The Cubic Curve is an Adding Machine

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The authors of [1] consider a special quadratic function property which involves real numbers x, y, and $x + y$. This property is used to prove a well-known identity about Fibonacci numbers. They then proceed to show that under mild restrictions, this property characterizes this type of function. Here, we consider a property which connects certain types of cubic polynomials with the addition of real numbers. In the spirit of [1], this is followed by a proof that under very simple conditions (e.g., that the function is bounded on some open interval (c, d)), this property characterizes certain polynomials of degree at most 3.

Our first theorem describes this additive connection. It is not new (see [2], [3], [4]) but the second part spells out in detail a special case which is usually omitted in the literature. We believe that our other results are essentially new. Figure 1 goes here.

THEOREM 1. Let $f(x) = A + Bx + Cx^3$, with $C \neq 0$.

(a) If $u \neq v$ and L is the line through the points $(-u, f(-u))$ and $(-v, f(-v))$, then L intersects the graph of $f(x)$ also at the point $(u + v, f(u + v))$.

(b) If L is the tangent line to the graph of $f(x)$ at the point $(-u, f(-u))$, then L intersects the graph of $f(x)$ also at the point $(2u, f(2u))$.

Proof.

(a) If the equation of L is $y = mx + b$, then $g(x) = f(x) - mx - b$ is a cubic polynomial with quadratic term 0 and two distinct roots at $x = -u$

and $x = -v$. Since the quadratic term of $g(x)$ is 0, the sum of the roots of $g(x)$ is 0. Therefore, the third root w of $g(x)$ satisfies $w - u - v = 0$ or $w = u + v$. It follows that L intersects the graph of $f(x)$ also at the point $(u + v, f(u + v))$.

(b) The equation of the tangent line L to the graph of $f(x)$ at $(-u, f(-u))$ is

$$y_T = f(-u) + f'(-u)(x + u).$$

By Taylor's Formula,

$$\begin{aligned} f(x) &= \frac{f'''(-u)}{3!}(x+u)^3 + \frac{f''(-u)}{2!}(x+u)^2 + \frac{f'(-u)}{1!}(x+u) + f(-u) \\ &= (x+u)^2 [C(x+u) - 3Cu] + y_T \\ &= C(x+u)^2 (x - 2u) + y_T. \end{aligned}$$

Then, since $C \neq 0$ and $g(x) = f(x) - y_T = C(x+u)^2(x-2u)$, $g(x)$must have a double root at $x = -u$ and the remaining root at $x = 2u$. Hence, L intersects the graph of $f(x)$ also at the point $(2u, f(2u))$. ■

Part (b) of THEOREM 1 is intended to extend Part (a) to the case where $u = v$. Also, it should be noted that in Part (a), the point $(u + v, f(u + v))$ might coincide with either $(-u, f(-u))$ or $(-v, f(-v))$. This occurs when $v = -2u$ or $u = -2v$, respectively. When this happens, we get a situation similar to Part (b).

By THEOREM 1, functions of the form $f(x) = A + Bx + Cx^3$ satisfy:

PROPERTY (★). For all real x and y, the points $(-x, f(-x))$, $(-y, f(-y))$, and $(x + y, f(x + y))$ are collinear.

This property is a little more general than the result of THEOREM 1 in that it also clearly applies to straight lines (i.e., functions of the above type with $C = 0$). Our second result shows that under a rather minimal additional restriction on $f(x)$, PROPERTY (★) characterizes functions of the type $f(x) = A + Bx + Cx^3$, for any constants A, B, C.

THEOREM 2. If $f(x)$ satisfies PROPERTY (★) and $f'(0)$ exists, then there are constants A, B, and C such that $f(x) = A + Bx + Cx^3$ for all real x.

Proof. Let $f(x)$ be a function which satisfies PROPERTY (★) and assume that $f'(0)$ exists. Then, for all real x and y such that $-x$, $-y$, and $x + y$ are distinct, PROPERTY (★) implies that we may equate the slopes determined by the pair $(-x, f(-x))$ and $(x+y, f(x+y))$ and the pair $(-y, f(-y))$ and $(x+y, f(x+y))$ to obtain

$$\frac{f(x+y)-f(-x)}{2x+y}=\frac{f(x+y)-f(-y)}{x+2y}$$

or

$$(y-x)\,f(x+y)=(x+2y)\,f(-x)-(2x+y)\,f(-y)\,. \tag{1}$$

Since it is easily checked that condition (1) is also true if at least two of $-x$, $-y$, and $x+y$ are equal, condition (1) is true for all real x and y. Further, a straightforward computation shows that condition (1) holds for any function of the form $f(x)+M$, where M is a fixed constant. As a result, if we let $g(x)=f(x)-f(0)$, then $g(0)=0$, $g'(0)=f'(0)$, and

$$(y-x)\,g(x+y)=(x+2y)\,g(-x)-(2x+y)\,g(-y) \tag{1*}$$

for all real x and y.

