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DISTRIBUTIVE LATTICES GENERATED FROM SINGLE POLYNOMIALS ON GENERALIZED BASES

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Abstract. For $n = 1, 2, 3, \ldots$ let us define $R^n = r_1 r_2 \ldots r_n$ where each $0 < r_i = p_i/q_i$ and each $p_i, q_i \in \{1, 2, 3, \ldots\}$ and each p_i/q_i is not necessarily reduced to lowest terms. We call $R^0 = 1$, R, R^2, R^3, $\ldots$ a generalized base.

In this paper we study both finite and infinite distributive lattices on sets of polynomials

$$p = \sum_{i=0}^{n} a_i R^i = N$$

where $N \in \{0, 1, 2, 3, \ldots\}$ or $N \in \mathbb{Q}^+$ is fixed and each $a_i \in \{0, 1, 2, \ldots\}$.

Using the algorithm defined in section 10, we show that all of our lattices have a fractal type property that allows the entire lattice to be completely generated (ie, constructed) from every polynomial element of the lattice. This is very important. It means that our lattices are qualitatively different from ordinary lattices. In a closing section, we show how to extend the base further. We create infinite lattices and we also generalize other parts of the paper.

1. Introduction. During the last 10 years, the concept called Exploding Dots [2] has taken the pre-college mathematics education community by storm. The idea easily enables teachers to understand ordinary place value and to build number representations and arithmetic which use fractional and negative bases. It's creator James Tanton has spoken worldwide on the topic to enthusiastic audiences. This paper represents our effort to understand these ideas and to begin to ask some technical questions about the underlying structure.

Let us define $R^0 = 1$ and for $n = 1, 2, 3, \ldots$ define $R^n = r_1 r_2 \ldots r_n$ where each $0 < r_i = p_i/q_i$ and each $p_i, q_i \in \{0, 1, 2, \ldots\}$ and where p_i/q_i is not necessarily reduced to lowest terms. Also no restrictions are added to r_i, r_j other that the facts that all p_i, q_i are positive integers and $R^0 = 1$, R, R^2, $\ldots$ satisfies a certain convergence condition which we specify later. It is also possible to drop this convergence condition. We call $R^0 = 1$, R, R^2, R^3, $\ldots$ a generalized base (or a regular generalized base).

If we place more restrictions on $R^0 = 1$, R, R^2, R^3, $\ldots$ we can prove a larger number of base related theorems. On the other hand if we place almost no restrictions on $R^0 = 1$, R, R^2, R^3, $\ldots$ then we can create an extremely large number of distributive lattices (both finite and infinite) on the following sets of polynomials. In this paper we have chosen the later option.

These distributive lattices are created on sets of polynomials

$$p = \sum_{i=0}^{n} a_i R^i = N$$

where $N \in \{0, 1, 2, \ldots\}$ or $N \in \mathbb{Q}^+$ is fixed and each $a_i \in \{0, 1, 2, \ldots\}$. Q^+ is the set of positive rationals.

All of our lattices have a fractal type property that allows the entire lattice to be completely generated from every polynomial element of the lattice.

If we require each r_i to satisfy, $1 < r_i = p_i/q_i$ where p_i/q_i is reduced to lowest terms and $(p_i p_j, q_i q_j)$ are relatively prime for all i, j then we call $R^0 = 1$, R, R^2, $\ldots$ a strong generalized base. In this paper we have almost no interest in strong generalized bases.

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We need to mention that the collection of all finite distributive lattices is equivalent to the collection of all finite topological spaces and we only need to use the t_1- spaces. We discuss this near the end of the paper. We have not been able to prove that our methods will generate all finite or all countably infinite distributive lattices. In light of Section 10 this conjecture seems reasonable. In a closing section we generalized our methods and state how to construct infinite distributive lattices.

2. A Basic Definition.

DEFINITION 1. *Suppose $R^0 = 1$, R, R^2, $\ldots$ is a generalized base where $R^n = r_1r_2\ldots r_n$ for $n = 1,2,3,\ldots$. Suppose $N \in \{0,1,2,\ldots\}$ or $N \in \mathbb{Q}^+$ is arbitrary but fixed. If*

$$p = \sum_{i=0}^{n} a_i R^i = N$$

and each $a_i \in \{0,1,2,\ldots,p_{i+1}-1\}$ we call this polynomial p a base R expansion of N.

