Applications of the Cevian Group in a Triangle

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1 Abstract

We first define and develop the geometric properties of the Cevian group in a triangle. This Abelian group distributes over the plane Euclidean projective space.

We also develop the properties of the basic, the isotomic and the isogonal conjugates of a point in a triangle.

We later proceed to transfer four points on the Euler line to the harmonic axis of the centroid of a triangle. Then using the properties of the Cevian group, we proceed to generate an infinite collection of three or more colinear points in a triangle.

2 Introduction

We first define and develop the basic theory of the Cevian group in a triangle. We are especially interested in the definitions of the harmonic pole and the harmonic axis with emphasis on the harmonic axis of the centroid G.

Next, we define and develop some of the basic properties of the isotomic conjugate, the isogonal conjugate and the basic conjugate of a point in a triangle.

We then apply the isotomic conjugate and the isogonal conjugate to the centroids and the incenters of the medial and the anti- complementary triangles of a triangle $\triangle ABC$ to define an infinite collection of points in $\triangle ABC$. This infinite collection includes many standard points in a triangle such as the orthocenter, centroid, incenter, circumcenter, Lemoine point, Gergonne point and Nagel point. We then focus our attention on four points on the Euler line three of which are the well known centroid, orthocenter and circumcenter. We then transfer these four points to the harmonic axis of the centroid, and there is a very special reason for doing this. From this translation, we proceed to generate an infinite collection of three or more colinear points in $\triangle ABC$.

This is far more than what one would expect from four points on the Euler line, and this is not even remotely close to being exhaustive.

3 The Cevian Group in a Triangle.

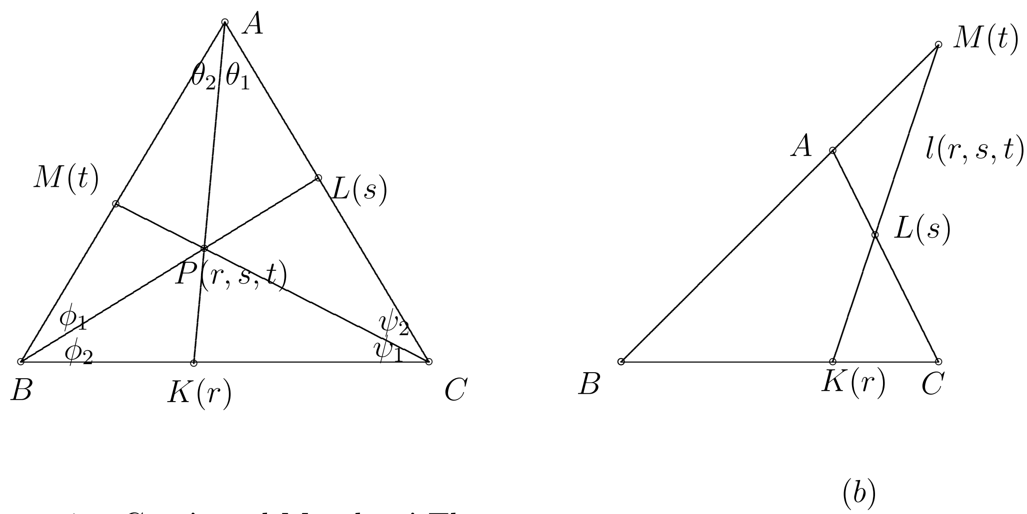

Fig.1 Illustrating Ceva's and Menelaus' Theorems

In $\triangle ABC$, suppose point K lies on the directed line segment BC, p.151, [1]. We say that K has a Cevian coordinate of r (which we write $K = K(r)$) if $\frac{BK}{KC} = r$ is true in both magnitude and sign. In directed line segments, we note that $BC = -CB, BK = -KB, KC = -CK$, etc. Thus, r is positive if K lies strictly between B and C and r is negative if K lies strictly outside of the line segment BC. Also, $r=0$ if $K=B$ and $r=\infty$ if $K=C$. In Fig. 1 (a), (b), suppose point $K=K(r)$ lies on BC, point $L=L(s)$ lies on CA and point $M=M(t)$ lies on AB where $\frac{BK}{KC}=r, \frac{CL}{LA}=s, \frac{AM}{MB}=t$. That is, r,s,t are the Cevian coordinates of points K,L,M. Ceva's Theorem states that AK,BL,CM are concurrent if and only if $\frac{BK}{KC}\cdot\frac{CL}{LA}\cdot\frac{AM}{MB}=rst=1$ is true in both magnitude and sign, p.159-163,[1]. Menelaus' Theorem states that K,L,M are colinear if and only if $\frac{BK}{KC}\cdot\frac{CL}{LA}\cdot\frac{AM}{MB}=rst=-1$ is true in both magnitude and sign, p.159-163,[1]. Thus, in Fig. 1, (a), if $rst=1$ we write $P=P(r,s,t), rst=1$, and say that (r,s,t) are the Cevian coordinates of the point of concurrency P of AK,BL,CM. Also, in Fig. 1, (b), if $rst=-1$, we say that the line l through K,L,M has Menelaus' coordinates of $(r,s,t), rst=-1$,and we write this as $l=l(r,s,t), rst=-1$. For line l we usually use the notation $l=l(l,m,n), lmn=-1$, in the place of (r,s,t) where $\frac{BK}{KC}=l, \frac{CL}{LA}=m, \frac{AM}{MB}=n$.

If $P(r,s,t), rst=1, Q(\overline{r},\overline{s},\overline{t}), \overline{rst}=1$, are the Cevian coordinates of the points P,Q then the point $R=P\cdot Q$ is defined by $R=P(r,s,t)\cdot Q(\overline{r},\overline{s},\overline{t})=R(r\overline{r},s\overline{s},t\overline{t})$, where $(r\overline{r},s\overline{s},t\overline{t}), (r\overline{r})(s\overline{s})(t\overline{t})=1$, are the Cevian coordinates of R. Of course, $(r,s,t)\cdot(\overline{r},\overline{s},\overline{t})=(r\overline{r},s\overline{s},t\overline{t})$ is the standard inner product vector of the two vectors $(r,s,t), (\overline{r},\overline{s},\overline{t})$ and the operator $(\cdot)$ is a group when $r,s,t,\overline{r},\overline{s},\overline{t}\in R\backslash\{0\}$. We will call $R=P\cdot Q$ the Cevian product or the inner product point of the points P and Q. If $P(r,s,t), rst=1$, is a point and $l(l,m,n), lmn=-1$, is a line, then $P(r,s,t)\cdot l(l,m,n)=(rl,sm,tn)$ is a line since $(rl)(sm)(tn)=-1$. Also, if $l(l,m,n), lmn=-1$, and $l^*(l^*,m^*,n^*)$, $l^*m^*n^*=-1$, are lines then $l(l,m,n)\cdot l^*(l^*,m^*,n^*)=(ll^*,mm^*,nn^*)$ is a point since $(ll^*)(mm^*)(nn^*)=1$.

Thus, we can expand our group $(\cdot)$ to deal with $(r,s,t)\cdot(r^*,s^*,t^*)=(rr^*,ss^*,tt^*)$ when $rst=\pm1, r^*s^*t^*=\pm1$. We continue to call this group the Cevian group, and we call the multiplication of two terms of the group the Cevian product.

If $P(r,s,t), rst=1$, is a point, the harmonic axis of P is the line $l(-r,-s,-t)$, where $(-r)(-s)(-t)=-1$. Also, if $l(l,m,n), lmn=-1$, is a line, then the point $P(-l,-m-n), (-l)(-m)(-n)=1$, is the harmonic pole of $l(l,m,n)$. In this paper, we are especially interested in the harmonic axis of the centroid $G(1,1,1)$ which has Menelaus' coordinates of $(-1,-1,-1)$. This line $(-1,-1,-1)$ lies at infinity.

Lemma 1 *In $\triangle ABC$ of Fig. 1.(a), suppose $|AB|=c, |AC|=b$, where c,b are the lengths of sides AB,AC of $\triangle ABC$. Then* $\frac{\sin\theta_2}{\sin\theta_1}=\frac{BK}{KC}\cdot\frac{b}{c}$.

Proof. From $\triangle AKB$, $\frac{\sin\theta_2}{BK} = \frac{\sin B}{AK}$. Also, from $\triangle AKC$, $\frac{\sin\theta_1}{KC} = \frac{\sin C}{AK}$. Therefore, $\frac{\sin\theta_2}{\sin\theta_1} = \frac{BK}{KC} \cdot \frac{\sin B}{\sin C} = \frac{BK}{KC} \cdot \frac{b}{c}$. ■

Corollary 1 In Fig. 1.(a), AK, BL, CM are concurrent if and only if $\frac{\sin\theta_1}{\sin\theta_2} \cdot \frac{\sin\phi_1}{\sin\phi_2} \cdot \frac{\sin\psi_1}{\sin\psi_2} = 1$.