By setting $y=0$ in (1*), we get

$$-xg(x)=xg(-x)$$

for all x and hence (since $g(0)=0$),

$$g(-x)=-g(x) \tag{2}$$

for all real x. Then, when $x\neq y$, (1*) becomes

$$g(x+y)=\frac{x+2y}{x-y}g(x)+\frac{2x+y}{y-x}g(y)$$

and we have

$$\begin{aligned}\frac{g(x+y)-g(x)}{y}&=\frac{3}{x-y}g(x)+\frac{2x+y}{y-x}\cdot\frac{g(y)}{y}\\&=\frac{3}{x-y}g(x)+\frac{2x+y}{y-x}\cdot\frac{g(y)-g(0)}{y}\end{aligned} \tag{3}$$

when $y\neq x,0$. Therefore, for $x\neq 0$ and $B=g'(0)$,

$$\begin{aligned}g'(x)&=\lim_{y\to 0}\left[\frac{3}{x-y}g(x)+\frac{2x+y}{y-x}\cdot\frac{g(y)-g(0)}{y}\right]\\&=\frac{3}{x}g(x)-2B\end{aligned} \tag{4}$$

For $x > 0$, (4) is a homogeneous differential equation which can be solved by using the substitution $g(x) = x^3 z$. Its general solution is

$$g(x) = Bx + Cx^3, \tag{5}$$

for some arbitrary constant C. Then, condition (2) and the fact that $g(0) = 0$ imply that (5) is true for all real x. Finally, if $A = f(0)$, then for all real x,

$$f(x) = g(x) + A = A + Bx + Cx^3.$$

■

In this proof, the key step is to show that condition (1) and the existence of $f'(0)$ imply that $g(x) = f(x) - f(0)$ satisfies a first order differential equation. For another simple example of this approach, see [5].

The next few results show some ways to relax the condition that $f'(0)$ exists and still get the desired conclusion.

COROLLARY 3. If $f(x)$ satisfies PROPERTY (★) and $f'(a)$ exists for at least one real number a, then there are constants A, B, and C such that $f(x) = A + Bx + Cx^3$ for all real x.

Proof. We may suppose that $a \neq 0$. If $g(x) = f(x) - f(0)$, then as in the proof of THEOREM 2, condition (3) holds for $x = a$. Therefore,

$$\frac{g(a+y) - g(a)}{y} = \frac{3}{a-y} g(a) + \frac{2a+y}{y-a} \cdot \frac{g(y) - g(0)}{y}$$

or equivalently,

$$\frac{g(y) - g(0)}{y} = \frac{y-a}{2a+y} \cdot \frac{g(a+y) - g(a)}{y} + \frac{3}{2a+y} \cdot g(a)$$

for $y \neq 0$, a, or $-2a$. As a result,

$$\begin{aligned} f'(0) = g'(0) &= \lim_{y \to 0} \left[\frac{y-a}{2a+y} \cdot \frac{g(a+y) - g(a)}{y} + \frac{3}{2a+y} \cdot g(a) \right] \\ &= -\frac{1}{2} g'(a) + \frac{3}{2a} g(a). \end{aligned}$$

The conclusion now follows from THEOREM 2. ■

LEMMA 4. If $f(x)$ satisfies PROPERTY (★) and $f(x)$ is continuous at $x = a$, where $a \neq 0$, then $f'\left(-\frac{a}{2}\right)$ exists.

Proof. Since $f(x)$ is continuous at $x = a$, $\lim_{h\to 0} f(a-h) = f(a)$. Then, apply PROPERTY (★) to $x = \dfrac{a}{2}$ and $y = \dfrac{a}{2} - h$ (with h chosen so that $-x$, $-y$, and $x+y$ are distinct) to obtain

$$\frac{f(-y) - f(-x)}{x-y} = \frac{f(x+y) - f(-x)}{2x+y}$$

or

$$\frac{f\left(-\frac{a}{2}+h\right) - f\left(-\frac{a}{2}\right)}{h} = \frac{f(a-h) - f\left(-\frac{a}{2}\right)}{\frac{3a}{2}-h}.$$

Therefore,

$$f'\left(-\frac{a}{2}\right) = \lim_{h\to 0} \frac{f(a-h) - f\left(-\frac{a}{2}\right)}{\frac{3a}{2}-h} = \frac{2}{3}\cdot\frac{f(a) - f\left(-\frac{a}{2}\right)}{a}. \qquad \blacksquare$$

LEMMA 5. If $f(x)$ satisfies PROPERTY (★) and $f(x)$ is bounded on some open interval (c,d), then for any non-zero point $a \in (c,d)$, $f(x)$ is continuous at $x = -\dfrac{a}{2}$.