Note that each $a_i \in \{0,1,2,\ldots,p_{i+1}-1\}$. Thus, $a_0 \in \{0,1,2,\ldots,p_1-1\}$, $a_1 \in \{0,1,2,\ldots,p_2-1\}$, etc.

If $R^0 = 1$, R, R^2, $\ldots$ is a strong generalized base we can show that if $N \in \{0,1,\ldots\}$ or $N \in \mathbb{Q}^+$ and $p = \sum_{i=0}^{n} a_i R^i = N$ where each $a_i \in \{0,1,2,\ldots,p_{i+1}-1\}$ then the a_i's must be unique. Also if $R^0 = 1$, R, R^2, $\ldots$ is a strong generalized base and $N \in \{0,1,2,3,\ldots\}$ we can show that a polynomial $p = \sum_{i=0}^{n} a_i R^i = N, a_i \in \{0,1,2,\ldots,p_{i+1}-1\}$ will always exist. Also, when we add the convergence condition to the regular generalized base $R^0 = 1$, R, R^2, $\ldots$ we soon show that any $N \in \{0,1,2,3,\ldots\}$ can be represented by $p = \sum_{i=0}^{n} a_i R^i = N, a_i \in \{0,1,2,\ldots,p_{i+1}-1\}$ in any such generalized base. However, as Theorem 2 shows this representation of N may not always be unique.

THEOREM 2. *Let $R^0 = 1$, R, R^2, $\ldots$ be a generalized base and $N \in \{0,1,2,\ldots\}$. Suppose $p = \sum_{i=0}^{n} a_i R^i = N$ where each $a_i \in \{0,1,2,\ldots,p_{i+1}-1\}$. Then this polynomial p may not necessarily be unique.*

Proof. The following is an example of where p is not unique.

Let $r_1 = \frac{3}{1}, r_2 = \frac{4}{3}, r_3 = \frac{7}{2}, r_4 = \frac{3}{2}$ and $N = 15$. Then

$$15 = 1 + 2\left(\frac{3}{1}\right) + 2\left(\frac{3}{1}\right)\left(\frac{4}{3}\right) = 1 + 1\left(\frac{3}{1}\right)\left(\frac{4}{3}\right)\left(\frac{7}{2}\right) = 1\left(\frac{3}{1}\right) + 3\left(\frac{3}{1}\right)\left(\frac{4}{3}\right)$$

.

The reader can check that each $a_i \in \{0,1,2,\ldots,p_{i+1}-1\}$. □

3. Representing $N \in \{0,1,2,\ldots\}$ in the Generalized Base and the Derived Partial Order $(\overline{p}_N, \leq)$.

As always $0 < r_i = \frac{p_i}{q_i}$, $p_i, q_i \in \{1,2,3,\ldots\}$, $R^n = r_1r_2\ldots r_n, n = 1,2,3,\ldots$ is a generalized base. Suppose we wish to represent $N \in \{0,1,2,3,\ldots\}$ as a polynomial $p = \sum_{i=0}^{n} a_i R^i = N, a_i \in \{0,1,2,\ldots,p_{i+1}-1\}$.

Of course, we know that this representation of N may not always be unique. However, the algorithm that we now give produces a unique representation.

First we observe that $p_{k+1}r_1r_2\ldots r_k = q_{k+1}r_1r_2\ldots r_{k+1}$ since $\frac{p_{k+1}}{q_{k+1}} = r_{k+1}$.

Therefore, $p_{k+1}R^k = q_{k+1}R^{k+1}$.

Let us now arrange containers called $R^0 = 1$, R, R^2, R^3, $\ldots$ sequentially in a row starting with the $R^0 = 1$ container on the left and going sequentially to the right with containers $R^0 = 1$, R, R^2, R^3, $\ldots$. At any time we will have a_0, a_1, a_2, $\ldots$ markers in containers $R^0 = 1$, R, R^2, $\ldots$ respectively where a_0, a_1, $a_2 \ldots \in \{0, 1, 2, \ldots\}$.

