The Commutative Equihoop and the Card Game SET

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1 Introduction

In the fall of 1972 at Davidson College, Dr. Howard Eves "tossed out" unpublished work on a *point algebra* that he called an equihoop. This paper was intended for undergraduate research and since Dr. Eves never "tossed out" the same material at too many places, he never published this work.

Arthur Holshouser (a non-student), Dr. Ben Klein (a professor) and Brian White (a student) classified all commutative equihoops in 1972 and all non-commutative equihoops in the summer of 1973. They never published this work since they discovered that D.C. Murdoch [3] in 1939 had already classified all medial quasigroups, a much larger class of structures. This classification uses an underlying Abelian group and two commutative automorphisms on this group.

In 1974 Marsha Jean Falco [1] invented an 81 card game called SET. The 81 cards of SET contain all possible combinations of the following four attributes. Number: {one, two, three}, shading: {solid, striped, open}, color: {red, green, purple}, shape: {ovals, squiggles, diamond}. For example, one card displays $\{one, solid, purple, ovals\}$. The goal of SET is to find collections of cards satisfying the SET rule: Three cards are called a SET if, with respect to each of the four attributes, the cards are either all the same or all different. See [1], [2] . One example of a SET is {one, solid, purple, ovals}, {two, solid, purple, squiggles}, {three, solid, purple, diamonds}.

If a, b are cards, let us define $a \cdot b = c$ where c is the third card in the SET. We also define $a \cdot a = a$. Then (SET cards, $\cdot$) is a commutative equihoop having 81 elements, and all commutative equihoops of 81 elements are isomorphically identical. In general all finite commutative equihoops, $(S, \cdot)$, have $|S| = 3^k$ elements and for each k, $(S, \cdot)$ is unique up to isomorphism. In addition to proving this, we study the structure of the commutative equihoop and in another paper we use this structure to find the maximum number of cards that contains no SETS. We also define isomorphic subsets in a commutative equihoop, and show in that later paper that all the maximum collections of cards that contain no SETS are isomorphic. At the end of this paper, we summarize some of our results.

A web page of David Van Brink [6] states a SET-free collection cannot have more than 20 cards. He says that he proved this in 1997 with a computer program that took about one week to run on a 90MHz Pentium computer.

The result that 20 is the largest size of a SET-free collection of cards was actually proved in much stronger form by G. Pellegrino, [4], without using computers. Pellegrino showed that any set of 21 points in the projective space of $81 + 27 + 9 + 3 + 1$ elements, represented by nonzero 5-tuples in which x and $-x$ are considered equivalent, has three collinear points. This would correspond to sets of three distinct points in which the third is the sum or difference of the first two.

This paper has two main sections. The next section builds the basic theory of Commutative Equihoops, and the final section builds the more specialized theory needed to study the 81 card game of SET.

2 Commutative Equihoops

Basic Definitions. Some of the following definitions are taken unchanged from a "musty" copy of Dr. Eves' 1972 handout at Davidson College.

A Point Algebra, Basis: "A nonempty set S of elements $a, b, c, d, \cdots$ and a binary operation (herein denoted by juxtaposition) is called a *Point Algebra* if the following three properties are satisfied: .

P1. $a(ba) = b$ for all a, b of S. (left central property of x).

P2. $aa = a$ for all a of S. (idempotent property of x).

P3. $(ab)(cd) = (ac)(bd)$ for all a, b, c, d of S. (medial property of x)."

Dr. Eves listed other definitions, theorems, and conjectures that he labeled $D_1, D_2, \cdots D_6$; $T_1, T_2, \cdots, T_{77}$ and C_1, C_2, some of which we discuss below.

For example, theorem T_3 states $(ab)a = a(ba) = b$. This is proved by $b = a(ba) = (aa)(ba) = (ab)(aa) = (ab)a$. This is the right central property of x.

Definition 1 (Eves' D5). "A *hoop* is a nonempty set S of elements $a, b, c, d, \cdots$ and a binary operation (herein denoted by juxtaposition) such that

Q1. For all a, b of $S, ax = b$ and $ya = b$ have unique solutions x and y in S.

Q2. $aa = a$ for all a of S.

Q3. $(ab)(cd) = (ac)(bd)$ for all a, b, c, d of S."

Theorem 1. (Eves' T77) *Every point algebra is a hoop.*

Definition 2. (Eves' D6) A point algebra is called an *equihoop*.

Dr. Eves provided a model for thinking about equihoops. If a, b are points in the Euclidean plane, then $a \cdot b = c$ where a, b, c is an equilateral triangle and $a \to b \to c \to a$ has a counter clockwise orientation.

Conjecture 1. (Eves' C1) *If $ab = ba$ for all a, b of S in the equihoop $(S, \cdot)$, then S is a singleton set.*

Dr Eves' conjecture is wrong, and he knew this, of course. He was just getting the students started.

Definition 3. (ours 1972) A commutative equihoop $(S, \cdot)$ is an equihoop that satisfies $P4$. $ab = ba$ for all a, b of S.

Lemma 1. *(SET cards,$\cdot$) is a commutative equihoop where $(\cdot)$ was defined in the introduction. Therefore, Eves' Conjecture 1 is wrong.*

To prove Lemma 1, simply check to see that properties P1 to P4 are satisfied.

Definition 4. (Quasigroups). A *left quasigroup* is a binary operator $(S, \cdot)$ such that $xa = b$ has a unique solution $(x = b/a)$ for all a, b of S. $(S, \cdot)$ is a *right quasigroup* if $ax = b$ has a unique solution $(x = b \setminus a)$ for all a, b of S. Also, $(S, \cdot)$ is a *quasigroup* if it is both a left and a right quasigroup. A table of a quasigroup is called a *Latin square*. See [5].

Definition 5 (Medial Quasigroup). A medial quasigroup is a quasigroup satisfying $(ab)(cd) = (ac)(bd)$ for all a, b, c, d of S. See [3]. Some authors use the term "abelian" quasigroup.

Note that the hoop and the equihoop are medial quasigroups.

Murdoch's Theorem (1939) [3]. *Suppose $(S, \cdot)$ is a medial quasigroup that has at least one idempotent element $0 \in S$. That is, $0 \cdot 0 = 0$. Then $\exists$ an Abelian group $(S, 0, +)$ and two automorphisms $A : (S, 0, +) \to (S, 0, +)$, $B : (S, 0, +) \to (S, 0, +)$ satisfying $A \circ B = B \circ A$ (where $\circ$ denotes composition of functions) such that $\forall x, y \in S,\ xy = A(x) + B(y)$.*

Notation 1. *In the rest of this paper, we deal only with commutative equihoops, which we usually call the CEH. Also, we always denote the CEH by* $(E, \cdot)$. *If we wish to single out a particular* $0 \in E$, *we will denote the structure by* $(E, 0, \cdot)$. *In a CEH, any element can be so distinguished.*

The CEH can be studied either directly or by first defining the underlying Abelian group. We use the easier Abelian group approach, and we start the reader off on the direct approach. Holshouser, Klein, and White discovered Theorems 2 - 14 in 1972 - 1973. Theorems 2 through 5 show that the Abelian group structure and the CEH structure are retrievable from one another in a natural way.

Theorem 2. *Suppose* $(E, 0, \cdot)$ *is a CEH where* $0 \in E$ *is arbitrary but fixed.* $\forall a, b \in E$ *define* $a + b = 0\,(ab)$. *Then* $(E, 0, +)$ *is an Abelian group with identity* 0 *satisfying* $3a = a + a + a = 0, \forall a \in E$.

Proof.

1. To see the associativity of $+$, note that $(a + b) + c = a + (b + c)$ if and only if $0\,[(0\,(ab))\,c] = 0\,[a\,(0\,(bc))]$. Using the central property, this is true if and only if $(0(ab))c = a(0(bc))$. Now $(0(ab))c = (0(ab))(0(0c)) = (00)((ab)(0c)) = (00)((a0)(bc)) = (0(a0))(0(bc)) = a(0(bc))$.
2. Commutativity. $\forall a, b \in E, a + b = 0\,(ab) = 0\,(ba) = b + a$.
3. Identity. $\forall a \in E, 0 + a = 0\,(0a) = 0\,(a0) = a$.
4. $\forall a \in E, 3a = a + (a + a) = a + 0\,(aa) = a + 0a = 0\,(a\,(0a)) = 00 = 0$.

   Therefore, $\forall a \in E, a + (a + a) = 0$. Note that this implies $(-a) = a + a = 2a$.

□

Theorem 3. *Suppose* $(E, 0, +)$ *is an Abelian group with identity* 0 *satisfying* $3a = 0, \forall a \in E$. $\forall a, b \in E$, *define* $ab = -a - b$. *Then* $(E, \cdot) = (E, 0, \cdot)$ *is a CEH with* 0 *singled out.*

Proof. The easy proof is left to the reader. □

Theorem 4. *Suppose* $(E, 0, \cdot)$ *is a CEH with arbitrary* $0 \in E$ *singled out. Define* $(E, 0, +)$ *as in Theorem 2. Define* $(E, 0, \odot)$ *by* $a \odot b = -a - b = 2a + 2b$. *Then* $(E, 0, \cdot) = (E, 0, \odot)$ *with* 0 *singled out.*

Proof. $a \odot b = -a - b = 2a + 2b = 0\,(aa) + 0\,(bb) = (0a) + (0b) = 0\,((0a)\,(0b)) = 0\,((00)\,(ab)) = 0\,((ab)\,0) = ab$. □

Theorem 5. *Suppose* $(E, 0, +)$ *is an Abelian group with identity* 0 *satisfying* $3a = 0, \forall a \in E$. *Define* $(E, 0, \cdot)$ *as in Theorem 3. As in Theorem 2, define* $\left(E, 0, \dot{+}\right)$ *by* $a \dot{+} b = 0\,(ab)$. *Then* $(E, 0, +) = \left(E, 0, \dot{+}\right)$.