Proof. Let $a \in (c,d)$ with $a \neq 0$. As in LEMMA 4, for appropriately chosen values of h, we may apply PROPERTY (★) to $x = \dfrac{a}{2}$ and $y = \dfrac{a}{2} - h$ to get

$$\frac{f\left(-\frac{a}{2}+h\right) - f\left(-\frac{a}{2}\right)}{h} = \frac{f(a-h) - f\left(-\frac{a}{2}\right)}{\frac{3a}{2}-h}$$

or

$$f\left(-\frac{a}{2}+h\right) = f\left(-\frac{a}{2}\right) + h\cdot\frac{f(a-h) - f\left(-\frac{a}{2}\right)}{\frac{3a}{2}-h}.$$

Therefore,

$$\lim_{h\to 0} f\left(-\frac{a}{2}+h\right) = \lim_{h\to 0}\left[f\left(-\frac{a}{2}\right) + h\cdot\frac{f(a-h) - f\left(-\frac{a}{2}\right)}{\frac{3a}{2}-h}\right] = f\left(-\frac{a}{2}\right)$$

since

$$\lim_{h\to 0}\frac{h}{\frac{3a}{2}-h} = 0$$

and $f(a-h) - f\left(-\frac{a}{2}\right)$ is bounded for small values of h. It follows that $f(x)$ is continuous at $x = -\frac{a}{2}$. ■

By combining COROLLARY 3 and LEMMAS 4 and 5, we have

THEOREM 6. If $f(x)$ satisfies PROPERTY (★) and $f(x)$ is bounded on some open interval (c,d), then there are constants A, B, and C such that $f(x) = A + Bx + Cx^3$ for all real x.

Note that if $f(x)$ satisfies PROPERTY (★) and $f(x)$ is not a polynomial of degree at most 3, then $f(x)$ must be unbounded on every open interval (c,d). Such a function would have to be quite wild.

Our final results show that THEOREMS 1 and 6 can be extended in a natural way to general cubic polynomials.

THEOREM 7. Let $f(x) = A + BX + Cx^2 + Dx^3$, with $D \neq 0$.

(a) If $u \neq v$ and L is the line through the points $(-u, f(-u))$ and $(-v, f(-v))$, then L intersects the graph of $f(x)$ also at the point $\left(u + v - \frac{C}{D}, f\left(u + v - \frac{C}{D}\right)\right)$.

(b) If L is the tangent line to the graph of $f(x)$ at the point $(-u, f(-u))$, then L intersects the graph of $f(x)$ also at the point $\left(2u - \frac{C}{D}, f\left(2u - \frac{C}{D}\right)\right)$.

Proof. We leave the details to the reader. One possibility is to imitate the approach used in THEOREM 1, using the fact that if $g(x) = \bar{A} + \bar{B}x + \bar{C}x^2 + \bar{D}x^3$, with $\bar{D} \neq 0$, then the sum of the roots of $g(x)$ is $-\frac{\bar{C}}{\bar{D}}$. Another suggestion is to show that the function $\bar{f}(x) = f\left(x - \frac{C}{D}\right)$ satisfies THEOREM 1 and use this to produce the desired conclusions.

As in our previous situation, arbitrary cubic polynomials satisfy the following more general property.

PROPERTY (★★). There is a constant K such that for all real x and y, the points $(-x, f(-x))$ $(-y, f(-y))$, and $(x + y - K, f(x + y - K))$ are collinear.

The last result shows that PROPERTY (★★) essentially characterizes cubic polynomials.

THEOREM 8. If $f(x)$ satisfies PROPERTY (★★) and $f(x)$ is bounded on some open interval (c,d), then there are constants A, B, C, and D such that $f(x) = A + BX + Cx^2 + Dx^3$ for all real x.