In the following process if $a_0, a_1, a_2, \ldots, a_n$ markers are in containers $R^0 = 1, R, R^n, \ldots, R^n$ respectively then these markers correspond to the polynomial $p = a_0 + a_1R + a_2R^2 + \ldots + a_nR^n = N$.We initially place $a_0 = N$ markers in container $R^0 = 1$. This corresponds to the constant polynomial $p = a_0 = N$.

If there are at least p_1 markers in the first container $R^0 = 1$ we take p_1 of the markers out of container $R^0 = 1$ and place q_1 markers in the R container which is the 2nd container. This corresponds to $p_1R^0 = q_1R$.

We continue to follow this same general pattern over and over in any arbitrary order that we choose so that in general if there are ever at least p_{k+1} markers in the R^k container we take p_{k+1} of these markers out of the R^k container and place q_{k+1} markers in the R^{k+1} container. This corresponds to the equality $p_{k+1}R^k = q_{k+1}R^{k+1}$. We call this transfer $p_{k+1}R^k \to q_{k+1}R^{k+1}$ a unit forward move or a unit forward transfer and we call these transformations the unit forward transformations.

We continue this process over and over in any arbitrary order until we cannot continue the process any further. This occurs when the number of markers a_k in each container R^k satisfies $0 \le a_k \le p_{k+1} - 1$. We now make the convergence assumption on our generalized base $0 < r_i = \frac{p_i}{q_i}, R^n = r_1r_2 \ldots r_n$.

This assumption states that the above process must always come to an end no matter in what arbitrary order we carry the process out. If we want to stop the above process at some R^k we can let $r_{k+1} = \frac{p_{k+1}}{q_{k+1}} = \frac{\infty}{1}$. When we reach the end of the process we will have defined a polynomial $\overline{p} = \sum_{i=0}^{n} a_iR^i = N$ where each $a_i \in (0, 1, 2, \ldots, p_{i+1} - 1)$. Of course, we actually only need the convergence condition to hold for the specific N in question.

Observation 1: If each $1 < r_i = \frac{p_i}{q_i}$ then the convergence assumption will automatically be true. The reason for this is that each unit forward transfer $p_{k+1}R^k \to q_{k+1}R^{k+1}$ will give us a smaller total number of markers since $1 < r_i = \frac{p_i}{q_i}$ implies that $q_{k+1} < p_{k+1}$. Obviously the total number of markers cannot decrease forever.

We now show that when we carry out the above process to the end that no matter in what arbitrary order we carry out the process we will always end with the exact same polynomial $\overline{p} = N$. Also, the total number of forward unit transfers will always be the same and the total number of forward unit transfers from each $p_{k+1}R^k \to q_{k+1}R^{k+1}$ will always be the same. We call the following argument the standard argument and we call the order $R^0 \to R^1, R^1 \to R^2, R^2 \to R^3$ the standard order. This standard order and standard type argument is used over and over in this paper. To prove the above we see that the total number of different markers that lie in container $R^0 = 1$ (at some time) during the course of our forward transformation is obviously N since we start with N markers in container $R^0 = 1$. Of course, during the course of our forward transfers we are going to take out exactly $\left\lfloor \frac{N}{p_1} \right\rfloor p_1$ markers from the first container $R^0 = 1$ and we are going to end up with exactly $N - \left\lfloor \frac{N}{p_1} \right\rfloor p_1$ markers in the first container $R^0 = 1$, where $\lfloor\ \rfloor$ is the floor function. Also, the total number of forward unit transfers from container $R^0 = 1$ to container $R^1 = R$ is $\left\lfloor \frac{N}{p_1} \right\rfloor$.

Therefore, the number of different markers that lie in the 2nd container R (at some time) during the course of our forward transformations will be $N_1 = q_1 \left\lfloor \frac{N}{p_1} \right\rfloor$. Using this N_1 and reasoning the same way, we see that during the course of our forward unit moves we are going to take out exactly $\left\lfloor \frac{N_1}{p_2} \right\rfloor p_2$ markers from the 2nd container