Proof. $a \dot{+} b = 0\,(ab) = 0\,(-a - b) = -0 - (-a - b) = a + b$. □

Theorems 4, 5 show that the CEH's $(E, 0, \cdot)$, with 0 singled out, can be paired $1 - 1$ with the Abelian groups $(E, 0, +)$ with identity 0 and satisfying $3a = 0, \forall a \in E$, by defining $\forall a, b \in E, a + b = 0\,(ab)$ and $ab = -a - b$.

Theorem 6. *Suppose* $(E, \cdot) = (E, 0, \cdot)$ *is a CEH and* $(E, 0, +)$ *is the corresponding Abelian group. Also, suppose* $0 \in H \subseteq E$. *Then* $(H, \cdot) = (H, 0, \cdot)$ *is a CEH if and only if* $(H, 0, +)$ *is an Abelian group. In other words* $(H, \cdot)$ *is closed under* $(\cdot)$ *if and only if* $(H, 0, +)$ *is closed under* $+$. *We call such a* $(H, \cdot)$ *a sub-CEH of* $(E, \cdot)$.

Proof. The proof is obvious. □

Notation 2. *Suppose* $x_1, x_2, \cdots, x_k \in E$ *where* $(E, \cdot)$ *is a CEH. Then* $g(x_1, x_2, \cdots, x_k)$ *denotes the set of all members of* E *that can be generated from* $x_1, x_2, \cdots, x_k$ *using the binary operator* $(\cdot)$*. This is the same as* $g(x_1, x_2, \cdots, x_k) = \bigcap_{i \in I} H_i$*, where* $(H_i, \cdot), i \in I$ *are all sub-commutative equihoops of* $(E, \cdot)$ *that satisfy* $\{x_1, x_2, \cdots, x_k\} \subseteq H_i$*. Of course,* $(g(x_1, x_2, \cdots, x_k), \cdot)$ *is a CEH.*

Notation 3. *Suppose* $x_1, x_2, \cdots, x_k \in E$ *where* $(E, 0, +)$ *is an Abelian group satisfying* $3a = 0, \forall a \in E$*. Then* $\overline{g}(x_1, x_2, \cdots, x_k)$ *denotes the set of all members of* E *that can be generated from* $x_1, x_2, \cdots, x_k$ *using the group operator* $(+)$*. Of course,* $(\overline{g}(x_1, x_2, \cdots, x_k), 0, +)$ *is an Abelian group and* $\overline{g}(x_1, x_2, \cdots, x_k) = \left\{ \sum_{i=1}^{k} c_i x_i : c_i \in \{0, 1, 2\}, i = 1, 2, \cdots, k \right\}$.

Theorem 7. ***A***. *Let* $(E, 0, \cdot)$ *be a CEH with an arbitrary* $0 \in E$ *being singled out, and let* $(E, 0, +)$ *be the corresponding Abelian group satisfying* $3a = 0, \forall a \in E$*. Suppose*
$\{x_0 = 0, x_1, x_2, \cdots, x_k\} \subseteq E, k \geq 1$*. Then* $g(0, x_1, x_2, \cdots, x_k) = \overline{g}(x_1, x_2, \cdots, x_k)$ *where* $g(0, x_1, x_2, \cdots, x_k)$ *is generated in* $(E, 0, \cdot)$ *and* $\overline{g}(x_1, x_2, \cdots, x_k)$ *is generated in* $(E, 0, +)$*.*

Proof. We show that (a) $g(0, x_1, x_2, \cdots, x_k) \subseteq \overline{g}(x_1, x_2, \cdots, x_k)$ and
(b) $g(0, x_1, x_2, \cdots, x_k) \supseteq \overline{g}(x_1, x_2, \cdots, x_k)$.

(a) Since $\forall a, b \in E, a \cdot b = -a - b = 2a + 2b$, and since $\forall a \in E, 0 + a = a$ and $3a = 0$, it is obvious that each $x \in g(x_0 = 0, x_1, x_2, \cdots, x_k)$ can be written as $x = \sum_{i=1}^{k} c_i x_i$ where each $c_i \in \{0, 1, 2\}$. Therefore, $x \in \overline{g}(x_1, x_2, \cdots, x_k)$.

(b) Since $\forall a, b \in E, a + b = 0\,(ab)$, it is obvious that each $\overline{x} \in \overline{g}(x_1, x_2, \cdots, x_k)$ satisfies $\overline{x} \in g(0, x_1, x_2, \cdots, x_k)$.

□

Definition 6. Suppose $\{x_1, x_2, \cdots, x_k\} \subseteq E$ where $k \geq 2$ and $(E, \cdot)$ is a CEH. Then $\{x_1, x_2, \cdots, x_k\}$ is *independent* in $(E, \cdot)$ if $\forall i = 1, 2, \cdots, k, x_i \notin g(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k)$. Also, if $k = 1, \{x_1\}$ is automatically independent in $(E, \cdot)$.

Note that if $\{x_1, x_2, \cdots, x_k\}, k \geq 2$, is independent in $(E, \cdot)$, then $x_1, x_2, \cdots, x_k$ are distinct.

Definition 7. Suppose $\{x_1, x_2, \cdots, x_k\} \subseteq E \setminus \{0\}$ where $k \geq 2$ and $(E, 0, +)$ is an Abelian group satisfying $3a = 0, \forall a \in E$. Then $\{x_1, x_2, \cdots, x_k\}$ is independent in $(E, 0, +)$ if $\forall i = 1, 2, \cdots, k, x_i \notin \overline{g}(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k)$. If $k = 1, \{x_1\} \subseteq E \setminus \{0\}$ is automatically independent in $(E, 0, +)$.

Note that if $\{x_1, x_2, \cdots, x_k\} \subseteq E \setminus \{0\}$ is independent in $(E, 0, +)$, then $x_1, x_2, \cdots, x_k$ are distinct.

Theorem 7B. $(E, 0, \cdot)$ is a CEH and $(E, 0, +)$ is the corresponding Abelian group satisfying $3a = 0, \forall a \in E$. Suppose $x_0 = 0, x_1, x_2, \cdots, x_k$ are distinct members of E where $k \geq 1$. Then $\{0, x_1, x_2, \cdots, x_k\}$ is independent in $(E, \cdot) = (E, 0, \cdot)$ if and only if $\{x_1, x_2, \cdots, x_k\}$ is independent in $(E, 0, +)$. Of course, $\{x_1, x_2, \cdots, x_k\} \subseteq E \setminus \{0\}$.

Proof. Suppose that $\{0, x_1, x_2, \cdots, x_k\}$ is not independent in $(E, \cdot) = (E, 0, \cdot)$.

First, suppose $x_0 = 0 \in g(x_1, x_2, \cdots, x_k)$. This means that there is some specific expression $e(x_1, x_2, \cdots, x_k)$ involving $x_1, x_2, \cdots, x_k$ and $(\cdot)$ such that $0 = e(x_1, x_2, \cdots, x_k)$. Using $ab = -a - b$ and $3a = 0$ repeatedly, write the formal expression $e(x_1, x_2, \cdots, x_k) = \sum_{i=1}^{k} \overline{c}_i x_i$ where each $\overline{c}_i \in \{0, 1, 2\}$. Suppose in this formal expansion of $e(x_1, x_2, \cdots, x_k)$, it is true that each $\overline{c}_i = 0$. If each $\overline{c}_i = 0$, then since this is a formal expression, if we let $x_1 = x_2 = \cdots = x_k = x$, we would have $e(x_1, x_2, \cdots, x_k) = e(x, x, \cdots, x) = 0$. However, since $x \cdot x = x$, it is easy to use induction to show that $e(x, x, \cdots, x) = x$. Therefore, we have a contradiction, which means $\overline{c}_i \neq 0$ for at least one i. Therefore, $(*)\ 0 = \sum_{i=1}^{k} \overline{c}_i x_i$ where each $\overline{c}_i \in \{0, 1, 2\}$, at least one $\overline{c}_i \neq 0$ and each $x_i \neq 0$. Alternatively, if $e(x_1, x_2, \cdots, x_k) = \sum_{i=1}^{k} \overline{c}_i x_i$ where each $\overline{c}_i \in \{0, 1, 2\}$, the reader can use induction to prove that $\sum_{i=1}^{k} \overline{c}_i \equiv 1 \pmod 3$. Now since $2a = -a, \forall a \in E$, and since each $x_i \neq 0$, we see that $(*)$ implies that one x_i satisfies $x_i \in \overline{g}(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k)$ which implies that $\{x_1, x_2, \cdots, x_k\}$ is not independent in $(E, 0, +)$.

Next, suppose $\{0, x_1, x_2, \cdots, x_k\}$ is not independent in $(E, \cdot) = (E, 0, \cdot)$ and $x_i \in g(0, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k)$ where $i \neq 0$. Therefore, $\exists$ a specific expression in $(E, 0, \cdot)$ such that $x_i = e(x_0 = 0, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k) = \sum_{j \in \{1,2,\cdots,k\} \setminus \{i\}} \overline{c}_j\, x_j$ and each $\overline{c}_j \in \{0, 1, 2\}$. Since $\{x_1, x_2, \cdots, x_k\} \subseteq E \setminus \{0\}$, we see that $\{x_1, x_2, \cdots, x_k\}$ is not independent in $(E, 0, +)$. This proves the implication in one direction.

Next, suppose $\{x_1, x_2, \cdots, x_k\}$ is not independent in $(E, 0, +)$. Therefore, $k \geq 2$ and $\exists i \in \{1, 2, \cdots, k\}$ such that $x_i \in \overline{g}(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k)$.
Therefore, $x_i = \overline{e}(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k)$ where $\overline{e}$ is a specific expression in $(E, 0, +)$. Using $a + b = 0\,(ab)$ repeatedly and noting that $x_i \neq 0$, we see that
$x_i \in g(0, x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k)$. Therefore, $\{0, x_1, x_2, \cdots, x_k\}$ is not independent in $(E, 0, \cdot) = (E, \cdot)$. $\square$

Standard Theorem 1. Suppose $(E, 0, +)$ is an Abelian group satisfying $3a = 0, \forall a \in E$. Then $(E, 0, +)$ can be written as the direct sum of order 3 cyclic groups.

Corollary 1. *If* $(E, \cdot) = (E, 0, \cdot)$ *is a finite CEH then* $|E| = 3^k$.