Proof. The proof uses Lemma 1 with Ceva's theorem. ■

4 Three Conjugates in a Triangle

We first define the isotomic conjugate of a point P. Using Fig. 1,(a), suppose $P(r,s,t), rst = 1$, are the Cevian coordinates of a point P in $\triangle ABC$. Define points $\overline{K}$ on BC,$\overline{L}$ on CA and $\overline{M}$ on AB such that $BK = \overline{K}C, CL = \overline{L}A$ and $AM = \overline{M}B$ are true in both magnitude and sign. Then it is obvious that $\overline{K}\left(\frac{1}{r}\right), \overline{L}\left(\frac{1}{s}\right), \overline{M}\left(\frac{1}{t}\right)$ are the Cevian coordinates of the points $\overline{K}, \overline{L}, \overline{M}$. Also, $A\overline{K}, B\overline{L}, C\overline{M}$ are concurrent since $\left(\frac{1}{r}\right)\left(\frac{1}{s}\right)\left(\frac{1}{t}\right) = 1$. Calling $\overline{P}$ the point of concurrency of $A\overline{K}, B\overline{L}, C\overline{M}$ we have $\overline{P} = \overline{P}\left(\frac{1}{r}, \frac{1}{s}, \frac{1}{t}\right)$ and we define $\overline{P}$ to be the isotomic conjugate of the point $P(r,s,t)$. It is easy to see that $\overline{P} = P^{-1}$ where P^{-1} is the inverse of P in the Cevian group. Let I be the incenter of $\triangle ABC$. Also, in $\triangle ABC$ let $|BC| = a, |CA| = b, |AB| = c$. It is standard that the Cevian coordinates of I are $I\left(\frac{c}{b}, \frac{a}{c}, \frac{b}{a}\right)$ since $\frac{BK}{KC} = \frac{c}{b}$, $\frac{CL}{LA} = \frac{a}{c}$, $\frac{AM}{MB} = \frac{b}{a}$.

Also, $G(1,1,1)$ are the Cevian coordinates of the centroid G. $G(1,1,1)$ is the identity element in the Cevian group.

We now define the isogonal conjugate of a point P. From Fig. 1, (a), we now use the angles $\theta_1, \theta_2, \phi_1, \phi_2, \psi_1, \psi_2$. Suppose we reverse the two angles θ_1, θ_2, reverse the two angles ϕ_1, ϕ_2 and reverse the two angles ψ_1, ψ_2 to define new points $K', L'.M'$ on sides BC, CA, AB respectively. In other words we find points K' on BC, L' on CA and M' on AB such that $\angle BAK' = \theta_1, \angle K'AC = \theta_2, \angle CBL' = \phi_1, \angle L'BA = \phi_2, \angle ACM' = \psi_1, \angle M'CB = \psi_2$.

From Fig. 1, (a) and Lemma 1 we know that $r = \frac{BK}{KC} = \frac{\sin\theta_2}{\sin\theta_1} \cdot \frac{c}{b}$.

If $K'(r'), L'(s'), M'(t')$ are the Cevian coordinates of (K', L', M'), then $r' = \frac{BK'}{K'C} = \frac{\sin\theta_1}{\sin\theta_2} \cdot \frac{c}{b}$. Therefore, $r' = \left(\frac{c}{b}\right)^2 \cdot \frac{1}{r}$. Likewise, $s' = \left(\frac{a}{c}\right)^2 \cdot \frac{1}{s}$ and $t' = \left(\frac{b}{a}\right)^2 \cdot \frac{1}{t}$. Therefore, AK', BL', CM' are concurrent since $r's't' = 1$. If we call $\theta(P)$ the point of concurrent of AK', BL', CM', we define $\theta(P)$ to be the isogonal conjugate of the point $P(r,s,t)$ and it is obvious that $\theta(P) = \left(\left(\frac{c}{b}\right)^2 \frac{1}{r}, \left(\frac{b}{c}\right)^2 \frac{1}{s}, \left(\frac{b}{a}\right)^2 \frac{1}{t}\right) = I \cdot I \cdot \overline{P} = I^2 \cdot \overline{P} = I^2 \cdot P^{-1}$ where I is the incenter of $\triangle ABC, \overline{P} = P^{-1}$ is isotomic conjugate of P and multiplication is in the Cevian group. We also mention that

$I^2 = \theta(G) = K$ is called the Lemoine point of $\triangle ABC$ and it is usually denoted by K.

For completeness, we define $\phi(P) = I \cdot \overline{P}$ to be the basic conjugate of the point P. We do not know of a geometric meaning of $\phi(P)$. If $l(l,m,n), lmn = -1$ is a line, we can also define $\bar{l}(l,m,n) = \left(\frac{1}{l}, \frac{1}{m}, \frac{1}{n}\right), \phi(l) = I \cdot \bar{l}$ and $\theta(l) = I^2 \cdot \bar{l}$ to be the isotomic, basic and isogonal conjugates of the line l. Even though we do not know the geometric meaning of ϕ, it is used endlessly in developing the deeper properties of the triangle. As an example, if I_a, I_b, I_c are the three excenters of $\triangle ABC$ and X, Y, Z are the three points of contact of the incircle with the sides of $\triangle ABC$, then the two triangles $\triangle I_a I_b I_c$ and $\triangle XYZ$ are homothetic and the homothetic center of these two triangles is $\phi(\overline{M}) = \phi(N)$, where N is the Nagel point and M is the Gergonne point, two terms defined in Section 7.

If P is a point, it is easy to prove by induction that we can construct $I^{2n} \cdot P$ and $I^{2n} \cdot \overline{P}$ when $n \in Z$ by using only the isotomic conjugate $\overline{P}$ and the isogonal conjugate $\theta(P)$.

(a) Note that for $n \in N, \overline{I^{2n} \cdot P} = \overline{I}^{2n} \cdot \overline{P}$ and $\overline{I^{2n} \cdot \overline{P}} = \overline{I}^{2n} \cdot P$.

(b) If by induction we can construct $I^{2n} \cdot P$ and $I^{2n} \cdot \overline{P}$ for some $n \in N$, then $\theta\left(\overline{I^{2n} \cdot P}\right) = \theta\left(\overline{I}^{2n} \cdot \overline{P}\right) = I^{2n+2} \cdot P$ and $\theta\left(\overline{I^{2n} \cdot \overline{P}}\right) = \theta\left(\overline{I}^{2n} \cdot P\right) = I^{2n+2} \cdot \overline{P}$. By combining (a) and (b), we can construct $I^{2n} \cdot P$ and $I^{2n} \cdot \overline{P}$ for any $n \in Z$.

As examples $I^4 \cdot \overline{P} = \theta\left(\overline{\theta(P)}\right)$ and $I^4 \cdot P = \theta\left(\overline{\theta(\overline{P})}\right)$.

From this we see that by starting with the centroid $G(1,1,1)$ and the incenter $I\left(\frac{c}{b}, \frac{a}{c}, \frac{b}{a}\right)$ we can construct any point $I^n, n \in Z$, by using only $G, I, \theta(P), \overline{P}$. We will call $I^n, n \in Z$, the generalized incenter.

Also, by induction we can construct $I^n \cdot P, I^n \cdot \overline{P}$ for any $n \in Z$ by using only the isotomic conjugate $\overline{P}$ and the basic conjugate $\phi(P)$.

If we use all three conjugate $\overline{P}$, $\theta(P), \phi(P)$, then we can construct the points $I^n \cdot P, I^n \cdot \overline{P}, n \in Z$, in different ways. Indeed, $\theta(P) = \phi\left(\overline{\phi(P)}\right) = I^2 \cdot \overline{P}$. Also, $\overline{(\overline{P})} = P, \theta(\theta(P)) = P$ and $\phi(\phi(P)) = P$.

For this reason, we will always leave our points in the form $I^n \cdot P$ and $I^n \cdot \overline{P}$, and we will not use θ and ϕ at all.

5 Basic Properties of Lines and Points in the Cevian Group

We will make use of the following three theorems.

Theorem 1 Suppose $P(r,s,t), rst=1$, is a point and $l(l,m,n), lmn=-1$, is a line written with respect to $\triangle ABC$. Then P lies on l (written $P\in l$) if and only if (a) is true or (b) is true or (c) is true where (a),(b),(c) are logically equivalent.

(a) $mt(r-l)=1$,

(b) $nr(s-m)=1$,

(c) $ls(t-n)=1$.