$R^1 = R$ and we are going to end with exactly $N_1 - \left\lfloor \frac{N_1}{p_2} \right\rfloor p_2$ markers in container R. Also, we are going to make $\left\lfloor \frac{N_1}{p_2} \right\rfloor$ unit transfers from $p_2 R \to q_2 R^2$. Therefore, the number of different markers that lie in container R^2 (at same time) during the course of our forward transformation of $N \to \overline{p}$ is $N_2 = \left\lfloor \frac{N_1}{p_2} \right\rfloor q_2$. This pattern (called the standard order) continues which reveals the action and proves that no matter in what order we carry out the forward unit transfers we will always end with the exact same polynomial $\overline{p}$. Also, for each $k \geq 0$ we will always carry out the same number of forward unit transfers $p_{k+1} R^k \to q_{k+1} R^{k+1}$ from container $R^k \to R^{k+1}$ and this will imply that the total number of forward unit transfers will always be the same no matter in what arbitrary order we carry out the forward transformations of $N \to \overline{p}$.

Suppose this number of forward unit transfers from $N \to \overline{p}$ is m. Then m is called the level of the final polynomial $\overline{p}$.

The above reasoning also shows that if the convergence condition holds for just one particular sequence of unit forward transfers then the convergence condition also holds for any arbitrary sequence of unit forward transfers.

Suppose in forward transforming $N \to \overline{p}$ we transform $N \to Q \to \overline{p}$. Then if we forward transform $Q \to \overline{p}$ in any arbitrary order the total number of forward unit transfers $p_{k+1} R^k \to q_{k+1} R^{k+1}$ of $Q \to \overline{p}$ will always be the same. Also for each i the total number of unit forward transfers $p_{i+1} R^i \to q_{i+1} R^{i+1}$ will always be the same. We deal much more with this later. Since the number of unit forward transfers of $N \to \overline{p}$ is always the same we see that when we forward transform from N to the final polynomial $N \to \overline{p}$ it is obvious that there are no cycles. In other words the forward transformations cannot go back and repeat a position since each unit forward transfer takes us one step closer to $\overline{p}$. Suppose we now start with our final polynomial $\overline{p} = \sum_{i=0}^{n} a_i R^i = N$ that we have generated where each $a_i \in \{0, 1, 2, \ldots, p_{i+1} - 1\}$ and make unit backward transfers as far as we can go. A unit backward transfer uses the identity $q_k R^k = p_k R^{k-1}, k \geq 1$. In a unit backward transfer if $a_k \geq q_k, k \geq 1$, we take out q_k markers from container $R^k, k \geq 1$, and place p_k markers in container R^{k-1}.

If $a_k < q_k$ we cannot make a unit backward transfer $p_k R^{k-1} \leftarrow q_k R^k$. We cannot unit backward transfer from container R^0. To backward transform $\overline{p} \to \underline{p} = \sum_{i=0}^{m} \underline{a_i} R^i$ as far as we can go means that each $\underline{a}_i, i \geq 1$, satisfies $\underline{a}_i \in \{0, 1, 2, \ldots, q_i - 1\}$ and $a_0 \in \{0, 1, 2, \ldots\}$.

We initially have $a_0,\ a_1,\ \ldots a_n$ markers in containers $R,\ R,\ \ldots R^n$. When we make our backward transformations from $\overline{p}$ we can start our backward reasoning by starting with the a_n markers in the R^n container of $\overline{p} = \sum_{i=0}^{n} a_i R^i = N$ and work back step by step by going sequentially from containers

$$R^n \to R^{n-1} \to R^{n-2} \to \cdots \to R^0 = 1$$

the same way that we started with the N markers in the $R^0 = 1$ container and worked forward step by step $R^0 \to R^1 \to R^2 \to \cdots$ in the forward transformation of $N \to \overline{p}$.

This order $R^n \to R^{n-1} \to \cdots$ will always be called the standard order and this standard order allows us to see the entire action.

When we reach the $R^0 = 1$ container we cannot take out any markers from the $R_0 = 1$ container. Using the standard order and using an argument (called the standard argument) that is almost exactly the same as for the forward unit transfers of $N \to \overline{p}$ we see that no matter in what arbitrary order we carry out the unit backward transfers that we will always end with the same polynomial $\underline{p}$. We now show that

$\underline{p}$ is N markers in container $R^0 = 1$ which corresponds to the constant polynomial $\underline{p} = a_0 = N$. To see that $\underline{p} = N$ we know that at least one back path from $\overline{p}$ leads to N since N was forward transformed into $\overline{p}$. Since N cannot be back transformed any further and since all back paths from $\overline{p}$ lead to the same $\overline{p} \to \underline{p}$ we see that $\underline{p} = N$.