Note. The standard theorem is true when E has an arbitrary cardinality.

Notation 4. $(\{0, 1, 2, \}, 0, +)$ *denotes the order-3 cyclic group defined using modular 3 addition.*

Definition. The *basic* CEH, $(\{0, 1, 2, \}, \cdot)$ is the three element CEH defined by $0 \cdot 0 = 0,\ 1 \cdot 1 = 1,\ 2 \cdot 2 = 2,\ 0 \cdot 1 = 1 \cdot 0 = 2, 0 \cdot 2 = 2 \cdot 0 = 1, 1 \cdot 2 = 2 \cdot 1 = 0$, and it is denoted by E_3.

Suppose $(\{0, 1, 2\}, 0, +)$ is the order 3 cyclic group defined by modular 3 addition. Then the basic CEH, $(\{0, 1, 2, \} \cdot)$, is defined by $\forall a, b \in \{0, 1, 2\}, a \cdot b = -a - b = 2a + 2b$.

Note also that if $(\{0, 1, 2\}, 0, +)$ is expanded to the entire modular 3 field, we have the field $(\{0, 1, 2\}, 0, 1, +, -, \times, \div)$, and the theory of determinants, matrices, vector spaces, etc., can be used. See [2] for a different notation of this field.

Applying Standard Theorem 1. Suppose $(E, \cdot)$ is a CEH. Then $(E, \cdot) = (E, 0, \cdot)$ can be written as a direct product of basic CEH's. Also, any two CEH's of the same order are isomorphic. Therefore, if $(E, \cdot)$ is any finite CEH, then $|E| = 3^k$, and we denote $(E, \cdot)$ by E_3^k where E_3 is the basic CEH. We will later indicate how this can also be proved directly without using the underlying Abelian group. The reader may wish to practice some CEH multiplication. For example, in E_3^2, $(0, 0)(0, 1) = (0, 2)$, $(0, 1)(0, 1) = (0, 1)$ and $(0, 1)(1, 2) = (2, 0)$.

Theorem 8. *Suppose* $((E, 0, \cdot), (E, 0, +))$ *and also* $\left(\left(\overline{E}, \overline{0}, \odot\right), \left(\overline{E}, \overline{0}, \dot{+}\right)\right)$ *are associated CEH's and Abelian groups satisfying* $3a = 0, 3\overline{a} = \overline{0}$. *Also, suppose* $|E| = \left|\overline{E}\right|$.

Let $f : E \to \overline{E}$ be a 1-1 onto function such that $f(0) = \overline{0}$. Then f is an isomorphism of $(E, 0, \cdot)$ onto $\left(\overline{E}, \overline{0}, \odot\right)$ if and only if f is isomorphism of $(E, 0, +)$ onto $\left(\overline{E}, \overline{0}, \dot{+}\right)$.

Proof. Very briefly to say that f is an isomorphism of one structure to another means that the two structures are identical except that the elements x in E have been renamed $f(x)$ in $\overline{E}$. Therefore, the conclusion is fairly obvious since each of $(\cdot, +)$ defines the other and each of $\left(\odot, \dot{+}\right)$ defines the other. We leave the easy formal details to the reader. □

In this paper we use the terms isomorphism and automorphism interchangeably.

Standard Theorem 2. *Suppose $(E, 0, +)$ is an Abelian group satisfying $3a = 0, \forall a \in E$ and $|E| = 3^k$ where $k \in N$. Suppose $\{x_1, x_2, \cdots, x_k\} \subseteq E \backslash \{0\}$ and $x_1, x_2, \cdots, x_k$ are distinct and independent in $(E, 0, +)$. Also, suppose $\left(\overline{E}, \overline{0}, \dot{+}\right)$ is an Abelian group satisfying $3\overline{a} = \overline{0}, \forall \overline{a} \in \overline{E}$ and $\left|\overline{E}\right| = 3^k$. Also, suppose $\{\overline{x}_1, \overline{x}_2, \cdots \overline{x}_k\} \subseteq \overline{E} \backslash \left\{\overline{0}\right\}$ and $\overline{x}_1, \overline{x}_2, \cdots \overline{x}_k$ are distinct and independent in $\left(\overline{E}, \overline{0}, \dot{+}\right)$. Then $\exists$ an isomorphism $f : (E, 0, +) \to \left(\overline{E}, \overline{0}, \dot{+}\right)$ such that $f(0) = \overline{0}$ and $\forall i = 1, 2, \cdots, k, f(x_i) = \overline{x}_i$. Of course, f is uniquely determined by $f(x_i) = \overline{x}_i, i = 1, 2, \cdots, k$.*

There are also other related standard results for $(E, 0, +)$ when $|E| = 3^k$. For example, if $\{x_1, x_2, \cdots, x_e\} \subseteq E \backslash \{0\}$ are distinct and independent then $e \leq k$ and $|\overline{g}(x_1, x_2, \cdots, x_e)| = 3^e$. Also, if $e < k$ we can always add $\{x_{e+1}, x_{e+2}, \cdots, x_k\} \subseteq E \backslash \{0\}$ such that $\{x_1, x_2, \cdots, x_k\}$ are distinct and independent. Therefore, if $|E| = 3^k$ and $\{x_1, x_2, \cdots, x_k\} \subseteq E \backslash \{0\}$ are distinct and independent, then $\overline{g}(x_1, x_2, \cdots, x_k) = E$.

Theorem 9. *Suppose $(E, \cdot)$ is a CEH and $\{x_1, x_2, \cdots, x_k\} \subseteq E$ are distinct and independent in $(E, \cdot)$. Then $(g(x_1, x_2, \cdots, x_k), \cdot)$ is a CEH and $|g(x_1, x_2, \cdots, x_k)| = 3^{k-1}$.*

Proof. Suppose $k \geq 2$. Let $x_1 = 0$ and consider the associated $(E, 0, +)$. Now $x_1 = 0, x_2, \cdots, x_k$ are independent in $(E, \cdot)$ if and only if $x_2, x_3, \cdots, x_k$ are independent in $(E, 0, +)$ by Theorem 7B. Also, by Theorem 7A, $g(x_1 = 0, x_2, \cdots, x_k) = \overline{g}(x_2, x_3, \cdots, x_k)$ where g is generated in $(E, 0, \cdot)$ and $\overline{g}$ is generated in $(E, 0, +)$. By standard Abelian group theory, $|\overline{g}(x_2, x_3, \cdots, x_k)| = 3^{k-1}$. Therefore, $|\overline{g}(x_1, x_2, \cdots, x_k)| = 3^{k-1}$. □

Corollary 2. *$(E, \cdot)$ is a CEH and $\{x_1, x_2, \cdots, x_k\} \subseteq E$. Then $|g(x_1, x_2, \cdots, x_k)| = 3^e$ where $0 \leq e \leq k - 1$, and $\{x_1, x_2, \cdots, x_k\}$ is independent in $(E, \cdot)$ if and only if $e = k - 1$. Also, $\exists \left\{x'_1, x'_2, \cdots, x'_{e+1}\right\} \subseteq \{x_1, x_2, \cdots, x_k\}$ such that $\left\{x'_1, x'_2, \cdots, x'_{e+1}\right\}$ is independent in $(E, \cdot)$, and, therefore, $g(x_1, x_2, \cdots, x_k) = g\left(x'_1, x'_2, \cdots, x'_{e+1}\right)$.*

Proof. We first observe that if $y_{t+1} \in g(y_1, y_2, \cdots, y_t)$ then $g(y_1, y_2, \cdots, y_t) = g(y_1, y_2, \cdots, y_t, y_{t+1})$. We now let the reader finish the proof. □

Construction. Let $(E, \cdot)$ be a finite CEH and $|E| = 3^k$. We wish to construct $\{x_1, x_2, \cdots, x_{k+1}\} \subseteq E$ such that $x_1, x_2, \cdots, x_{k+1}$ are independent in $(E, \cdot)$ which implies $g(x_1, x_2, \cdots, x_{k+1}) = E$. Choose $x_1 \in E$, $x_2 \in E \backslash g(x_1)$, $x_3 \in E \backslash g(x_1, x_2)$, $x_4 \in E \backslash g(x_1, x_2, x_3), \cdots$, $x_{k+1} \in E \backslash g(x_1, x_2, \cdots, x_k)$. Of course, $(x_1, x_2, \ldots, x_{k+1})$ can be constructed in $3^k(3^k - 1)(3^k - 3) \cdots (3^k - 3^{k-1})$ different ways.

Theorem 10. *$(E, \cdot)$ is a finite CEH with $|E| = 3^k$. Suppose $\{x_1, x_2, \cdots, x_{k+1}\} \subseteq E$ and $x_1, x_2, \cdots, x_{k+1}$ are distinct and independent in $(E, \cdot)$. Also, suppose $\{\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_{k+1}\} \subseteq E$ and $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_{k+1}$ are distinct and independent in $(E, \cdot)$. Also, $x_1, x_2, \cdots, x_{k+1}$ and $\overline{x}_1, \overline{x}_2, \cdots \overline{x}_{k+1}$ are arranged in an arbitrary way. Then $\exists$ an automorphism $f : (E, \cdot) \to (E, \cdot)$ such that $\forall i = 1, 2, \cdots, k + 1, f(x_i) = \overline{x}_i$.*

Note. Since $g(x_1, x_2, \cdots, x_{k+1}) = g(\overline{x}_1, \overline{x}_2, \cdots \overline{x}_{k+1}) = E$, we see that f is uniquely determined by $f(x_i) = \overline{x}_i, i = 1, 2, \cdots, k+1$. Also, the above construction shows how to find such $\{x_1, x_2, \cdots, x_{k+1}\}$ and $\{\overline{x}_1, \overline{x}_2, \cdots \overline{x}_{k+1}\}$.

Proof. Let $x_1 = 0$ and $\overline{x}_1 = \overline{0}$ and consider the usual pairs $((E, 0, \cdot), (E, 0, +))$ and $\left(\left(E, \overline{0}, \cdot\right), \left(E, \overline{0}, \dot{+}\right)\right)$. Now $x_1 = 0, x_2, \cdots, x_{k+1}$ are independent in $(E, 0, \cdot)$ which implies $x_2.x_3, \cdots, x_{k+1}$ are independent in $(E, 0, +)$ by Theorem 7B.