Theorem 2 Suppose $\triangle A'B'C'$ is the medial triangle of $\triangle ABC$. Thus, A',B',C' are the midpoints of sides BC,CA,AB respectively. Suppose a point P has Cevian coordinates of $(r,s,t), rst=1$, with respect to $\triangle ABC$ and P has Cevian coordinates of $(r',s',t'), r's't'=1$, with respect to $\triangle A'B'C'$. Then

(a) $(r,s,t)=\left(\frac{t'+1}{\frac{1}{s'}+1},\frac{r'+1}{\frac{1}{t'}+1},\frac{s'+1}{\frac{1}{r'}+1}\right)$ and

(b) $(r',s',t')=\left(\frac{t-\frac{1}{s}+1}{-t+\frac{1}{s}+1},\frac{r-\frac{1}{t}+1}{-r+\frac{1}{t}+1},\frac{s-\frac{1}{r}+1}{-s+\frac{1}{r}+1}\right)$.

Note 1 The anti-complementary $\triangle A''B''C''$ of $\triangle ABC$ is the triangle such that $\triangle ABC$ is the medial triangle of $\triangle A''B''C''$. Thus, the formulas (a), (b) of Theorem 2 can also be used when $P=(r,s,t)$ are the Cevian coordinates of a point P with respect to $\triangle ABC$ and $P=(r'',s'',t'')$ are the Cevian coordinates of P with respect to $\triangle A''B''C''$.

Theorem 3 $P(r,s,t), rst=1$, are the Cevian coordinates of a point P in $\triangle ABC$. Also, in vector form $P=xA+yB+zC, x+y+z=1$. Then (a), (b) are true.

(a). $(r,s,t)=\left(\frac{z}{y},\frac{x}{z},\frac{y}{x}\right)$.

(b) $(x,y,z)=\left(\frac{1}{1+t+rt}\frac{t}{1+t+rt}\frac{rt}{1+t+rt}\right)=\left(\frac{s}{1+s+st},\frac{st}{1+s+st},\frac{1}{1+s+st}\right)=\left(\frac{rs}{1+r+rs},\frac{1}{1+r+rs}\frac{r}{1+r+rs}\right)$.

Proof of Theorem 3 Formula (a) is almost obvious from the definition of Cevian coordinates if we write $P=xA+(y+z)\left[\frac{yB}{y+z}+\frac{zC}{y+z}\right]=yB+(x+z)\left[\frac{xA}{x+z}+\frac{zC}{x+z}\right]=zC+(x+y)\left[\frac{xA}{x+y}+\frac{yB}{x+y}\right]$.

Also, the three formulas in (b) are obviously equivalent, and they are easy to derive from (a). The proofs of (b) are also self evident since from the first formula in (b) we easily see that $x+y+z=\frac{1+t+rt}{1+t+rt}=1$ and $\left(\frac{z}{y},\frac{x}{z},\frac{y}{x}\right)=\left(\frac{rt}{t},\frac{1}{rt},\frac{t}{1}\right)=(r,s,t)$. ■

The proofs of Theorems 1, 2 are now easy applications of Theorem 3 and the details are left to the reader. To prove Theorem 1, we write $P = xA + yB + zC = B + x(A - B) + z(C - B), x + y + z = 1$. We then set up an oblique co-ordinate system with $B = (0,0)$ as the origin, BC as the x-axis with $C = (1,0)$ and BA as the y-axis with $A = (0,1)$. Therefore, $P = (z,x)$ are the coordinates of P in this co-ordinate system. From line $l(l,m,n), lmn = -1$, we can find real numbers x^*, y^* such that $l = \left\{\frac{x}{x^*} + \frac{y}{y^*} = 1 : x, y \in R\right\}$ in this oblique co-ordinate system. The condition $P \in l$ is now a simple problem in analytic geometry and the proof is exactly the same as in rectangular co-ordinate analytic geometry.

The proof of Theorem 2 uses Theorem 3 and the obvious facts that $A' = \frac{1}{2}(B + C), B' = \frac{1}{2}(A + C), C' = \frac{1}{2}(A + B), A = -A' + B' + C', B = A' - B' + C', C = A' + B' - C'$.

Lemma 2 Suppose distinct points $P(r,s,t), rst = 1, Q(\overline{r}, \overline{s}, \overline{t}), \overline{r}\overline{s}\overline{t} = 1$, lie on line $l(l,m,n),\ lmn = -1$. Then $l = \frac{\frac{1}{s} - \frac{1}{\overline{s}}}{t - \overline{t}}, m = \frac{\frac{1}{t} - \frac{1}{\overline{t}}}{r - \overline{r}}, n = \frac{\frac{1}{r} - \frac{1}{\overline{r}}}{s - \overline{s}}$.

Proof. Since $(r,s,t) \in l$ and $(\overline{r}, \overline{s}, \overline{t}) \in l$, from Theorem 1 we know that $mt(r - l) = 1$ and $m\overline{t}(\overline{r} - l) = 1$. Therefore, $r - l = \frac{1}{mt}$ and $\overline{r} - l = \frac{1}{m\overline{t}}$. Therefore, $r - \overline{r} = \frac{1}{m}\left(\frac{1}{t} - \frac{1}{\overline{t}}\right)$ and $m = \frac{\frac{1}{t} - \frac{1}{\overline{t}}}{r - \overline{r}}$. Likewise, we have the formulas for l and n. ■

Lemma 2′(optional) The lines $l(l,m,n), lmn = -1$, and $\overline{l}(\overline{l}, \overline{m}, \overline{n}), \overline{l}\overline{m}\overline{n} = -1$, intersect at the point $P(r,s,t), rst = 1$. Then $r = -\frac{\left(\frac{1}{n} - \frac{1}{\overline{n}}\right)}{m - \overline{m}}, s = -\frac{\left(\frac{1}{l} - \frac{1}{\overline{l}}\right)}{n - \overline{n}}, t = -\frac{\left(\frac{1}{m} - \frac{1}{\overline{m}}\right)}{l - \overline{l}}$.

Proof. From Theorem 1, since $P \in l$ and $P \in \overline{l}$ we have $mt(r - l) = 1$ and $\overline{m}t\left(r - \overline{l}\right) = 1$. Therefore, $r - l = \frac{1}{mt}$ and $r - \overline{l} = \frac{1}{\overline{m}t}$. Therefore, $l - \overline{l} = \frac{1}{t}\left(\frac{1}{\overline{m}} - \frac{1}{m}\right)$.

Therefore, $t = -\frac{\left(\frac{1}{m} - \frac{1}{\overline{m}}\right)}{l - \overline{l}}$.

The formulas for r, s are proved the same way. ■

Corollary 2 In $\triangle ABC$ the three distinct points $P(r,s,t), rst = 1, Q(\overline{r}, \overline{s}, \overline{t})$, $\overline{r}\overline{s}\overline{t} = 1, R(r^*, s^*, t^*), r^*s^*t^* = 1$, are colinear if and only if any one of the following logically equivalent conditions (a), (b), (c) is satisfied. We use $Q(\overline{r}, \overline{s}, \overline{t})$ as the anchor point in (a), (b), (c). By symmetry we could also use $P(r,s,t)$ or $R(r^*, s^*, t^*)$ as the anchor point.

(a) $\frac{\frac{1}{r} - \frac{1}{\overline{r}}}{s - \overline{s}} = \frac{\frac{1}{r^*} - \frac{1}{\overline{r}}}{s^* - \overline{s}}$,

(b) $\frac{\frac{1}{s} - \frac{1}{\overline{s}}}{t - \overline{t}} = \frac{\frac{1}{s^*} - \frac{1}{\overline{s}}}{t^* - \overline{t}}$,

(c) $\frac{\frac{1}{t} - \frac{1}{\overline{t}}}{r - \overline{r}} = \frac{\frac{1}{t^*} - \frac{1}{\overline{t}}}{r^* - \overline{r}}$.

Proof. Let line QP intersect side BC in the point x and let line QR intersect the side BC in the point y. Then the three points Q, P, R are colinear if and only if $x = y$. Thus, from

Lemma 2, we see that formula (b) gives a necessary and sufficient condition such that Q, P, R are colinear. Likewise, formulas (a) and (c) give necessary and sufficient conditions such that Q, P, R are colinear. $\blacksquare$

Lemma 3 Suppose $P(r,s,t), rst = 1$, is a point and $l\,(l,m,n)\,, lmn = -1$, is a line. Also, suppose $C\,(x,y,z)\,, xyz = 1$, is a point. Then $P \in l$ if and only if $C \cdot P \in C \cdot l$.

Proof. We prove $P \in l$ implies $C \cdot P \in C \cdot l$. The proof that $C \cdot P \in C \cdot l$ implies $P \in l$ is similar. From Theorem 1, $P \in l$ if and only if $mt\,(r - l) = 1$. Also, $C \cdot P = (xr, ys, zt) \in C \cdot l = (x \cdot l, y \cdot m, z \cdot n)$ if and only if $(ym)\,(zt)\,(xr - xl) = 1$. Now $(ym)\,(zt)\,(xr - xl) = (xyz)\,(mt)\,(r - l) = (mt)\,(r - l) = 1$. Therefore, $C \cdot P \in C \cdot l$. $\blacksquare$

Corollary 3 Suppose line $l = \{P_i = (r_i, s_i, t_i) : i \in I\}$ where $l\,(l,m,n)\,, lmn = -1$, are the Menelaus coordinates of l and $\{P_i = (r_i, s_i, t_i) : i \in I\}$, each $r_i s_i t_i = 1$, are the Cevian coordinates of the points on line l. If $C\,(x,y,z)\,, xyz = 1$, is a point then $C \cdot l = \{C \cdot P_i : i \in I\}$. That is, $\{C \cdot P_i : i \in I\}$ where $C \cdot P_i = (x,y,z) \cdot (r_i, s_i, t_i) = (xr_i, ys_i, zt_i)$ are the points on line $C \cdot l = (x,y,z) \cdot (l,m,n) = (xl, ym, zn)$.