Also, from the standard order and the standard argument we see that in general no matter in what order we make the unit backward transfers from $\overline{p} \to \underline{p} = N$ we will always make the same number of backward unit transfers from each $q_k R^k \to p_k R^{k-1}$ and this implies that the total number of unit backward transfers will always be the same. Of course, we do not need a convergence condition for the backward unit transfers because when we get to the $R^0 = 1$ container we have hit a wall in the sense that we cannot make any more backward unit transfers from the $R^0 = 1$ container.

The proceeding paragraph is obvious anyway since the backward unit transfers $\overline{p} \to N$ are just the reverse order of forward unit transfers of $N \to \overline{p}$ and we already know all of this for the $N \to \overline{p}$ forward unit transfers.

Again note that the unit forward transformations of $N \to \overline{p}$ do not have any cycles. Let $\underline{\overline{p}}_N$ be the set of all polynomials that we get (along the way) when we forward transform N into the final polynomial $\overline{p} = \sum_{i=0}^{n} a_i R^i = N, a_i \in \{0, 1, 2, \ldots, p_{i+1} - 1\}$, in all possible orders. For $p, Q \in \underline{\overline{p}}_N$ we say that $p \prec Q$ if p can be forward transformed into Q in one unit forward transfer. Since the number of unit forward transfers $N \to \overline{p}$ is fixed we see that $\left(\underline{\overline{p}}_N, \prec\right)$ is a Hasse diagram.

The transitive closure of $\left(\underline{\overline{p}}_N, \prec\right)$ defines a partial order $\left(\underline{\overline{p}}_N, \leq\right)$. For $p, Q \in \underline{\overline{p}}_N$ we can also say that $p \leq Q$ if p can be forward transformed into Q in a sequence of unit forward transfers $p_{k+1} R^k \to q_{k+1} R^{k+1}$. Also, $\underline{p} = N$ is the first (or least) element and $\overline{p}$ is the final (or greatest) element of the partial order $\left(\underline{\overline{p}}_N, \leq\right)$. Also, the different levels of the polynomials $p \in \left(\underline{\overline{p}}_N, \leq\right)$ can be defined by the number of unit forward transfers needed to go from $N \to p$. This is fixed since the number of unit forward transfers of $N \to \overline{p}$ is fixed. Of course, the unit backward transfers $q_k R^k \to p_k R^{k-1}$ are equivalent to following the Hasse diagram $\left(\underline{\overline{p}}_N, \prec\right)$ of the partial order $\left(\underline{\overline{p}}_N, \leq\right)$ backwards.

For all $p, Q \in \left(\underline{\overline{p}}_N, \leq\right)$ we note that $p = N, Q = N$ are also true numerically.

We show in Sections 6, 9 that $\left(\underline{\overline{p}}_N, \leq\right)$ is a distributive lattice. We will generalize Section 3 in Section 4 and also go into much greater detail.

Suppose $p, Q \in \left(\underline{\overline{p}}_N, \leq\right)$ and $p \leq Q$. Let

$$p = \sum_{i=0}^{n} a_i R^i, \quad Q = \sum_{i=0}^{m} \overline{a}_i R^i, \quad n \leq m.$$

As always as we explain in Section 4 the standard order in forward transforming

$$p = (a_0, a_1, \ldots, a_n) \to Q = (\overline{a}_0, \overline{a}_1, \ldots, \overline{a}_m)$$

is to transform in the order

$$p_1 R^0 \to q_1 R^1, p_2 R \to q_2 R^2, p_3 R^2 \to q_3 R^3 \to \cdots .$$

That is we make all of the unit forward transfers $p_1 R^0 \to q_1 R^1$ first, then make all of the unit forward transfers $p_2 R^1 \to q_2 R^2$ second, etc. This allows us to see the action when we forward transform $p \to Q$ in any arbitrary order. The same definitions are used in the back transformations of $Q \to p$. In Section 4 we explain in detail all of the ways to forward transform $p \to Q$.

From all of the previous information we can see the following.