Also, $\overline{x}_1 = \overline{0}, \overline{x}_2, \cdots, \overline{x}_{k+1}$ are independent in $\left(E, \overline{0}, \cdot\right)$ which implies $\overline{x}_2, \overline{x}_3, \cdots \overline{x}_{k+1}$ are independent in $\left(E, \overline{0}, \dot{+}\right)$. Therefore, by Standard Theorem 2, $\exists$ an isomorphism $f : (E, 0, +) \to \left(E, \overline{0}, \dot{+}\right)$ such that $f(x_1) = f(0) = \overline{0} = \overline{x}_1$ and $\forall i = 2, 3, \cdots, k+1, f(x_i) = \overline{x}_i$.

By Theorem 8, $f : (E, 0, \cdot) \to \left(E, \overline{0}, \cdot\right)$ is also an automorphism such that $f(x_1) = f(0) = \overline{0} = \overline{x}_1$ and $\forall i = 2, 3, \cdots, k+1, f(x_i) = \overline{x}_i$. □

The homogeneous nature of the automorphisms makes Theorem 10 the most remarkable automorphism theorem we have seen. It will later help us to find, by trivial arithmetic calculations, the maximum number of cards in the game SET that contains no SETS and also show that any two such maximum collections are isomorphic. Of course, by fixing $\{x_1, x_2, \ldots, x_{k+1}\}$ we immediately know that if $(F, \circ)$ is the group of all automorphisms $f : (E, \cdot) \to (E, \cdot)$, then $|F| = 3^k(3^k - 1)(3^k - 3) \cdots (3^k - 3^{k-1})$. Some readers might have observed that $(F, \circ)$ is also the group of affine transformations on the abelian group $(E, \circ, +)$, a concept used in reference [2].

Definition 8. Suppose $(E, \cdot)$ is a CEH, $S \subseteq E$ and $\overline{S} \subseteq E$. We say that S and $\overline{S}$ are *isomorphic* in $(E, \cdot)$, denoted $S \cong \overline{S}$, if $\exists$ an automorphism $f : (E, \cdot) \to (E, \cdot)$ such that $f(S) = \overline{S}$.

Of course, $\cong$ is an equivalence relation. The paper [2], which does not consider the CEH, uses the term "similar" instead of isomorphic.

From group theory, we know that the number of distinct $\overline{S} \subseteq E$ such that $\overline{S}$ is isomorphic to S equals the number of automorphisms $f : (E, \cdot) \to (E, \cdot)$ divided by the number of these automorphisms that satisfy $f(S) = S$. In general, suppose $(F, \circ) = (\{f_1, f_2, \ldots, f_n\}, \circ)$ is a group of permutations on a set X, where $\circ$ denotes the composition of functions. Suppose $A \subseteq X$. Then $stabilizer(A) = (\{f_{i_1}, f_{i_2}, \ldots, f_{i_t}\}, \circ)$ where $f_{i_1}, f_{i_2}, \ldots, f_{i_t}$ are those member of F satisfying $f_{i_j}(A) = A$. Also, $orbit(A) = \{f_i(A) : i = 1, 2, \ldots n\}$. Note that the relation ARB if $B \in orbit(A)$ is an equivalence relation on 2^X, the power set of X. It is easy to use Lagrange's (coset) theorem to prove that $|F| = |stabilizer(A)| \cdot |orbit(A)|$.

Definition 9. Suppose $(E, \cdot)$ is a CEH and $S, \overline{S} \subseteq E$. Then the *product* of two sets $S, \overline{S}$ is defined as follows: $S \cdot \overline{S} = \left\{xy : x \in S, y \in \overline{S}\right\}$. Also if $a \in E, a \cdot \overline{S} = \left\{ay : y \in \overline{S}\right\}$.

Remark. The product $S \cdot \overline{S}$ in definition 9 is a true set. However, it is sometimes useful to consider $S, \overline{S}$, and $S \cdot \overline{S}$ to be multisets, which are sets with repetition of the elements allowed. As an example, suppose $S, \overline{S} \subseteq E_3^4$ are true sets with $|S| = |\overline{S}|$, and we wish to quickly guess whether S and $\overline{S}$ are isomorphic in E_3^4. In our work, we create the *profiles* of these sets as follows: $profile(S) = ((S \cdot S)(S \cdot S))((S \cdot S)(S \cdot S))$ and $profile(\overline{S}) = ((\overline{S} \cdot \overline{S})(\overline{S} \cdot \overline{S}))((\overline{S} \cdot \overline{S})(\overline{S} \cdot \overline{S}))$, where $S \cdot S$, etc are multisets. We then express each profile as an 81-vector. If S and $\overline{S}$ are isomorphic, then it is necessary that these two 81-vectors be similar in the same way that $(1, 2, 2, 4, 4, 4, 4, 9, 9)$ and $(2, 9, 1, 4, 4, 2, 4, 9, 4)$ are similar. However, since these profiles are so complex, we assume that similarity of profiles is also sufficient for $S, \overline{S}$ to be isomorphic in E_3^4.

Definition 10. Suppose $(E, \cdot)$ is a CEH and $(H, \cdot)$ is a sub-commutative equihoop. That is, $H \subseteq E$ and $(H, \cdot)$ is a CEH. We define $E/H = \{xH : x \in E\}$. The sets in E/H are called the *cosets* of H. They can also be called the CEH cosets of H

In definition 10, let $0 \in H$ and consider the usual pairs $(E, 0, \cdot), (E, 0, +)$ where $\forall a \in E, 3a = 0$. Since $(H, 0, \cdot)$ is closed under $(\cdot)$, we see that $(H, 0, +)$ is closed under $(+)$. Therefore, $(H, 0, +)$ is a subgroup. Now the (group) cosets of $(H, 0, +)$ are $\{x + H : x \in E\}$. Also, the CEH cosets of $(H, \cdot) = (H, 0, \cdot)$ are $\{-x + H : x \in E\}$ since $xH = -x - H = -x + H$. Therefore, the CEH cosets of H in $(E, \cdot)$ are identical to the group cosets of H in $(E, 0, +)$. From this we know that the members of E/H partition E, and $\forall K \in E/H, |H| = |K|$.

Theorem 11. *In Definition 10, suppose $K \in E/H$. Then $(K, \cdot)$ is a CEH.*

Proof. Let $0 \in H$ and consider the usual pairs $(E, 0, \cdot), (E, 0, +)$ where $\forall a \in E, 3a = 0$. Now $\exists x \in E$ such that $K = xH = -x - H = -x + H$. We show that $(K, \cdot)$ is closed. Suppose $a, b \in K$. Then $a = -x + h,\ b = -x + \overline{h},\ h,\ \overline{h} \in H$. Now $ab = -a - b = (x - h) + \left(x - \overline{h}\right) = 2x - h - \overline{h} = -x + \left(-h - \overline{h}\right)$ since $3x = 0$. Now $-h - \overline{h} \in H$ since H is closed under both $(\cdot)$ and $(+)$. □

Note. The reader can easily prove Theorem 11 directly from $(\cdot)$ without using $(+)$.

Theorem 12. *Suppose $K, L \in E/H$ and define $KL = \{xy : x \in K, y \in L\}$, as in Definition 9. Then $KL \in E/H$. This means in the CEH $(E, \cdot)$, the product of two cosets of H is a coset of H.*

Proof. We use $0 \in H$ and $(E, 0, \cdot), (E, 0, +)$. Now $KL = (-x + H) \cdot (-y + H) = -(-x + H) - (-y + H) = x + y - H - H = x + y + H = -(-x - y) + H$, which is a member of E/H. □

Notation 5. $(E/H, \cdot)$ *denotes the coset multiplication on the cosets of H.*

Theorem 13. ***A*** $(E/H, \cdot)$ *is a CEH, which we call the factor or quotient CEH defined from H. We also call $(E/H, \cdot)$ a coset CEH.*

Proof. By choosing representatives of the cosets, it is obvious that $(E/H, \cdot)$ satisfies the central, idempotent, medial and commutative properties. □

Observation Define $E/H = \{H = H_1, H_2, H_3, \cdots, H_{3^t}\}$. Then $\forall i, j \in \left\{1, 2, \cdots 3^t\right\}$, $E/H_i = E/H_j$ and $(E/H_i, \cdot) = (E/H_j, \cdot)$. In other words, $\forall i = 1, 2, \cdots 3^t$, H_i generates the same cosets as $H = H_1$. This homogeneous property is very different from the cosets in group theory.

If $E/H = \{H = H_1, H_2, \cdots, H_{3^t}\}$, then we refer to this collection as a family of cosets since each coset generates the other cosets in a homogeneous way.

Definition 11. Suppose H, K, L are distinct cosets in a family of cosets. We say that H, K, L are *mutual* cosets if $H = KL$, which is equivalent to $K = HL$ and also equivalent to $L = HK$ by the central property.

If H, K, L are mutual cosets (and therefore distinct), then $(H \cup K \cup L, \cdot)$ is a CEH having $3|H|$ elements since $(\cdot)$ is closed on $H \cup K \cup L$. Also, if H, K are distinct cosets in a family of cosets, then H, K, and $L = H \cdot K$ are mutual cosets.

Theorem 13B. Suppose $(E, \cdot)$ and $\left(\overline{E}, \odot\right)$ are CEHs and $f : (E, \cdot) \to \left(\overline{E}, \odot\right)$ is an isomorphism. Suppose $\{H_1, H_2, H_3, \cdots\}$ is a family of cosets in $(E, \cdot)$. Then $\{f(H_1), f(H_2), f(H_3), \cdots\}$ is a family of cosets in $\left(\overline{E}, \odot\right)$.