Proof. Obvious from Lemma 3 and the proof of Lemma 3. $\blacksquare$

Corollary 4 Suppose P, Q, R, C are points. Then P, Q, R are colinear if and only if $C \cdot P, C \cdot Q, C \cdot R$ are colinear.

Proof. Obvious from Lemma 3. $\blacksquare$

Note 2 If a is a fixed point, then the mapping $f\,(x) = a \cdot x, x$ is a point, maps lines into lines and preserves cross-ratios. Thus, the Cevian group distributes over the Euclidean projective space.

Lemma 4 (optional) Point $P\,(r,s,t)\,, rst = 1$, lies on line $l\,(l,m,n)\,, lmn = -1$. Let $Q\,(-l,-m,-n)\,, (-l)\,(-m)\,(-n) = 1$, be the harmonic pole of $l\,(l,m,n)$. Now $\overline{P}\,(r,s,t) = \left(\frac{1}{r}, \frac{1}{s}, \frac{1}{t}\right)$ and $\overline{Q}\,(-l,-m,-n) = \left(-\frac{1}{l}. -\frac{1}{m}, -\frac{1}{n}\right)$ are the isotomic conjugates of P, Q. Then $\overline{Q}$ lies on the harmonic axis of $\overline{P}$.

Proof. The easy proof uses Theorem 1. $\blacksquare$

Lemma 5 Let $G\,(1,1,1)$ be the centroid of $\triangle ABC$ and let $l\,(-1,-1,-1)$ be the harmonic axis of G.

Let $(r,s,t) = \left(\frac{z}{y}, \frac{x}{z}, \frac{y}{x}\right)$ be a point. Then (r,s,t) lies on $(-1,-1,-1)$ if and only if any one of the logically equivalent conditions (a), (b), (c), (d) is true.

(a) $1 + r + rs = 0$.

(b) $1 + t + tr = 0$.

(c) $1 + s + st = 0$.

(d) $x + y + z = 0$.

Proof. We show that $\left(\frac{z}{y}, \frac{x}{z}, \frac{y}{x}\right) \in (-1, -1, -1)$ if and only if $x + y + z = 0$. The logically equivalent conditions (a), (b), (c) easily follow from this.

From Theorem 1, we know that $\left(\frac{z}{y}, \frac{x}{z}, \frac{y}{x}\right) \in (-1, -1, -1)$ if and only if $(-1)\left(\frac{y}{x}\right)\left(\frac{z}{y} + 1\right) = 1$. This is equivalent to $-\frac{z}{x} - \frac{y}{x} = 1$ which is equivalent to $x + y + z = 0$. ■

Problem 1 Suppose $P(r, s, t), rst = 1$, is a point in $\triangle ABC$. Find necessary and sufficient conditions on $(r, s, t), rst = 1$, so that (r, s, t) lies at infinity.

Solution In Fig.1(a), let $|BC| = a, |CA| = b, |AB| = c$ be the lengths of sides BC, CA, AB. Also, in order for Fig.1(a) to be realistic, we place K to the left of B and C so that B now lies between K and C. Now since $\frac{BK}{KC} = r, \frac{CL}{LA} = s$, we see that $KC = \frac{a}{1+r}$ and $CL = \frac{sb}{1+s}$. Now $P(r, s, t)$ lies at infinity, if and only if $AK||BL$ which is equivalent to $\frac{BC}{CL} = \frac{KC}{CA}$. This is equivalent to $\frac{a}{\left(\frac{sb}{1+s}\right)} = \frac{\left(\frac{a}{1+r}\right)}{b}$. This is equivalent to $(1 + r)(1 + s) = s$ which is equivalent to $1 + r + rs = 0$. Using this with Lemma 5, we now see that $P(r, s, t), rst = 1$, lies at infinity if and only if $P(r, s, t) \in (-1, -1, -1)$. ■

6 Properties of the Medial and Anti-complementary Triangles

Problem 2 In Problems 2, 3, we let $|BC| = a, |CA| = b, |AB| = c$. Suppose I' is the incenter of the medial triangle $\triangle A'B'C'$ of $\triangle ABC$. Since $\triangle ABC \sim \triangle A'B'C'$ and $I = \left(\frac{c}{b}, \frac{a}{c}, \frac{b}{a}\right)$ we see that the Cevian coordinates (r', s', t') of I' with respect to $\triangle A'B'C'$ are also $I'(r', s', t') = \left(\frac{c}{b}, \frac{a}{c}, \frac{b}{a}\right)$. Let $(I')^n = (r'_n, s'_n, t'_n) = \left(\left(\frac{c}{b}\right)^n, \left(\frac{a}{c}\right)^n, \left(\frac{b}{a}\right)^n\right)$, be the n^{th} power of I' with respect to the Cevian group of $\triangle A'B'C'$ where $n \in Z$. We wish to compute the Cevian coordinates (r_n, s_n, t_n) of $(I')^n$ with respect to $\triangle ABC$.

Solution Using formula (a) of Theorem 2. We see that $(r_n, s_n, t_n) = \left(\frac{a^n+b^n}{a^n+c^n}, \frac{b^n+c^n}{b^n+a^n}, \frac{c^n+a^n}{c^n+b^n}\right)$. ■

Problem 3 Let I'' be the incenter of the anti-complementary $\triangle A''B''C''$ of $\triangle ABC$. Since $\triangle ABC \sim \triangle A''B''C''$, the Cevian coordinates (r'', s'', t'') of I'' with respect to $\triangle A''B''C''$ are $I''(r'', s'', t'') = \left(\frac{c}{b}, \frac{a}{c}, \frac{b}{a}\right)$.

Let $(I'')^n = \left(\left(\frac{c}{b}\right)^n, \left(\frac{a}{c}\right)^n, \left(\frac{b}{a}\right)^n\right)$, be the n^{th} power of I'' with respect to the Cevian group of $\triangle A''B''C''$ where $n \in Z$. We wish to compute the Cevian coordinates (r_n, s_n, t_n) of $(I'')^n$ with respect to $\triangle ABC$.

Solution Since $\triangle ABC$ is the medial triangle of $\triangle A''B''C''$ using formula (b) of Theorem 2 we see that $(r_n, s_n, t_n) = \left(\frac{a^n+b^n-c^n}{a^n-b^n+c^n}, \frac{-a^n+b^n+c^n}{a^n+b^n-c^n}, \frac{a^n-b^n+c^n}{-a^n+b^n+c^n}\right)$. ■

7 Generalized Points in a Triangle

$\triangle ABC$ is a triangle with $|BC| = a, |CA| = b, |AB| = c$. Using Problems 2, 3 we define for all $n \in Z, (I'')^n = \overline{H_n} = \left(\frac{a^n+b^n-c^n}{a^n-b^n+c^n}, \frac{-a^n+b^n+c^n}{a^n+b^n-c^n}, \frac{a^n-b^n+c^n}{-a^n+b^n+c^n}\right), (I')^n = \overline{h}_n = \left(\frac{a^n+b^n}{a^n+c^n}, \frac{b^n+c^n}{a^n+b^n}, \frac{a^n+c^n}{b^n+c^n}\right)$, where $\overline{H}_n, \overline{h}_n$ are the isotomic conjugates of H_n, h_n in $\triangle ABC$.

Of course, $I^n = \left(\left(\frac{c}{b}\right)^n, \left(\frac{a}{c}\right)^n, \left(\frac{b}{a}\right)^n\right)$.

Also, $I^0 = G(1,1,1)$ is the centroid and $I^1 = I\left(\frac{c}{b}, \frac{a}{c}, \frac{b}{a}\right)$ is the incenter of $\triangle ABC$.

Also, $I^2 = \theta(G) = K$ is the Lemoine point. Also, $H_2 = H$ is the orthocenter and $\theta(H_2) = \theta(H) = I^2 \cdot \overline{H} = O$ is the circumcenter.

Also, $H_1 = M$ is the Gergonne point and $\overline{M} = \overline{H}_1 = N$ is the Nagel point. (N usually denotes the 9-point center).

