqrt{R_1 R_2}\right)^2}\right) \\ &= \frac{16 R_1^2}{16\left(R_1 + R_2 + \sqrt{R_1 R_2}\right)^2}\left(4R_2\left(R_1 + R_2 + \sqrt{R_1 R_2}\right) - 3R_2^2\right) \\ &= \frac{R_1^2}{\left(R_1 + R_2 + \sqrt{R_1 R_2}\right)^2}\left(4R_1 R_2 + R_2^2 + 4R_2\sqrt{R_1 R_2}\right) \\ &= \frac{R_1^2\left(R_2 + 2\sqrt{R_1 R_2}\right)^2}{\left(R_1 + R_2 + \sqrt{R_1 R_2}\right)^2} \end{aligned}$$

Taking square roots,

$$\sqrt{16R_1 x - 48x^2} = \frac{R_1\left(R_2 + 2\sqrt{R_1 R_2}\right)}{R_1 + R_2 + \sqrt{R_1 R_2}}.$$

Likewise,

$$\sqrt{16R_2x - 48x^2} = \frac{R_2\left(R_1 + 2\sqrt{R_1R_2}\right)}{R_1 + R_2 + \sqrt{R_1R_2}}.$$

Therefore

$$\sqrt{16R_1x - 48x^2} + \sqrt{16R_2x - 48x^2} = \frac{2R_1R_2 + 2R_1\sqrt{R_1R_2} + 2R_2\sqrt{R_1R_2}}{R_1 + R_2 + \sqrt{R_1R_2}} = 2\sqrt{R_1R_2},$$

and we are done.

References

[1] A Tale of Two Fractals (in Russian, published by MCCME, Moscow, 2009) `http://www.math.upenn.edu/~kirillov/`