Proof. An isomorphism of a structure is the same structure with a renaming of the elements of the structure. We leave the formal details to the reader. □

Corollary 3. *Suppose* $\{H_1, H_2, \cdots\}, \{\overline{H}_1, \overline{H}_2, \cdots\}$ *are two families of cosets in* $(E, \cdot)$ *and* $f : (E, \cdot) \to (E, \cdot)$ *is an automorphism. Then* $\{\overline{H}_1, \overline{H}_2, \cdots\} = \{f(H_1), f(H_2), \cdots\}$ *if and only if* $\exists H_i, \overline{H}_j$ *such that* $f(H_i) = \overline{H}_j$*. This is true since any coset in a family of cosets completely determines the other cosets in that family of cosets and since* $\{f(H_1), f(H_2), \cdots\}$ *is also a family of cosets. Also, of course,* $\{H_1, H_2, \cdots\} = \{f(H_1), f(H_2), \cdots\}$ *if and only if* $\exists H_i, H_j$ *such that* $f(H_i) = H_j$*.*

Proving our Theorems directly from the CEH structure. Any theorem that we have stated that does not involve the underlying Abelian group can be proved directly from the binary operator of $(E, \cdot)$. Let $(E, \cdot)$ be a CEH and let $(H, \cdot)$ be a sub-commutative equihoop. Let $0 \in H, 0^* \notin H$. Let us write H as $H = \{x_1, x_2, \cdots x_t\} = \{0(x_1 0), 0(x_2 0), \cdots, 0(x_t 0)\}$. Let $K = \{(00^*)(x_1 0), (00^*)(x_2 0), \cdots, (00^*)(x_t 0)\}$ and $L = \{0^*(x_1 0), 0^*(x_2 0), \cdots, 0^*(x_t 0)\}$.

The reader can use mathematical induction on $|E|$ with the above idea to prove many of our theorems directly when $|E|$ is finite. The above idea led Arthur Holshouser, Ben Klein and Brian White to classify all finite CEH's in the form given in the observation preceding Theorem 8 in the few weeks after Dr Eves' 1972 lecture.

We invite the reader to use the group theory ideas of orbits and stabilizer to prove the answer to the following problem, which is related to proposition 4 of [2]. Problem. Let $(E, \cdot)$ be a finite CEH of cardinality 3^n and let $R \subseteq E$ be a fixed sub-CEH of E of cardinality 3^r, where $0 \le r \le n$. Let m be fixed such that $r < m \le n$. Find the number of distinct sub-CEHs $H \subseteq E$ satisfying $R \subseteq H$ and $|H| = 3^m$. The answer is

$$\frac{(3^n - 3^r)(3^n - 3^{r+1}) \cdots (3^n - 3^{m-1})}{(3^m - 3^r)(3^m - 3^{r+1}) \cdots (3^m - 3^{m-1})}.$$

The reader might also like to solve problem 11, page 173, of reference [7]: If m, n are positive integers, then $(x-1)(x^2-1)\cdots(x^m-1)$ divides $(x^n-1)(x^{n+1}-1)\cdots(x^{n+m-1}-1)$.

3 Using the CEH to study SET

In section 3 we develop the additional machinery on the structure of the CEH that is needed in part II to study the 81 card game SET. There are many interesting questions one can ask about the 81 card SET game. One popular problem is to determine the largest possible collection of cards that contains no SET. In the language of the CEH, three cards a, b, c is a SET if $a = bc$, which is equivalent to $b = ac$ and $c = ab$. Of course, all cards in SET are distinct. However, we are also interested in the deeper problem of finding up to isomorphism (Def. 8) all maximum collections of cards that contain no SET. We show here that the solution is unique when $|E| \in \{1, 3, 9, 27\}$, and in a part II we deal with the isomorphism problem for $|E| = 81$. In part II we show that any 21 cards in the 81-card SET deck contain a SET and there exists only one collection (up to isomorphism) of 20 cards that do not contain a SET. We also show how to construct all automorphisms on the 81 cards that map a given SET-free, 20-card collection onto itself. Recall that this is called the stabilizer of the collection.

Definition 12. $(E, \cdot)$ is a CEH and $S \subseteq E$. We say that S is SET-free if $\forall$ distinct $a, b, c \in S, ab \neq c$.

Definition 13. $(E, \cdot)$ is a finite CEH and $S \subseteq E$. We say that S is a *maximum* SET-free subset of E if S is SET-free and if $\overline{S} \subseteq E$ is any SET-free subset of E then $|\overline{S}| \le |S|$.

An alternate but not equivalent definition would be to say that S is a *maximal* SET-free subset of E if S is SET-free and if $S \subseteq \overline{S} \subseteq E$ together with $\overline{S}$ is SET-free implies that $S = \overline{S}$

Problem 1. *$(E, \cdot)$ is a finite CEH and $|E| = 3^k$. Therefore, $(E, \cdot) \cong E_3^k$ where E_3 is the basic CEH defined earlier. Find $|S|$ where S is a maximum SET-free subset of E. We will denote this maximum cardinality if $M(k)$.*

Problem 2. *For $|E| = 3^k$, let $\{S_1, S_2, \cdots, S_t\}$ be the collection of all maximum SET-free subsets of E. Partition $\{S_1, S_2, \cdots, S_t\}$ into isomorphic equivalence classes.*

Problem 3. *For $|E| = 3^k$, let $\mathcal{S}$ be the collection of* all *SET-free subsets of E. Partition $\mathcal{S}$ into isomorphic equivalence classes and construct the Hasse diagram for the partially ordered set of equivalence classes, where we use the partial ordering $A \leq B$ if A is isomorphic to a subset of B.*

Before solving these problems 1 and 2 for $k = 1, 2, 3$, we state one more theorem.

Theorem 14. *Suppose $(E, \cdot)$ is a CEH and $(H, \cdot)$ is a sub-CEH where H is a proper subset of E. Suppose $x \in E \setminus H$. Then $(g(H, x), \cdot)$ contains $3|H|$ elements where $(g(H, x), \cdot)$ is the sub-CEH generated by H and x.*

Proof. Consider the 3 mutual cosets, H, xH and $(xH)H$. It is easy to see that $g(x, H) = H \cup (xH) \cup ((xH)H)$ and also $H \cup (xH) \cup ((xH)H)$ is closed under $(\cdot)$. □

Figure 1 is used repeatedly in both this paper and part II in solving the problems we pose.

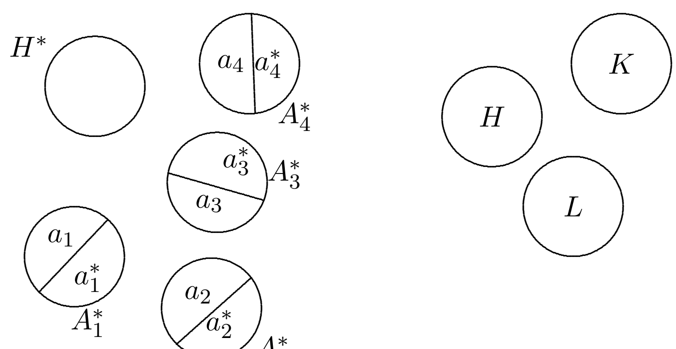


Fig. 1 Cosets in $(E, \cdot)$. $|E| = 9|H^*|, |E| = 3|H|$.

In figure (a), $E = H^* \bigcup_{i=1}^{4} A_i^*$. In (b) $H \cup K \cup L = E$. In figure (b), H, K, L are three mutual coset in $(E, \cdot)$. In figure (a), H^* is a sub-commutative equihoop of $(E, \cdot)$ and $a_i, a_i^*, i = 1, 2, 3, 4$ are the 8 cosets of H^* in $(E, \cdot)$. Also, $\forall i = 1, 2, 3, 4, H^*, a_i, a_i^*$ are mutual cosets. Also, $\forall i = 1, 2, 3, 4, A_i^* = a_i \cup a_i^*$. Of course, $\forall i = 1, 2, 3, 4$, if $x \in A_i^*$ then $H^* \cup A_i^* = g(x, H)$ where $g(x, H)$ denotes the elements of $(E, \cdot)$ that are generated from x and H.

For convenience as the solutions to the problems progress, the reader should draw (a) or (b) on paper and place the number of dots in H^*, a_i, a_i^*, H, K, L that the proof calls for.

Theorem 15. *Suppose $(E, \cdot) \cong E_3^k$ and $S \subseteq E$ is maximum SET-free. Then $g(S) = E$.*

Proof. Suppose $g(S) \neq E$. Then $\exists x \in E \backslash g(S)$. We show that $\overline{S} = S \cup \{x\}$ is SET-free, contradicting the assumption that S is maximum SET-free. Now $\forall a, b \in S, ab \in g(S)$. Therefore, if $a \neq b$ then $ab \notin S$ since S is SET-free and also $ab \neq x$ since $x \notin g(S)$. Therefore, $\overline{S}$ is SET-free. □

Corollary 4. Suppose $(E,\cdot)$ is a $CEH, |E| = 3^k$ and $S \subseteq E$ is maximum SET-free. Then there exists $\{x_1, x_2, \cdots, x_{k+1}\} \subseteq S$ such that the members of $\{x_1, x_2, \cdots, x_{k+1}\}$ are distinct and independent in $(E,\cdot)$ and, therefore, $g(x_1, x_2, \cdots, x_{k+1}) = E$.

Proof. Now $g(S) = E$. Using the construction given after the corollary of theorem 9, it is easy to sequentially produce subsets of S, namely $\{x_1\}, \{x_1, x_2\}, \cdots \{x_1, x_2, \cdots, x_{k+1}\}$ such that each $\{x_1, x_2, \cdots, x_i\}$ has distinct and independent elements and, therefore, $g(x_1, x_2, \cdots, x_i) = 3^{i-1}$. □

Problems 1, 2. $(k = 1)$ Obviously $M(1) = 2$. Also, it is easy to see that the following is a representative of the isomorphic equivalence class of maximum SET-free S's.

•		•
0	1	2

Problems 1, 2. $(k = 2)$ Now $(E,\cdot) = E_3^2$. Now if $S \subseteq E$ is maximum SET-free then $g(S) = E$. From corollary 4, let $X = \{x_0, x_1, x_2\} \subseteq S$ be independent in $(E,\cdot)$. Of course, $g(x_0, x_1, x_2) = E$. Now $(0,0), (0,2), (2,0)$ are independent in E_3^2. From Theorem 10 let $f : (E,\cdot) \to (E,\cdot)$ be the unique automorphism defined by $f(x_0) = (0,0), f(x_1) = (2,0), f(x_2) = (0,2)$. Now S is isomorphic to $f(S)$ and also $f(S)$ is SET-free. Therefore, $f(S)$ must also be maximum SET-free. We now focus on $f(S)$.