The Gergonne point M is the common point of concurrency of the lines joining the vertices of a triangle with the points of contact of the opposite sides with the inscribed circle, p.160, [1]. The Nagel point N is the common point of concurrency of the lines joining the vertices of a triangle to the points of contact of the opposite sides with the excircles relative to these sides, p. 160-162, [1].

For $n \in Z$, we call I^n the generalized incenter. We call H_n the generalized orthocenter and h_n the generalized little orthocenter. Suppose $l_n = (G, I^n), n \in Z$, is the line through the centroid G and the point I^n. Then from Lemma 2, we easily see that $l_n(l, m, n) = \left(\frac{c^n-a^n}{b^n-a^n}, \frac{a^n-b^n}{c^n-b^n}, \frac{b^n-c^n}{a^n-c^n}\right)$.

If $P_n(r, s, t)$ is the harmonic pole of line (G, I^n), then $P_n = \left(\frac{a^n-c^n}{b^n-a^n}, \frac{b^n-a^n}{c^n-b^n}, \frac{c^n-b^n}{a^n-c^n}\right)$.

From Lemma 5, it is obvious that for all $n \in Z, \overline{P_n} \in (-1,-1,-1)$ where $\overline{P}_n$ is the isotomic conjugate of P_n.

The following identities are used in Section 9 and they are easily proved by simple algebra.

(a) $I^n \cdot \overline{h}_n = \overline{h}_{-n}$, for all $n \in Z$.

(a′) $\overline{I}^n \cdot h_n = h_{-n}$, for all $n \in Z$.

(b) $P_{-n} = \overline{I}^n \cdot P_n$, for all $n \in Z$.

(b′) $\overline{P}_{-n} = I^n \cdot \overline{P}_n$, for all $n \in Z$.

(b″) $P_{-1} \cdot \overline{P}_{-n} = I^{n-1} \cdot P_1 \cdot \overline{P_n}$, for all $n \in Z$.

(c) $P_{2n} \cdot \overline{P}_n = h_n$, for all $n \in N$.

(c′) $P_{2^n} \cdot \overline{P}_1 = h_1 \cdot h_2 \cdot h_4 \cdots h_{2^{n-1}}$, for all $n \in N$.

For example, (a) is equivalent to $\left(\left(\frac{c}{b}\right)^n, \left(\frac{a}{c}\right)^n, \left(\frac{b}{a}\right)^n\right) \cdot \left(\frac{a^n+b^n}{a^n+c^n}, \frac{b^n+c^n}{a^n+b^n}, \frac{a^n+c^n}{b^n+c^n}\right) =$

$$\left(\frac{\left(\frac{1}{a}\right)^n + \left(\frac{1}{b}\right)^n}{\left(\frac{1}{a}\right)^n + \left(\frac{1}{c}\right)^n}, \frac{\left(\frac{1}{b}\right)^n + \left(\frac{1}{c}\right)^n}{\left(\frac{1}{a}\right)^n + \left(\frac{1}{b}\right)^n}, \frac{\left(\frac{1}{a}\right)^n + \left(\frac{1}{c}\right)^n}{\left(\frac{1}{b}\right)^n + \left(\frac{1}{c}\right)^n}\right)$$

which is obviously true.

8 Four Points on the Euler Line

The three points $G, H_2 = H, O = \theta(H_2) = I^2 \cdot \overline{H}_2$ are standard points on the Euler line.

From Corollary 2, if we use $G(1,1,1)$ as the anchor point, it is very easy and also fairly short to show that $G, H_2 = H$ and $I^2 \cdot \overline{H}_4 = \theta(H_4)$ are colinear. Thus, we now have the four points $G, H_2 = H, O = I^2 \cdot \overline{H}_2$ and $I^2 \cdot \overline{H}_4$ on the Euler line. The point $I^2 \cdot H_2 = \theta\left(\overline{H}\right)$ also lies on the Euler line, but this is derived in Section 10 along with endless other colinear points. (This point, by the way, is the homothetic center of the tangential triangle and the orthic triangle of a $\triangle ABC$.) Also, the harmonic pole of the Euler line is easily computed to be $P_2 \cdot H_2$ where $(\cdot)$ is multiplication in the Cevian group. To see this, define $x = -a^2 + b^2 + c^2, y = a^2 - b^2 + c^2, z = a^2 + b^2 - c^2$. Then the Euler line is $(G, H) = (G, H_2) = \left((1,1,1), \left(\frac{y}{z}, \frac{z}{x}, \frac{x}{y}\right)\right)$, and from Lemma 2 the Menelaus coordinates (l, m, n) of the

Euler line (G, H) are $(l, m, n) = \left(\frac{\frac{x}{z}-1}{\frac{x}{y}-1}, \frac{\frac{y}{x}-1}{\frac{y}{z}-1}, \frac{\frac{z}{y}-1}{\frac{z}{x}-1}\right) = \left(\left(\frac{y}{z}\right)\left(\frac{x-z}{x-y}\right), \left(\frac{z}{x}\right)\left(\frac{y-x}{y-z}\right), \left(\frac{x}{y}\right)\left(\frac{z-y}{z-x}\right)\right) = \left(\left(\frac{y}{z}\right)\left(\frac{c^2-a^2}{b^2-a^2}\right), \left(\frac{z}{x}\right)\left(\frac{a^2-b^2}{c^2-b^2}\right), \left(\frac{x}{y}\right)\left(\frac{b^2-c^2}{a^2-c^2}\right)\right) = H_2 \cdot P_2 \cdot (-1, -1, -1)$ where P_2 was computed in Section 7. Therefore, $H_2 \cdot P_2 \cdot$ is the harmonic pole of the Euler line (G, H). Before we go into the last four sections, we summarize the following easy facts.

If P, Q are two points, there is a unique points x such that $P \cdot x = Q$ namely $x = \overline{P} \cdot Q$ since $\overline{P} = P^{-1}$. If l, l^* are lines, there is a unique point x such that $l \cdot x = l^*$ namely $x = \bar{l} \cdot l^*$ since $\bar{l} = l^{-1}$.

If P is the harmonic pole of line l, then $l = P \cdot (-1, -1, -1)$ where $(-1, -1, -1)$ is the harmonic axis of $G(1, 1, 1)$. Thus, if P is the harmonic pole of line l and P^* is the harmonic pole of line l^*, then the unique point x such that $l \cdot x = l^*$ can also be written $x = \bar{l} \cdot l^* = \overline{P} \cdot P^*$ since $\bar{l} = \overline{P} \cdot (-1, -1, -1), l^* = P^* \cdot (-1, -1, -1)$ and $(-1, -1, -1) \cdot (-1, -1, -1) = (1, 1, 1)$.

9 Transferring the Four Points on the Euler Line to Line $(-1, -1, -1)$

Since the Euler line $\left(G, H_2, I^2 \cdot \overline{H}_2, I^2 \cdot \overline{H}_4\right) = P_2 \cdot H_2 \cdot (-1, -1, -1)$, we know that $(-1, -1, -1) = \overline{P}_2 \cdot \overline{H}_2 \cdot$ Euler line.

Therefore, from Corollary 3, $\overline{P}_2 \cdot \overline{H}_2 \cdot G = \overline{P}_2 \cdot \overline{H}_2 \in (-1, -1, -1), \overline{P}_2 \cdot \overline{H}_2 \cdot H_2 = \overline{P}_2 \in (-1, -1, -1), \overline{P}_2 \cdot \overline{H}_2 \cdot I^2 \cdot \overline{H}_2 = I^2 \cdot \overline{P}_2 \cdot \overline{H}_2 \cdot \overline{H}_2 \in (-1, -1, -1)$ and $\overline{P}_2 \cdot \overline{H}_2 \cdot I^2 \cdot \overline{H}_4 = I^2 \cdot \overline{P}_2 \cdot \overline{H}_2 \cdot \overline{H}_4 \in (-1, -1, -1)$.

In $\triangle ABC$ with $|BC| = a, |CA| = b, |AB| = c$, suppose $P(r(a, b, c), s(a, b, c), t(a, b, c)) \in (-1, -1, -1)$ where the Cevian coordinates $r(a, b, c), s(a, b, c), t(a, b, c), r(a, b, c) \cdot s(a, b, c) \cdot t(a, b, c) = 1$, and functions of a, b, c. From Lemma 5, it is fairly obvious that if we substitute any function $f(a, b, c)$ for a, substitute any function $g(a, b, c)$ for b, and substitute any function $h(a, b, c)$ for c in $r(a, b, c), s(a, b, c), t(a, b, c)$, then the new point will still lie on $(-1, -1, -1)$. (This simple fact is the reason for dealing with the line $(-1, -1, -1)$.)

Therefore, in particular, if $P(r(a, b, c), s(a, b, c), t(a, b, c)) \in (-1, -1, -1)$ then
$P(r(a^n, b^n, c^n), s(a^n, b^n, c^n), t(a^n, b^n, c^n)) \in (-1, -1, -1)$ for all $n \in R$. Therefore, we now know that the following 32 points lie on $(-1, -1, -1)$ where the $f(n), f(-n)$ columns are duals.