We see that N can be forward transformed into $N \to Q$ if and only if $\overline{p}$ can be back transformed into $\overline{p} \to Q$.

Now when N is forward transformed into $N \to \overline{p}$ in any order we know that the total number of unit forward transfers is always the same and the total number of unit forward transfers $p_{k+1}R^k \to q_{k+1}R^{k+1}$ is always the same for each k. Suppose $Q, S \in \left(\underline{\overline{p}}_N, \leq\right)$ and also $Q \leq S$. Now $N \leq Q \leq S \leq \overline{p}$. Suppose N can be forward transformed into $N \to Q$ in θ unit forward transfers. Also, Q can be forward transformed into $Q \to S$ in ϕ unit forward transfers. Also, S can be forward transformed into $S \to \overline{p}$ in ψ unit forward transfers.

From this, N can be forward transformed into $N \to \overline{p}$ in $\theta + \phi + \psi$ unit forward transfers. Now $\theta+\phi+\psi$ is fixed for all orders of $N \to \overline{p}$. Therefore, we see that θ, ϕ, ψ must be fixed. This means that all forward transformations of $N \to Q, Q \to S, S \to \overline{p}$ must always use exactly θ, ϕ, ψ unit forward transfers respectively. Also, the levels of $Q, S, \overline{p}$ must be $\theta, \theta, +\phi, \theta + \phi + \psi$ respectively. The same things are true for the backward transformations $\overline{p} \to S \to Q \to N$. Also, in $N \to Q, Q \to S, S \to \overline{p}$ the number of unit forward transfers $p_{k+1}R^k \to q_{k+1}R^{k+1}$ will be the same for all k no matter what the arbitrary order is. This is explained in greater detail in Section 4.

4. A More General Partial Order $\left(\overline{p}, \leq, \underline{p}, \overline{p}\right)$. As always $R^n = r_1 r_2 \ldots r_n$ is our generalized base. As always the unit forward transfers use $p_{k+1}R^k \to q_{k+1}R^{k+1}$ and the unit back transfers use $q_k R^k \to p_k R^{k-1}$. As always we assume that the generalized base satisfies the convergence condition.

Instead of starting with $N \in \{0, 1, 2, \ldots\}$ we can start with any arbitrary polynomial $p = \sum_{i=0}^{m} a_i R^i = \frac{a}{b}$ where $a_i \in \{0, 1, 2, \ldots\}$ and where $\frac{a}{b} \in \mathbb{Q}^+$.

We back transform $p \to \underline{p}$ as far as we can go in any arbitrary order. Using the convergence condition we also forward transform $p \to \overline{p}$ as far as we can in any arbitrary order.

As always to back transform $p \to \underline{p}$ as far as we can go means that $\underline{p} = \sum_{i=0}^{m} \underline{a}_i R^i$ where $\underline{a}_0 \in \{0, 1, 2, \ldots\}$ and for $i \geq 1, \underline{a}_i \in \{0, 1, 2, \ldots, q_i - 1\}$. To forward transform $p \to \overline{p}$ as far as we can go means that $\overline{p} = \sum_{i=0}^{r} \overline{a}_i R^i, \overline{a}_i \in \{0, 1, 2, \ldots, p_{i+1} - 1\}$.

In this section we assume that p has been forward transformed as far as possible into $\overline{p}$ and backward transformed as far as possible into $\underline{p}$.

Our standard argument using the standard orders of transformation $R^m \to R^{m-1}$, $R^{m-1} \to R^{m-2}, \ldots$ and $R^0 \to R^1$, $R^1 \to R^2, \ldots$ reveals the action and shows that $\underline{p}, \overline{p}$ are always the same no matter what the arbitrary order is. The standard argument is analogous to Section 3. From our standard argument using the standard order $R^0 \to R^1$, $R^1 \to R^2, \ldots$ if $\underline{p}$ is forward transformed in any arbitrary order as far as we can go we will always end in the same polynomial T. We show $T = \overline{p}$. Now $\underline{p}$ can be forward transformed into p and when p is forward transformed in any order as far as we can go we will always end $p \to \overline{p}$. Therefore, $T = \overline{p}$.