2	$(0,2)$	a_1	a_2
1	$\times$	$\times$	a_3
0	$(0,0)$	$\times$	$(2,0)$
	0	1	2

Fig. 2 $f(X) = \{(0,0), (0,2), (2,0)\}$

Now $f(S)$ could not contain any of the 3 $\times$'ed squares since $f(S)$ has no SETs. It is obvious that each of $f(X) \cup \{a_i\}$ is maximum SET-free where by obvious geometric symmetry one of $f(X) \cup \{a_1\}$ and $f(X) \cup \{a_3\}$ is redundant. We now show that $f(X) \cup \{a_1\} \cong f(X) \cup \{a_2\}$, where $\cong$ denotes isomorphic and $\cong$ is an equivalence relation. Define the automorphism $\overline{f} : E_3^2 \to E_3^2$ by $\overline{f}(0,0) = (0,0), \overline{f}(2,0) = (2,0), \overline{f}(0,2) = a_1$. Therefore, $\overline{f}(a_2) = \overline{f}((0,0)((0,2)(2,0))) = \overline{f}(0,0)(\overline{f}(0,2)\overline{f}(2,0)) = (0,0)((a_1)(2,0)) = (0,0)(0,1) = (0,2)$. Therefore, $M(2) = 4$ and the following is a representative of the isomorphic equivalence class of maximum SET-free S 's:

2	•		•
1			
0	•		•
	0	1	2

. □

Note. Lemmas 2 and 3 are included only to illustrate the use of Fig. 1.

Problems 1, 2. $(k = 3)$

Lemma 2. *Suppose* $(E,\cdot) = E_3^3, S \subseteq E,\ |S| = 10$ *and* S *is SET-free. Also, suppose* $(H,\cdot)$ *is a sub-CEH of* $(E,\cdot)$ *and* $|H| = 9$. *Then* $|H \cap S| = 4$.

Proof. Consider the mutual cosets H, K, L in Fig. 1-b where $|H| = [K] = |L| = 9$. Now $|K \cap S| \leq 4,\ |L \cap S| \leq 4$ from the E_3^2 solution since S is SET-free. Therefore, $|H \cap S| \geq 2$. Therefore, $\exists$ a CEH $(H^*,\cdot)$ with $H^* \subseteq H, |H^*| = 3$ and $|H^* \cap S| = 2$. We now use Fig. 1-a with H^* and $A_4^* = H \backslash H^*$. As always for convenience one can place 2 dots in H^* since $|H^* \cap S| = 2$.

Now $(H^* \cup A_i^*, \cdot), i = 1,2,3,4$, are CEH's with each containing 9 elements. Since S is SET-free, then $|(H^* \cup A_i^*) \cap S| \leq 4, i = 1,2,3,4$. Since $|H^* \cap S| = 2$, this implies $|A_i^* \cap S| \leq 2,\ i = 1,2,3,4$. Now $|S \cap (\bigcup_{i=1}^{4} A_i^*)| = 8$. Therefore, $|(H^* \cup A_i^*) \cap S| = 4,\ i = 1,2,3,4$, and in particular $|(H^* \cup A_4^*) \cap S| = |H \cap S| = 4$. □

Lemma 3. *Suppose* $(E, \cdot) = E_3^3, S \subseteq E$, *and* $|S| = 10$. *Then* S *cannot be SET-free.*

Proof. Suppose S is SET-free. Consider Fig. 1-b with $|H| = |K| = |L| = 9$. From Lemma 2, $|S \cap H| = |S \cap K| = |S \cap L| = 4$, a contradiction. □

We now show that $M(3) = 9$ and simultaneously show that if $S, \overline{S}$ are maximum SET-free subsets of E_3^3 then $S \cong \overline{S}$. This proves lemma 3 directly since it is clear that for all a in $E \setminus S$, $S \cup \{a\}$ cannot be SET-free.

Lemma 4. *Suppose* $(E, \cdot)$ *is a CEH,* $|E| = 27$, $S \subseteq E, |S| = 7$ *and* S *is SET-free. Then* $\exists H \subseteq E$ *such that* $(H, \cdot)$ *is a CEH,* $|H| = 9$ *and* $|H \cap S| = 4$

Later, we will tighten lemma 4 to $|S| = 6$ after we develop our final machinery.

Proof. Consider Fig. 1-a with $|H^*| = 3$ and $|H^* \cap S| = 2$. Of course each $(H^* \cup A_i^*, \cdot), i = 1, 2, 3, 4$, is a CEH containing 9 elements. Now $|S| = 7, |H^* \cap S| = 2$ implies $\exists t \in \{1, 2, 3, 4\}$ such that $|A_t^* \cap S| \geq 2$. Of course, $|A_t^* \cap S| = 2$ since each $(H^* \cup A_t^*, \cdot)$ can contain at most 4 members of the SET-free S. We now let $H = H^* \cup A_t^*$. □

Lemma 5. $(E, \cdot) = E_3^3, S \subseteq E, |S| = 9$ *and* S *is SET-free. Suppose* $H \subseteq E, (H, \cdot)$ *is a CEH,* $|H| = 9$ *and* $|H \cap S| \geq 2$. *Then* $|H \cap S| \geq 3$.

Proof. Now $\exists H^* \subseteq H$ such that $(H^*, \cdot)$ is a CEH, $|H^*| = 3$ and $|H^* \cap S| = 2$. Consider Fig. 1-a with $H = H^* \cup A_4^*$. The reader may wish to place 2 dots in H^*.

Now $|A_i^* \cap S| \leq 2, i = 1, 2, 3$, since $|H^* \cap S| = 2, (H^* \cup A_i^*, \cdot)$ is a CEH,$|H^* \cup A_i^*| = 9$ and S is SET-free.

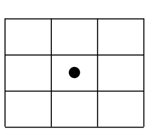
Now $|A_4^* \cap S| = 0$ and $|A_i^* \cap S| \leq 2, i = 1, 2, 3$ implies that $|S| \leq 8$, which contradiction the assumption that $|S| = 9$. □

Solution to Problems 1, 2 $(k = 3)$ Let us now suppose $S \subseteq E_3^3 = E, |S| = 9$ and S is SET-free.

Now by Lemma 4, $\exists H \subseteq E$ such that $(H, \cdot)$ is a CEH, $|H| = 9$ and $|H \cap S| = 4$. Using this H, consider the Fig. 1-b mutual cosets H, K, L. Now if both $|K \cap S| \geq 2$ and $|L \cap S| \geq 2$ then by Lemma 5, $|K \cap S| \geq 3$ and $|L \cap S| \geq 3$. However, since $|H \cap S| = 4$ and $|S| = 9$, this is impossible.

Therefore, by symmetry we may assume that $|H \cap S| = |K \cap S| = 4, |L \cap S| = 1$. Since $|H \cap S| = 4$, since $|H| = 9$, and since all maximum SET-free subsets of $(H, \cdot)$ are isomorphic, we can write $H \cap S = \{x_1, x_2, x_3, x_4\}$ where x_1, x_2, x_3 are independent in $(E, \cdot)$ and $x_4 = x_1(x_2, x_3)$. Also, $L \cap S = \{x_5\}$ and x_1, x_2, x_3, x_5 are independent in $(E, \cdot)$. Let $f : (E, \cdot) \to (E, \cdot)$ be the unique automorphism that satisfies $f(x_1) = (0, 0, 0), f(x_2) = (2, 0, 0), f(x_3) = (0, 2, 0)$ and $f(x_5) = (1, 1, 2)$. Of course, $f(x_4) = f(x_1(x_2x_3)) = f(x_1)\left((f(x_2))(f(x_3))\right) = (0, 0, 0)((2, 0, 0)(0, 2, 0)) = (0, 0, 0)(1, 1, 0) = (2, 2, 0)$. Since H, K, L are mutual cosets in $(E, \cdot)$, by Theorem 13-B we know that $f(H), f(K), f(L)$ are mutual cosets in $(E, \cdot)$. Now $f(H) = f(g(x_1, x_2, x_3)) = g(f(x_1), f(x_2), f(x_3)) = g((0, 0, 0), (2, 0, 0), (0, 2, 0)) = \{(x, y, 0) : x, y \in \{0, 1, 2\}\}$. Also, $f(K) = \{(x, y, 1) : x, y \in \{0, 1, 2\}\}$ and $f(L) = \{(x, y, 2) : x, y \in \{0, 1, 2\}\}$.

Of course, $f(S)$ is a maximum SET-free subset of $(E, \cdot)$ and $|f(H) \cap f(S)| = 4, |f(K) \cap f(S)| = 4, |f(L) \cap f(S)| = 1$. We now study Fig. 3.

2  $= f(L)$

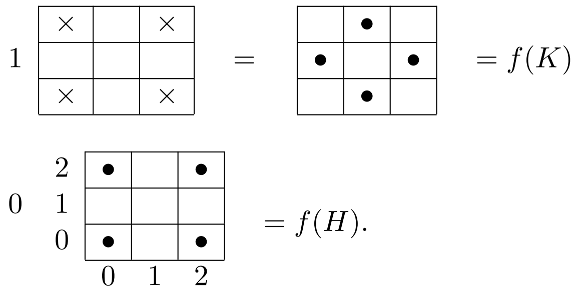


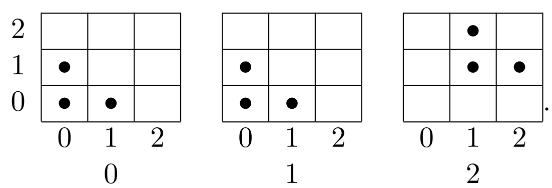
Fig. 3. $E_3^3 = (\{(x,y,z) : x,y,z \in \{0,1,2\}\}, \cdot)$

Now since $f(H \cap S) = \{(0,0,0),(0,2,0),(2,0,0),(2,2,0)\}$, $f(L \cap S) = \{(1,1,2)\}$ and since $f(S)$ is SET-free, we know that $f(K \cap S)$ cannot contain any of the 4 elements that we have ruled out. Also, since $|f(S) \cap f(K)| = 4$ and $f(S)$ is SET-free, it is obvious that $(1,1,1) \notin f(S)$. Therefore, $f(S) \cap f(K) = \{(1,0,1),(0,1,1),(1,2,1),(2,1,1)\}$.