We note that $(a), (b), (c), (d), (a'), (b'), (c'), (d')$ are generated from $\overline{P}_2 \cdot \overline{H}_2$.

Also, $(e), (f), (g), (h), (i), (j), (e'), (f'), (g'), (h'), (i'), (j')$ are generated from $\overline{P}_2$.

Also, $(k), (l), (m), (n), (k'), (l'), (m'), (n')$ are generated from $I^2 \cdot \overline{P}_2 \cdot \overline{H}_2 \cdot \overline{H}_2$.

Also, $(o), (p), (o'), (p')$ are generated from $I^2 \cdot \overline{P}_2 \cdot \overline{H}_2 \cdot \overline{H}_4$.

Note that we are restricting P_i, h_i, H_i so that $-4 \leq i \leq 4$. We also use the identities a, a', b, b', b'', c, c' of Section 7.

$f(n)$	$f(-n)$
$(a)\ \overline{P_1} \cdot \overline{H_1}$	$(a')\ \overline{P_{-1}} \cdot \overline{H}_{-1} = I \cdot \overline{P}_1 \cdot \overline{H}_{-1}$
$(b)\ \overline{P}_2 \cdot \overline{H}_2 = \overline{P}_1 \cdot \overline{h}_1 \cdot \overline{H}_2$	$(b')\ \overline{P}_{-2} \cdot \overline{H}_{-2} = I^2 \cdot \overline{P}_2 \cdot \overline{H}_{-2} = I^2 \cdot \overline{P}_1 \cdot \overline{h}_1 \cdot \overline{H}_{-2}$
$(c)\ \overline{P}_3 \cdot \overline{H}_3$	$(c')\ \overline{P}_{-3} \cdot \overline{H}_{-3} = I^3 \cdot \overline{P}_3 \cdot \overline{H}_{-3}$
$(d)\ \overline{P}_4 \cdot \overline{H}_4 = \overline{P}_1 \cdot \overline{h}_1 \cdot \overline{h}_2 \cdot \overline{H}_4$	$(d')\ \overline{P}_{-4} \cdot \overline{H}_{-4} = I^4 \cdot \overline{P}_4 \cdot \overline{H}_{-4} = I^4 \cdot \overline{P}_1 \cdot \overline{h}_1 \cdot \overline{h}_2 \cdot \overline{H}_{-4}$
$(e)\ \overline{P}_1$	$(e')\ \overline{P}_{-1} = I \cdot \overline{P}_1$
$(f)\ \overline{P}_2 = \overline{P}_1 \cdot \overline{h}_1$	$(f')\ \overline{P}_{-2} = I^2 \cdot \overline{P}_2 = I^2 \cdot \overline{P}_1 \cdot \overline{h}_1$
$(g)\ \overline{P}_3$	$(g')\ \overline{P}_{-3} = I^3 \cdot \overline{P}_3$
$(h)\ \overline{P}_4 = \overline{P}_1 \cdot \overline{h}_1 \cdot \overline{h}_2$	$(h')\ \overline{P}_{-4} = I^4 \cdot \overline{P}_4 = I^4 \cdot \overline{P}_1 \cdot \overline{h}_1 \cdot \overline{h}_2$
$(i)\ \overline{P}_6 = \overline{P}_3 \cdot \overline{h}_3$	$(i')\ \overline{P}_{-6} = I^6 \cdot \overline{P}_6 = I^6 \cdot \overline{P}_3 \cdot \overline{h}_3$
$(j)\ \overline{P}_8 = \overline{P}_1 \cdot \overline{h}_1 \cdot \overline{h}_2 \cdot \overline{h}_4$	$(j')\ \overline{P}_{-8} = I^8 \cdot \overline{P}_8 = I^8 \cdot \overline{P}_1 \cdot \overline{h}_1 \cdot \overline{h}_2 \cdot \overline{h}_4$
$(k)\ I \cdot \overline{P}_1 \cdot \overline{H}_1 \cdot \overline{H}_1$	$(k')\ \overline{I} \cdot \overline{P}_{-1} \cdot \overline{H}_{-1} \cdot \overline{H}_{-1} = \overline{P}_1 \cdot \overline{H}_{-1} \cdot \overline{H}_{-1}$
$(l)\ I^2 \cdot \overline{P}_2 \cdot \overline{H}_2 \cdot \overline{H}_2 = I^2 \cdot \overline{P}_1 \cdot \overline{h}_1 \cdot \overline{H}_2 \cdot \overline{H}_2$	$(l')\ \overline{I}^2 \cdot \overline{P}_{-2} \cdot \overline{H}_{-2} \cdot \overline{H}_{-2} = \overline{P}_1 \cdot \overline{h}_1 \cdot \overline{H}_{-2} \cdot \overline{H}_{-2}$
$(m)\ I^3 \cdot \overline{P}_3 \cdot \overline{H}_3 \cdot \overline{H}_3$	$(m')\ \overline{I}^3 \cdot \overline{P}_{-3} \cdot \overline{H}_{-3} \cdot \overline{H}_{-3} = \overline{P}_3 \cdot \overline{H}_{-3} \cdot \overline{H}_{-3}$
$(n)\ I^4 \cdot \overline{P}_4 \cdot \overline{H}_4 \cdot \overline{H}_4 = I^4 \cdot \overline{P}_1 \cdot \overline{h}_1 \cdot \overline{h}_2 \cdot \overline{H}_4 \cdot \overline{H}_4$	$(n')\ \overline{I}^4 \cdot \overline{P}_{-4} \cdot \overline{H}_{-4} \cdot \overline{H}_{-4} = \overline{P}_1 \cdot \overline{h}_1 \cdot \overline{h}_2 \cdot \overline{H}_{-4} \cdot \overline{H}_{-4}$
$(o)\ I \cdot \overline{P}_1 \cdot \overline{H}_1 \cdot \overline{H}_2$	$(o')\ \overline{I} \cdot \overline{P}_{-1} \cdot \overline{H}_{-1} \cdot \overline{H}_{-2} = \overline{P}_1 \cdot \overline{H}_{-1} \cdot \overline{H}_{-2}$
$(p)\ I^2 \cdot \overline{P}_2 \cdot \overline{H}_2 \cdot \overline{H}_4 = I^2 \cdot \overline{P}_1 \cdot \overline{h}_1 \cdot \overline{H}_2 \cdot \overline{H}_4$	$(p')\ \overline{I}^2 \cdot \overline{P}_{-2} \cdot \overline{H}_{-2} \cdot \overline{H}_{-4} = \overline{P}_1 \cdot \overline{h}_1 \cdot \overline{H}_{-2} \cdot \overline{H}_{-4}$

10 Using the 32 Points on $(-1,-1,-1)$ to Generate Colinear Points

If P is any point, then from Corollary 3, $\{P \cdot P_i : P_i \in (-1,-1,-1)\} = P \cdot (-1,-1,-1)$ which implies $\{P \cdot P_i : P_i \in (-1,-1,-1)\}$ are colinear points. We now choose the point P and also choose the points $P_1, P_2, \cdots, P_k$ from the 32 points $\{a, a', b, b', \cdots, p, p'\} \subseteq (-1,-1,-1)$ that are given in Section 9 so that $P \cdot P_1, P \cdot P_2, \cdots, P \cdot P_k$ are points of the form $I^n, I^n \cdot h_m, I^n \cdot \overline{h}_m, I^n \cdot H_m, I^n \cdot \overline{H}_m$ where $n \in Z$ and $m \in \{-4,-3,-2,-1,1,2,3,4\}$. Note that we are only allowing $-4 \leq m \leq 4$. Of course, if Q is any point, then the points $I^n \cdot Q, I^n \cdot \overline{Q}, n \in Z$, can be constructed by using various combinations of the basic conjugate $\phi(P)$, the isogonal conjugate $\theta(P)$ and the isotomic conjugate $\overline{P}$. Also, $I^{2n} \cdot Q, I^{2n} \cdot \overline{Q}, n \in Z$, can be constructed by using only the isogonal conjugate and the isotomic conjugate. In the following list, we mention a few of the inner relations of these lines.

1. Since $(b) = \overline{P}_1 \cdot \overline{h}_1 \cdot \overline{H}_2, (b') = I^2 \cdot \overline{P}_1 \cdot \overline{h}_1 \cdot \overline{H}_{-2}, (e) = \overline{P}_1, (e') = I \cdot \overline{P}_1, (f) = \overline{P}_1 \cdot \overline{h}_1, (f') = I^2 \cdot \overline{P}_1 \cdot \overline{h}_1, (h) = \overline{P}_1 \cdot \overline{h}_1 \cdot \overline{h}_2, (h') = I^4 \cdot \overline{P}_1 \cdot \overline{h}_1 \cdot h_2$ all lie on the line $(-1,-1,-1)$, we see that $P_1 \cdot h_1 \cdot (b, b', e, e', f, f', h, h') = \left(\overline{H}_2, I^2 \cdot \overline{H}_{-2}, h_1, I \cdot h_1, G, I^2, \overline{h}_2, I^4 \cdot \overline{h}_2\right)$ are colinear. Note that $\overline{H}_2 = \overline{H}, I^2 \cdot \overline{H}_{-2} = \theta(H_{-2}), I \cdot h_1 = \phi\left(\overline{h}_1\right), I^2 = K$, the Lemoine point, and $I^4 \cdot \overline{h}_2 = \theta\left(\overline{\theta(h_2)}\right)$.