Proof. Let $\bar{f}(x) = f\left(x - \frac{K}{3}\right)$ for all real x. Then, if we apply PROPERTY ($\bigstar\bigstar$) to the points $x + \frac{K}{3}$ and $y + \frac{K}{3}$, the result is that $\left(-x - \frac{K}{3}, f\left(-x - \frac{K}{3}\right)\right)$, $\left(-y - \frac{K}{3}, f\left(-y - \frac{K}{3}\right)\right)$, and $\left(x + y - \frac{K}{3}, f\left(x + y - \frac{K}{3}\right)\right)$ are collinear, i.e., $\left(-x - \frac{K}{3}, \bar{f}(-x)\right)$, $\left(-y - \frac{K}{3}, \bar{f}(-y)\right)$, and $\left(x + y - \frac{K}{3}, \bar{f}(x+y)\right)$ are collinear. Let L be a line containing these 3 points and L_K be the line obtained by translating L horizontally by $\frac{K}{3}$ units. Since $(\bar{x}, \bar{y})$ is on L if and only if $\left(\bar{x} + \frac{K}{3}, \bar{y}\right)$ is on L_K, it follows that the points $\left(-x, \bar{f}(-x)\right)$, $\left(-y, \bar{f}(-y)\right)$, and $\left(x+y, \bar{f}(x+y)\right)$ are on L_K and hence, these points are collinear also. Therefore, $\bar{f}(x)$ satisfies PROPERTY ($\bigstar$). Further, since $f(x)$ is bounded on (c, d), $\bar{f}(x)$ is bounded on $\left(c + \frac{K}{3}, d + \frac{K}{3}\right)$. By THEOREM 6, there are constants $\bar{A}$, $\bar{B}$, and $\bar{C}$ such that $\bar{f}(x) = \bar{A} + \bar{B}x + \bar{C}x^3$ for all real x. Finally, for all real x,

$$
\begin{aligned}
f(x) = \bar{f}\left(x + \frac{K}{3}\right) &= \bar{A} + \bar{B}\left(x + \frac{K}{3}\right) + \bar{C}\left(x + \frac{K}{3}\right)^3 \\
&= A + Bx + Cx^2 + Dx^3,
\end{aligned}
$$

for appropriate constants A, B, C, and D. ■

Observe that if the details of the last step of this proof are carried out, we get $C = \bar{C}K$ and $D = \bar{C}$. If $D \neq 0$, it follows that $K = \frac{C}{D}$, which connects our result with THEOREM 7. We might also note that there is an algebraic connection between THEOREMS 1 and 7. It is an exercise in many abstract algebra books to show that for any constant K, $x \oplus y = x + y - K$ is a group operation on $\mathbb{R}$ and that the groups $(\mathbb{R}, \oplus)$ and $(\mathbb{R}, +)$ are isomorphic. Hence, general cubic polynomials produce a group operation on $\mathbb{R}$ which reduces to real addition when the polynomial has no quadratic term.

THEOREM 1 can be generalized as follows. Suppose $p(x, y) = 0$ is a curve satisfying (1) $p(x, y)$ is a polynomial in x and y, (2) each term $cx^a y^b$ of $p(x, y)$ satisfies $c \neq 0$, $0 \leq a$, $0 \leq b$, and $0 \leq a + b \leq 3$, (3) at least one term $cx^a y^b$ satisfies $a + b = 3$, and (4) $p(x, y)$ cannot be factored as a product of two non-constant polynomials. These curves are called *elliptic curves* (see

[6]). Also, suppose $p(x, y) = 0$ has a true inflection point (as opposed to a cusp) which we call $\bar{0}$. It is possible, using a continuous monotone scale, to turn any such curve $p(x, y) = 0$ into an adding machine. Of course, this would normally be extremely clumsy as compared with our simple scale.

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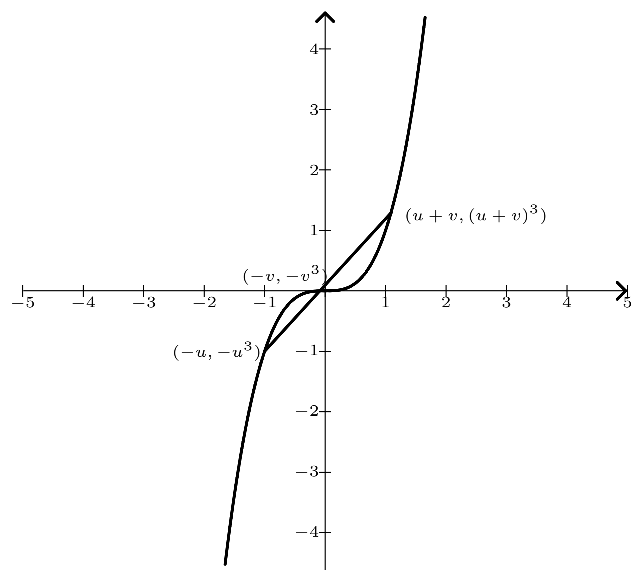

Fig. 1. The cubic curve is an adding machine.