Therefore, any S satisfying $S \subseteq E, |E| = 3^3, |S| = 9$ and S is SET free is isomorphic to the set of 9 points in E_3^3 shown in Fig. 3. □

Lemma 6. *Suppose $(E,\cdot)$ is a CEH, $|E| = 27$, $S \subseteq E$, $|S| = 9$ and S is SET-free. Also, suppose H, K, L are mutual cosets in $(E,\cdot)$ and $|H| = |K| = |L| = 9$. Then $\{|H \cap S|, |K \cap S|, |L \cap S|\}$ equals $\{4,4,1\}$ or $\{3,3,3\}$.*

Proof. First, suppose one of $|H \cap S|, |K \cap S|, |L \cap S|$ exceeds 3. By symmetry suppose $|H \cap S| > 3$. Therefore, $|H \cap S| = 4$. Also, by Lemma 5, it is easy to see that both $|K \cap S|$ and $|L \cap S|$ cannot exceed 1. Therefore, $\{|K \cap S|, |L \cap S|\} = \{1,4\}$. Now if none of $|H \cap S|, |K \cap S|, |L \cap S|$ exceeds 3, then $\{|H \cap S|, |K \cap S|, |L \cap S|\} = \{3,3,3\}$. The following also shows that $\{3,3,3\}$ is possible:



□

Note. The SET-free sets $S \subseteq E_3^3$, $|S| = 9$, have far more properties than we list here. We also list a few more at the end of the paper.

1. The sets S are 2-transitive. That is, $\forall a, b, \overline{a}, \overline{b} \in S$, if $a \neq b$, $\overline{a} \neq \overline{b}$, then there exists an automorphism $f : (E,\cdot) \to (E,\cdot)$ satisfying $f(S) = S$ and $f(a,b) = (\overline{a}, \overline{b})$.

2. From 1, it is easy to prove that $E \backslash S$ is transitive. That is, $\forall a, \overline{a} \in E \backslash S$, there exists an automorphism $f : (E,\cdot) \to (E,\cdot)$ satisfying $f(E \setminus S) = E \setminus S$ and $f(a) = \overline{a}$.

3. The reader can use the basic Lagrangian property of orbits to show that $E \setminus S$ cannot possibly be 2-transitive.

4. The number of automorphisms $f : (E,\cdot) \to (E,\cdot)$ satisfying $f(S) = S$ equals 144. That is, $|stabilizer(S)| = 144$.

5. Since $S \subseteq E = E_3^3$, $|S| = 9$, and S is SET-free is unique up to isomorphism, since the group of all automorphisms $f : (E, \cdot) \to (E, \cdot)$ has $27 \cdot 26 \cdot 24 \cdot 18 = 303{,}264$ elements and since $|stabilizer(S)| = 144$, we know from group theory that the number of distinct $S \subseteq E_3^3, |S| = 9$, and S is SET-free must equal $303{,}264 \div 144 = 2106$.

Problems 1, 2. ($k = 4$) In part II we show that if $(E, \cdot) = E_3^4$, $S \subseteq E, |S| = 20$ and S is SET-free, then S is unique up to isomorphism. It then follows that $M(4) = 20$ since for all $a \in E \setminus S$, $S \cup \{a\}$ is not SET-free.

In order to illustrate our final machinery, we show directly that $M(4) < 22$ and also that lemma 4 remains true when $|S| = 6$. However, we first prove Lemma 7 for $|S| = 21$.

Lemma 7. *Suppose $S \subseteq E, (E, \cdot)$ is a CEH, $|E| = 3^4, |S| = 21$ and S is SET-free. Then $\exists H \subseteq E$ such that $(H, \cdot)$ is a CEH, $|H| = 9$ and $|H \cap S| = 4$.*

Proof. Consider Fig. 1-b with mutual cosets H, K, L satisfying $|H| = |K| = |L| = 27$. Since $|S| = 21$, by symmetry we may suppose $|H \cap S| \geq 7$. By Lemma 4 $\exists H^* \subseteq H$ such that $(H^*, \cdot)$ is a CEH, $|H^*| = 9$ and $|H^* \cap S| = 4$. □

We will use Fig. 4 with the lettering $a, b, c, d, \cdots, i$ repeatedly in the rest of this paper and in the subsequent part II.

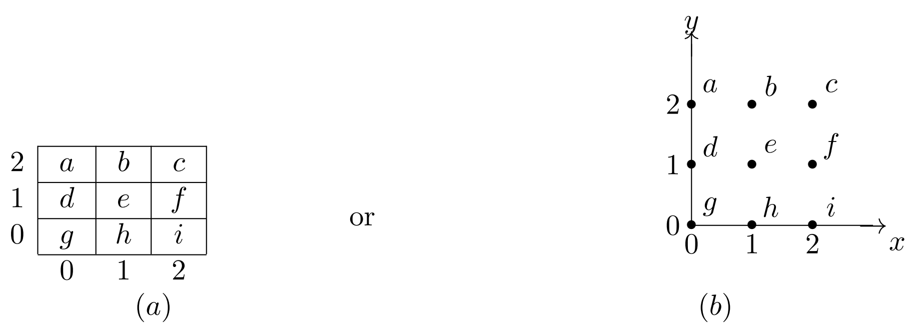


Fig. 4. Drawings of E_3^2, the CEH of 9 elements.

Using the letters $a, b, \cdots, i$, it is obvious that $ab = c, ge = c, bd = i, ah = f, fe = d$, etc. We use the proof of the easy Lemma 8 to introduce the reader to the remaining machinery that we will use. We explain the proof in detail so that we can slightly abbreviate the corresponding proofs later in the subsequent part II.

Lemma 8. *Suppose $(E, \cdot)$ is a CEH, $|E| = 3^4$, $S \subseteq E$ and $|S| = 22$. Then S is not SET-free.*

Proof. We assume that S is SET-free. By lemma 7 let us use Fig. 1-a with $|H^*| = |a_i| = |a_i^*| = 9, i = 1, 2, 3, 4,$and $|H^* \cap S| = 4$. Then $|A_i^*| = 18, i = 1, 2, 3, 4$.

Of course, $\forall i = 1, 2, 3, 4, (A_i^* \cup H^*, \cdot)$ is a CEH and $|A_i^* \cup H^*| = 27$. Therefore, $\forall i = 1, 2, 3, 4$, $|A_i^* \cap S| \leq 5$ since $|H^* \cap S| = 4$ and S is SET-free.

By symmetry, we can assume $\forall i \le j, |A_i^* \cap S| \ge |A_j^* \cap S|$ and $\forall i, |a_i \cap S| \ge |a_i^* \cap S|$.

Therefore, since $|S| = 22, |H^* \cap S| = 4$ and each $|A_i^* \cap S| \le 5$, we know that $|A_1^* \cap S| = |A_2^* \cap S| = 5$. Since each $(H^* \cup A_i^*, \cdot)$ is a CEH containing 27 elements, since $|H^* \cap S| = 4$, since $|A_1^* \cap S| = |A_2^* \cap S| = 5$, since $\forall i = 1, 2, 3, 4, |a_i \cap S| \ge |a_i^* \cap S|$ and since $\forall i = 1, 2, 3, 4, H^*, a_i, a_i^*$ are mutual cosets in $(E, \cdot)$, from Lemma 6 we know that $|a_1 \cap S| = |a_2 \cap S| = 4$ and $|a_1^* \cap S| = |a_2^* \cap S| = 1$.

Now the factor CEH $(\{H^*, a_1, a_1^*, \cdots, a_4, a_4^*\}, \cdot)$ contains 9 elements. Also, the cosets H^*, a_1, a_2 are independent in this 9 element coset CEH. Therefore, $\exists$ an isomorphism $f : (\{H^*, a_1, a_1^*, \cdots, a_4, a_4^*\}, \cdot) \to E_3^2$ of Fig. 4 such that $f(H^*) = a, f(a_1) = b$, and $f(a_2) = d$. Of course, $f(a_1^*) = f(H^* \cdot a_1) = f(H^*)f(a_1) = c$ and $f(a_2^*) = g$.

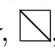
Of course, the set $\{f(a_3), f(a_3^*)\}$ equals one of the two sets $\{e, i\}$ or $\{f, h\}$ and $\{f(a_4), f(a_4^*)\}$ equals the other of these two sets. Define the non-negative integers $\overline{a}, \overline{b}, \cdots, \overline{i}$ as follows. $\overline{a} = |f^{-1}(a) \cap S| = |H^* \cap S|, \overline{b} = |f^{-1}(b) \cap S| = |a_1 \cap S|, \overline{c} = |f^{-1}(c) \cap S| = |a_1^* \cap S|, \overline{d} = |f^{-1}(d) \cap S| = |a_2 \cap S|$, etc. That is, $\forall j \in \{a, b, \ldots, i\},\ \overline{j} = |f^{-1}(j) \cap S|$. It will be convenient to place dots in the square of Fig. 4-a as we proceed. That is, we place $\overline{a}$ dots in square a, $\overline{b}$ in b, etc.. We know the following: $\overline{a} = \overline{b} = \overline{d} = 4, \overline{c} = \overline{g} = 1$. Also, $\overline{e} + \overline{f} + \overline{h} + \overline{i} = 8$. Now $f^{-1}(b), f^{-1}(d), f^{-1}(i)$ are mutual cosets in the original $(E, \cdot)$, which implies $\overline{b} + \overline{d} + \overline{i} \le 9$. Therefore, $\overline{i} \le 1$ since $\overline{b} = \overline{d} = 4$. Therefore, $\overline{e} + \overline{f} + \overline{h} \ge 7$. We show that $\overline{e} \ge 2, \overline{f} \ge 2, \overline{h} \ge 2$.