1′. Multiplying the points of (1) by $I^n, n \in Z$, we see that

$$\left(I^n \cdot \overline{H}_2, I^{n+2} \cdot \overline{H}_{-2}, I^n \cdot h_1, I^{n+1} \cdot h_1, I^n, I^{n+2}, I^n \cdot \overline{h}_2, I^{n+4} \cdot \overline{h}_2\right)$$

are colinear. When $n = 2$, line 1′ become $\left(I^2 \cdot \overline{H}_2, I^4 \cdot \overline{H}_{-2}, I^2 \cdot h_1, I^3 \cdot h_1, I^2, I^4, I^2 \cdot \overline{h}_2, I^6 \cdot \overline{h}_2\right)$. We note that $I^2 \cdot \overline{H}_2 = I^2 \cdot \overline{H} = \theta(H) = O, I^2 = K$ and $I^4 = \theta\left(\overline{K}\right)$. This line is called the Brogard diameter, and it is perpendicular to the Lemoine axis (which is the harmonic axis of K). The Brogard diameter is also perpendicular to the harmonic axis of $\theta(\overline{H})$.

2. $P_1 \cdot h_1 \cdot H_2 \cdot (b, f, f', l, p) = \left(G, H_2, I^2 \cdot H_2, I^2 \cdot \overline{H}_2, I^2 \cdot \overline{H}_4\right)$ are colinear. This is the Euler line which is the line that we started with. We note that we have also picked up the new point $I \cdot H_2 = I^2 \cdot H = \theta\left(\overline{H}\right)$, which was mentioned in Section 8. The Euler line is perpendicular to the harmonic axis of $\overline{K}$, and it is perpendicular to the harmonic axis of H.

2′. Multiplying the points of (2) by I^n, we see that $\left(I^n, I^n \cdot H_2, I^{n+2} \cdot H_2, I^{n+2} \cdot \overline{H}_2, I^{n+2} \cdot \overline{H}_4\right)$ are colinear. The line $\overline{I}^2 \cdot (2) = \left(\overline{I}^2, \overline{I}^2 \cdot H_2, H_2, \overline{H}_2, \overline{H}_4\right)$ is parallel to the Brogard diameter.

3. $P_1 \cdot (a, a', e, e', f, f') = \left(\overline{H}_1, I \cdot \overline{H}_{-1}, G, I, \overline{h}_1, I^2 \cdot \overline{h}_1\right)$ one colinear. Note that $\overline{H}_1 = \overline{M} = N$ is the Nagel point of a triangle.

3′. $I^n \cdot (3) \equiv \left(I^n \cdot \overline{H}_1, I^{n+1} \cdot \overline{H}_{-1}, I^n, I^{n+1}, I^n \cdot \overline{h}_1, I^{n+2} \cdot \overline{h}_1\right)$ are colinear.

4. $P_1 \cdot H_1 \cdot (a, e, e', k, o) = \left(G, H_1, I \cdot H_1, I \cdot \overline{H}_1, I \cdot \overline{H}_2\right)$ are colinear. Note that $H_1 = M$ is the Gergonne point of a triangle.

4′. $I^n \cdot (4) = \left(I^n, I^n \cdot H_1, I^{n+1} \cdot H_1, I^{n+1} \cdot \overline{H}_1, I^{n+1} \cdot \overline{H}_2\right)$ are colinear. If $n = -1$, we see that $\left(\overline{I}, \overline{I} \cdot H_1 = \overline{I} \cdot M, H_1 = M, \overline{H}_1 = N, \overline{H}_2 = \overline{H}\right)$ are colinear. This line is parallel to line $I \cdot (3) = \left(I \cdot \overline{H}_1, I^2 \cdot \overline{H}_{-1}, I, I^2, I \cdot \overline{h}_1, I^3 \cdot \overline{h}_1\right)$.

5. $P_1 \cdot H_2 \cdot (b, e, e', o) = \left(\overline{h}_1, H_2, I \cdot H_2, I \cdot \overline{H}_1\right)$ are colinear.

5′. $I^n \cdot (5) = \left(I^n \cdot \overline{h}_1, I^n \cdot H_2, I^{n+1} \cdot H_2, I^{n+1} \cdot \overline{H}_1\right)$ are colinear. When $n = -1$ we have $\overline{I} \cdot (5) = \left(\overline{I} \cdot \overline{h}_1, \overline{I} \cdot H_2, H_2, \overline{H}_1\right) = \left(\overline{I} \cdot \overline{h}_1, \overline{I} \cdot H, H, N\right)$.

This line is parallel to the line $I \cdot (4) = \left(I, I \cdot H_1, I^2 \cdot H_1, I^2 \cdot \overline{H}_1, I^2 \cdot \overline{H}_2\right)$.

6. $P_1 \cdot h_1 \cdot H_1 \cdot (a, f, f') = (h_1, H_1, I^2 \cdot H_1)$ are colinear.

6′. $I^n \cdot (6) = (I^n \cdot h_1, I^n \cdot H_1, I^{n+2} \cdot H_1)$ are colinear.

7. $P_1 \cdot h_1 \cdot h_2 \cdot (d, d', f, f', h, h', j, j') = \left(\overline{H}_4, I^4 \cdot \overline{H}_{-4}, h_2, I^2 \cdot h_2, G, I^4, \overline{h}_4, I^8 \cdot \overline{h}_4\right)$ are colinear.

7′. $I^n \cdot (7) = \left(I^n \cdot \overline{H}_4, I^{n+4} \cdot \overline{H}_{-4}, I^n \cdot h_2, I^{n+2} \cdot h_2, I^n, I^{n+4}, I^n \cdot \overline{h}_4, I^{n+8} \cdot \overline{h}_4\right)$ are colinear.

8. $P_1 \cdot h_1 \cdot H_4 \cdot (d, f, f', p) = \left(\overline{h}_2, H_4, I^2 \cdot H_4, I^2 \cdot \overline{H}_2\right)$ are colinear.

8′. $I^n \cdot (8) = \left(I^n \cdot \overline{h}_2, I^n \cdot H_4, I^{n+2} \cdot H_4, I^{n+2} \cdot \overline{H}_2\right)$ are colinear.

9. $P_3 \cdot (c, c', g, g', i, i') = \left(\overline{H}_3, I^3 \cdot \overline{H}_{-3}, G, I^3, \overline{h}_3, I^6 \cdot \overline{h}_3\right)$ are colinear.

9′. $I^n \cdot (9) = \left(I^n \cdot \overline{H}_3, I^{n+3} \cdot \overline{H}_{-3}, I^n, I^{n+3}, I^n \cdot \overline{h}_3, I^{n+6} \cdot \overline{h}_3\right)$ are colinear.

10. $P_3 \cdot H_3 \cdot (c, g, g', m) = \left(G, H_3, I^3 \cdot H_3, I^3 \cdot \overline{H}_3\right)$ are colinear.

10′. $I^n \cdot (10) = \left(I^n, I^n \cdot H_3, I^{n+3} \cdot H_3, I^{n+3} \cdot \overline{H}_3\right)$ are colinear.

11. $P_1 \cdot h_1 \cdot H_{-1} \cdot (a', f, f') = (I \cdot h_1, H_{-1}, I^2 \cdot H_{-1})$ are colinear.

11′. $I^n \cdot (11) = (I^{n+1} \cdot h_1, I^n \cdot H_{-1}, I^{n+2} \cdot H_{-1})$ are colinear.

12. $P_1 \cdot h_1 \cdot H_{-2} \cdot (b', f, f', l', p') = \big(I^2, H_{-2}, I^2 \cdot H_{-2}, \overline{H}_{-2}, \overline{H}_{-4}\big)$ are colinear.

12′. $I^n \cdot (12) = \big(I^{n+2}, I^n \cdot H_{-2}, I^{n+2} \cdot H_{-2}, I^n \cdot \overline{H}_{-2}, I^n \cdot \overline{H}_{-4}\big)$ are colinear.

13. $P_1 \cdot h_1 \cdot h_2 \cdot H_4 \cdot (d, h, h', n) = \big(G, H_4, I^4 \cdot H_4 . I^4 \cdot \overline{H}_4\big)$ are colinear.