The reasoning is the same (i.e., symmetric) in each case. Therefore, by this symmetry of reasoning, let us suppose $\overline{e} \le 1$. Then $\overline{f} + \overline{h} \ge 6$, which implies $\overline{a} + \overline{f} + \overline{h} \ge 10$. Now $f^{-1}(a), f^{-1}(f), f^{-1}(h)$ are mutual cosets in $(E, \cdot)$, which implies $\overline{a} + \overline{f} + \overline{h} \le 9$, a contradiction. Therefore, we now know that $\overline{e} \ge 2, \overline{f} \ge 2, \overline{h} \ge 2$.

Also, if two of $\overline{e}, \overline{f}, \overline{h}$ *exceeds* 2, we would also have a contradiction. Again the reasoning is the same (i.e., symmetric) in each case. So by symmetry let us suppose $\overline{e} \ge 3, \overline{h} \ge 3$. Now $f^{-1}(b), f^{-1}(e), f^{-1}(h)$ are mutual cosets in $(E, \cdot)$ and $\overline{b} = 4$. Therefore, $\overline{b} + \overline{e} + \overline{h} \ge 10$ is a contradiction since $\overline{b} + \overline{e} + \overline{h} \le 9$. Therefore, two of $\overline{e}, \overline{f}, \overline{h}$ must equal 2 and the third is 3 or more. We now show that this is impossible. Again, the reasoning is the same (i.e., symmetric) in each case. So by symmetry let us suppose $\overline{e} = \overline{f} = 2$ and $\overline{h} \ge 3$. Now $f^{-1}(b)f^{-1}(e), f^{-1}(h)$ are mutual cosets in $(E, \cdot)$. Therefore, $\overline{b} + \overline{e} + \overline{h} \le 9$, which implies that $\overline{e} = 2$ and $\overline{h} = 3$. But from Lemma 6, $\overline{b} = 4, \overline{e} = 2, \overline{h} = 3$ is impossible. $\square$

The ideas used in the above proof are used repeatedly in part II. The reader will observe that Lemma 8 was proved by using only trivial arithmetic calculations. Before we deal with $|S| = 20$ in another paper, we need to tighten up on Lemma 4. We do this now to further illustrate the machinery.

Lemma 9. *Suppose* $(E, \cdot)$ *is a CEH,* $|E| = 27, S \subseteq E, |S| = 6$ *and* S *is SET-free. Then* $\exists H \subseteq E$ *such that* $(H, \cdot)$ *is a CEH,* $|H| = 9$*, and* $|H \cap S| = 4$.

Proof. Let us assume that the conclusion of lemma 9 is false for S. Now $\exists H^* \subseteq E$ such that $(H^*, \cdot)$ is a CEH, $|H^*| = 3$ and $|H^* \cap S| = 2$. Use Fig 1-a with this H^*. Since $|S| = 6$, $|H^* \cap S| = 2$, and lemma 9 is false for S, it is easy to see that $\forall i = 1, 2, 3, 4, |A_i^* \cap S| = 1$. Also, assuming $|a_i \cap S| \ge |a_i^* \cap S|$, $i = 1, 2, 3, 4$, we see that $|a_i \cap S| = 1$ and $|a_i^* \cap S| = 0$, $i = 1, 2, 3, 4$. Let us define the unique isomorphism $f : (\{H^*, a_1, a_1^*, \ldots, a_4, a_4^*\}, \cdot) \to E_3^2$ (of Fig 4-a) such that $f(H^*) = a, f(a_1) = b$, and $f(a_2) = d$. Also, $\overline{a}, \overline{b}, \overline{c}, \ldots, \overline{i}$ are defined exactly as before. That is $\overline{j} = |f^{-1}(j) \cap S|, j = a, b, \ldots, i$. We now know the following about $\overline{a}, \overline{b}, \overline{c}, \ldots, \overline{i}$: $\overline{a} = 2, \overline{b} = \overline{d} = 1, \overline{c} = \overline{g} = 0, \overline{e} + \overline{i} = 1, \overline{f} + \overline{h} = 1$. By geometric symmetry, , we may assume that $\overline{f} = 1, \overline{h} = 0$. We consider two cases for $(\overline{e}, \overline{i})$: either $(\overline{e}, \overline{i}) = (1, 0)$ or $(\overline{e}, \overline{i}) = (0, 1)$. In case 1 we see that $\overline{b} + \overline{e} + \overline{h} = 2$ and in case 2 we see that $\overline{c} + \overline{f} + \overline{i} = 2$. Therefore in case 1, $(f^{-1}(b) \cup f^{-1}(e) \cup f^{-1}(h), \cdot)$ is a CEH of 9 elements containing exactly 2 elements of S. In case 2, $(f^{-1}(c) \cup f^{-1}(f) \cup f^{-1}(i), \cdot)$ is a CEH of 9 elements containing exactly 2 elements of S. In either case, $\exists \overline{H} \subseteq E$ such that $|\overline{H}| = 9, (\overline{H}, \cdot)$ is a

CEH and $|\overline{H} \cap S| = 2$. Now $\exists H^* \subseteq \overline{H}$ such that $|H^*| = 3$, $(H^*, \cdot)$ is a CEH and $|H^* \cap S| = 2$. We use Fig 1-a with $A_4^* = \overline{H} \backslash H^*$. Therefore, since $|S| = 6$, $|H^* \cap S| = 2$ and $|A_4^* \cap S| = 0$, we see that $\exists i \in \{1, 2, 3\}$ such that $|A_i^* \cap S| = 2$. Using $H = H^* \cup A_i^*$, we see that $|H| = 9, (H, \cdot)$ is a CEH and $|H \cap S| = 4$. □

4 Concluding Remarks

The two techniques of Fig 1 and Fig 4, when combined with the other ideas of this paper are quite effective in studying the 27 and 81 card SET games. The basic ideas of group theory can be added later, as well. The following is a sample of results that these techniques yield. Some of these will be proved in another paper.

1. The classification up to isomorphism and the construction of the Hasse diagram for the SET-free subsets of E_3^3.

2. The proof that for any $S \subseteq E_3^4$ if $|S| = 21$, then S is not SET-free.

3. Any two SET-free subsets of E_3^4 of cardinality 20 are isomorphic.

4. If S and $\overline{S}$ are two such SET-free subsets of E_3^4 of cardinality 20, and $a \in S$ and $\overline{a} \in \overline{S}$, there is an isomorphism f of E_3^4 such that $f(S) = \overline{S}$ and $f(a) = \overline{a}$. Letting $\overline{S} = S$, we see that S is transitive. It turns out that S is not 2-transitive, but it is "almost" 2-transitive.

5. Any two SET-free subsets of E_3^4 of cardinality 19 are isomorphic.

6. If S and $\overline{S}$ are two SET-free subsets of E_3^4 of cardinality 19, then $\exists x \in S, \overline{x} \in \overline{S}$ such that $\forall a \in S \setminus \{x\}, \forall \overline{a} \in \overline{S} \setminus \{\overline{x}\}$ there is an isomorphism f of E_3^4 such that $f(S) = \overline{S}$ and $f(a) = \overline{a}$. Letting $\overline{S} = S$, we might say that S is "almost" transitive.

7. If $S \subseteq E_3^4$ is SET-free and has cardinality 19, there is a unique $a \in E_3^4 \setminus S$ such that $S \cup \{a\}$ is SET-free.

8. If $S \subseteq E_3^4$ is SET-free and $|S| = k$, then there is a subCEH H of E_3^4 of cardinality 27 such that $|H \cap S| \geq h$ where (k, h) is any of the ordered pairs $(19, 9), (16, 8), (13, 7), (10, 6), (7, 5), (4, 4)$.

9. If $S \subseteq E_3^4$ is SET-free and $|S| = 9$, then there is a subCEH H of E_3^4 of cardinality 27 such that $|H \cap S| \geq 6$ if and only if there is a subCEH H^* of E_3^4 of cardinality 9 such that $|H^* \cap S| = 4$.

10. If S is a SET-free subset of E_3^4 of cardinality 20, and H, K, and L are mutual cosets of E_3^4 of cardinality 27, then $\{|H \cap S|, |K \cap S|, |L \cap S|\} = \{9, 9, 2\}$ or $\{|H \cap S|, |K \cap S|, |L \cap S|\} = \{6, 6, 8\}$.

11. If S is a SET-free subset of E_3^4 of cardinality 20 and $H_i, i = 1, 2, \ldots, 9$ is a family of nine cosets of E_3^4 all of cardinality 9, then the multiset $\{|H_i \cap S| \ : \ i = 1, 2, \ldots, 9\}$ is one of $\{0, 1, 1, 1, 1, 4, 4, 4, 4\}, \{0, 0, 2, 3, 3, 3, 3, 3, 3\}, \{2$ The reader might like to solve the analogous problem for E_3^3 when $|S| = 9$ and $|H_i| = 3$.

12. In 11, we have detailed information about the structure of $\{|H_i \cap S| \ : \ i = 1, 2, \ldots, 9\}$.

13. We have results analogous to those of 9, 10, and 11 for $S \subseteq E_3^4$ when S has cardinality 19 and SET-free.

14. In E_3^3 we found all theorems analogous to the following example. Suppose $S \subseteq E_3^3$, $|S| = 7$ and S is SET-free. Then there exists three mutual cosets H, K, and L all with cardinality 9 in E_3^3 such that $|H \cap S| = 4, |K \cap S| = 3$, and $|L \cap S| = 0$.

15. In E_3^3 we have found, up to isomorphism, all $S \subseteq E_3^3$ such that S is maximal SET-free, where maximal SET-free is defined in definition 13.

16. If S is a SET-free subset of E_3^4 of cardinality 20, there is a simple construction that finds all automorphisms $f : E_3^4 \to E_3^4$ that satisfy $f(S) = S$. The number of these isomorphisms equals $20 \cdot 18 \cdot 8 = 2880$.

17. Using 16, we know that the number of distinct set-free subsets of E_3^4 of cardinality 20 equals $(81 \cdot 80 \cdot 78 \cdot 72 \cdot 54) \div (20 \cdot 18 \cdot 8) = 682{,}344$.

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