13′. $I^n \cdot (13) = \big(I^n, I^n \cdot H_4, I^{n+4} \cdot H_4, I^{n+4} \cdot \overline{H}_4\big)$ are colinear.

14. $P_1 \cdot H_{-2} \cdot (b', e, e', o') = \big(I^2 \cdot \overline{h}_1, H_{-2}, I \cdot H_{-2}, \overline{H}_{-1}\big)$ are colinear.

14′. $I^n \cdot (14) = \big(I^{n+2} \cdot \overline{h}_1, I^n \cdot H_{-2}, I^{n+1} \cdot H_{-2}, I^n \cdot \overline{H}_{-1}\big)$ are colinear.

15. $P_1 \cdot h_1 \cdot h_2 \cdot h_4 \cdot (h, h', j, j') = (h_4, I^4 \cdot h_4, G, I^8)$ are colinear.

15′. $I^n \cdot (15) = (I^n \cdot h_4, I^{n+4} \cdot h_4, I^n, I^{n+8})$ are colinear.

16. $P_3 \cdot h_3 \cdot (g, g', i, i') = (h_3, I^3 \cdot h_3, G, I^6)$ are colinear.

16′. $I^n \cdot (16) = (I^n \cdot h_3, I^{n+3} \cdot h_3, I^n, I^{n+6})$ are colinear.

17. $P_1 \cdot h_1 \cdot h_2 \cdot H_{-4} \cdot (d', h, h', n') = \big(I^4, H_{-4}, I^4 \cdot H_{-4}, \overline{H}_{-4}\big)$ are colinear.

17′. $I^n \cdot (17) = \big(I^{n+4}, I^n \cdot H_{-4}, I^{n+4} \cdot H_{-4}, I^n \cdot \overline{H}_{-4}\big)$ are colinear.

18. $P_3 \cdot H_{-3} \cdot (c', g, g', m') = \big(I^3, H_{-3}, I^3 \cdot H_{-3}, \overline{H}_{-3}\big)$ are colinear.

18′. $I^n \cdot (18) = \big(I^{n+3}, I^n \cdot H_{-3}, I^{n+3} \cdot H_{-3} \cdot I^n \cdot \overline{H}_{-3}\big)$ are colinear.

19. $P_1 \cdot H_{-1} \cdot (a', e, e', k', o') = \big(I, H_{-1}, I \cdot H_{-1}, \overline{H}_{-1}, \overline{H}_{-2}\big)$ are colinear.

19′. $I^n \cdot (19) = \big(I^{n+1}, I^n \cdot H_{-1}, I^{n+1} \cdot H_{-1}, I^n \cdot \overline{H}_{-1}, I^n \cdot \overline{H}_{-2}\big)$ are colinear.

20. $P_1 \cdot h_1 \cdot H_{-4} \cdot (d', f, f', p') = \big(I^4 \cdot \overline{h}_2, H_{-4}, I^2 \cdot H_{-4}, \overline{H}_{-2}\big)$ are colinear.

20′. $I^n \cdot (20) = \big(I^{n+4} \cdot \overline{h}_2, I^n \cdot H_{-4}, I^{n+2} \cdot H_{-4}, I^n \cdot \overline{H}_{-2}\big)$ are colinear.

21. $P_1 \cdot h_1 \cdot h_2 \cdot H_{-2} \cdot (b', h, h') = (I^2 \cdot h_2, H_{-2}, I^4 \cdot H_{-2})$ are colinear.

21′. $I^n \cdot (21) = (I^{n+2} \cdot h_2, I^n \cdot H_{-2}, I^{n+4} \cdot H_{-2})$ are colinear.

22. $P_1 \cdot h_1 \cdot h_2 \cdot h_4 \cdot H_{-4} \cdot (d', j, j') = (I^4 \cdot h_4, H_{-4}, I^8 \cdot H_{-4})$ are colinear.

22′. $I^n \cdot (22) = (I^{n+4} \cdot h_4, I^n \cdot H_{-4}, I^{n+8} \cdot H_{-4})$ are colinear.

23. $P_3 \cdot h_3 \cdot H_{-3} \cdot (c', i, i') = (I^3 \cdot h_3, H_{-3}, I^6 \cdot H_{-3})$ are colinear.

23′. $I^n \cdot (23) = (I^{n+3} \cdot h_3, I^n \cdot H_{-3}, I^{n+6} \cdot H_{-3})$ are colinear.

24. $P_1 \cdot h_1 \cdot h_2 \cdot H_2 \cdot (b, h, h') = (h_2, H_2, I^4 \cdot H_2)$ are colinear.

24′. $I^n \cdot (24) = (I^n \cdot h_2, I^n \cdot H_2, I^{n+4} \cdot H_2)$ are colinear.

25. $P_3 \cdot h_3 \cdot H_3 \cdot (c, i, i') = (h_3, H_3, I^6 \cdot H_3)$ are colinear.

25′. $I^n \cdot (25) = (I^n \cdot h_3, I^n \cdot H_3, I^{n+6} \cdot H_3)$ are colinear.

26. $P_1 \cdot h_1 \cdot h_2 \cdot h_4 \cdot H_4 \cdot (d, j, j') = (h_4, H_4, I^8 \cdot H_4)$ are colinear.

26′. $I^n \cdot (26) = (I^n \cdot h_4, I^n \cdot H_4, I^{n+8} \cdot H_4)$ are colinear.

27. $P_1 \cdot h_2 \cdot (e, e', h, h') = \big(h_2, I \cdot h_2, \overline{h}_1, I^4 \cdot \overline{h}_1\big)$ are colinear.

27′. $I^n \cdot (27) = (I^n \cdot h_2, I^{n+1} \cdot h_2, I^n \cdot \overline{h}_1, I^{n+4} \cdot \overline{h}_1)$ are colinear.

28. $P_1 \cdot h_1 \cdot h_4 \cdot (f, f', j, j') = \big(h_4, I^2 \cdot h_4, \overline{h}_2, I^8 \cdot \overline{h}_2\big)$ are colinear.

28′. $I^n \cdot (28) = \big(I^n \cdot h_4, I^{n+2} \cdot h_4, I^n \cdot \overline{h}_2, I^{n+8} \cdot \overline{h}_2\big)$ are colinear.

11 Using the 32 Points on $(-1, -1, -1)$ in Unusual Ways

As we learn more about the triangle, we will discover unusual ways to use the 32 points on the line $(-1, -1, -1)$. We now give an example. Suppose $\triangle DEF$ is the orthic triangle of $\triangle ABC$ and let M_{DEF} be the Gergonne point of $\triangle DEF$. Using techniques similar to this paper, we can show $M_{DEF} = I^2 \cdot \overline{h_2} \cdot H_2 = \theta(h_2 \cdot \overline{H_2})$ where H_2, h_2 are the orthocenter and little orthocenter of triangle ABC. Using the 32 points $a, a', b, b', \ldots, p, p'$ on $(-1, -1, -1)$, we can now compute points on the Euler line of section 10 by $P_1 \cdot h_1 \cdot H_2 \cdot (b, f, f', l, p, h = \overline{P_1} \cdot \overline{h_1} \cdot \overline{h_2}) = (G, H_2, I^2 \cdot H_2, I^2 \cdot \overline{H_2}, I^2 \cdot \overline{H_4}, \overline{h_2} \cdot H_2 = \overline{I}^2 \cdot M_{DEF})$. Now, $\overline{I}^2 \cdot M_{DEF} = \overline{\theta(M_{DEF})}$ where $\overline{\theta(M_{DEF})}$ is the isotomic conjugate of the isogonal conjugate of M_{DEF}. Thus $\overline{\theta(M_{DEF})}$ lies on the Euler line of $\triangle ABC$.

12 Concluding Remarks

By considering other triangles in addition to the medial and anti- complementary triangles and also by allowing both $n, m \in Z$ in $I^n, I^n \cdot h_m, I^n \cdot \overline{h}_m, I^n \cdot H_m, I^n \cdot \overline{H}_m$ of Section 10 it is simple and straightforward to vastly expand the collection of colinear points given in Section 10. This means that the material in Section 10 is not even remotely close to being exhaustive. Another problem is to find more points on the Euler line. We are also researching the infinite number of perpendicular and parallel lines that exist in the triangle as well as other types of points. As one example, if $P(r,s,t)$,$rst = 1$, is a point, then the harmonic associates of $P(r,s,t)$ are the points $P_a(r,-s,-t), P_b(-r,s,-t), P_c(-r,-s,t)$. We are also researching the different substitution $(f(a,b,c), g(a,b,c), h(a,b,c))$ that we can use for (a,b,c). We conclude with the following example. We can show that line IH is parallel to line $N, \phi(H)$. Also, these two parallel lines are perpendicular to the harmonic axis of each of $N, \overline{\phi(M)}, \overline{\phi(N)}, \phi(O)$. This example also illustrates how the basic conjugate ϕ keep appearing in the triangle.

References

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