Likewise if $\overline{p}$ is back transformed as far as we can go in any order will always end in $\overline{p} \to \underline{p}$. Therefore, if $\underline{p}$ can be forward transformed into Q we see that when Q is forward transformed in any order as far as we can go then Q is always forward transformed into $Q \to \overline{p}$. This implies that $\overline{p}$ can be back transformed into Q. Likewise if $\overline{p}$ can be back transformed into Q then when Q is back transformed in any order as far as we can go then Q will always be back transformed into $Q \to \underline{p}$. Therefore $\underline{p}$ can be forward transformed into Q.

Again using our standard order $R^0 \to R^1$, $R^1 \to R^2$, ...and our standard argument we see that in forward transforming $\underline{p} \to \overline{p}$ that the total number of unit forward transfers $p_{i+1}R^i \to q_{i+1}R^{i+1}$ is always the same for each i no matter what the arbitrary order is. Therefore, in $\underline{p} \to \overline{p}$ the total number of unit forward transfers is always the same no matter what the arbitrary order is. The same is true for the back transformation of $\overline{p} \to \underline{p}$. The back unit transfers of $\overline{p} \to \underline{p}$ are also created by just reversing the unit forward transfers of $\underline{p} \to \overline{p}$.

Suppose $\underline{p} \to Q \to R \to \overline{p}$ is a forward transformation. Suppose the total number of unit forward transfers in $\underline{p} \to Q$ is θ in $Q \to R$ is ϕ and in $R \to \overline{p}$ is ψ. Suppose we now forward transform $\underline{p} \to Q$ in any arbitrary order, forward transform $Q \to R$ in any arbitrary order, and forward transform $R \to \overline{p}$ in any arbitrary order. Then the total number of unit forward transfers in $\underline{p} \to Q, Q \to R, R \to \overline{p}$ will always be θ, ϕ, ψ respectively since the number of unit forward transfers of $\underline{p} \to \overline{p}$ is $\theta+\phi+\psi$. We call θ the level (or the bottom level) of Q, ϕ the level of R with respect to Q and ψ the top (or back) level of R. Again note that $\theta+\phi+\psi$ is the total number of unit forward transfers of any arbitrary forward transformation $\underline{p} \to \overline{p}$. Also, in $\underline{p} \to Q, Q \to R, R \to \overline{p}$ the total number of unit forward transfers $p_{i+1}R^i \to q_{i+1}R^{i+1}$ is always the same for all i. The same is true for the back transformation of $\overline{p} \to R \to Q \to \underline{p}$.

We emphasize that in forward transforming $\underline{p} \to \overline{p}$ in all possible orders that there can be no cycles since all forward transformations of $\underline{p} \to \overline{p}$ always use the same number of unit forward transfers. Using the proceeding information we now define $\left(\underline{\overline{p}}, \leq, \underline{p}, \overline{p}\right) = \left(\underline{\overline{p}}, \leq\right)$ as follows. $\underline{\overline{p}}$ is the set of all polynomials that $\underline{p}$ can be forward transformed into. $\underline{\overline{p}}$ is also the set of all polynomials that $\overline{p}$ can be back transformed into.

For $R, S \in \underline{\overline{p}}$, we say that $R \leq S$ if R can be forward transformed into S. Note that for all $R \in \underline{\overline{p}}, R = \sum_{i=0}^{n} a_i R^i = \frac{a}{b}$ is also numerically true. Since $\underline{p} \to \overline{p}$ has no cycles, it is obvious that $\left(\underline{\overline{p}}, \leq, \underline{p}, \overline{p}\right)$ is a partial order with least element $\underline{p}$ and greatest element $\overline{p}$. However, much more than this, for each $Q \in \left(\underline{\overline{p}}, \leq, \underline{p}, \overline{p}\right)$ we say that the level (or bottom level) of Q is the total number of unit forward transfers $p_{k+1}R^k \to q_{k+1}R^{k+1}$ that are needed to forward transform $\underline{p} \to Q$. This level is the same for all forward transformations of $\underline{p} \to Q$.

The Hasse diagram $\left(\underline{\overline{p}}, \prec, \underline{p}, \overline{p}\right)$ of $\left(\underline{\overline{p}}, \leq, \underline{p}, \overline{p}\right)$ is defined by $Q \prec R$ if $Q, R \in \underline{\overline{p}}$ and Q can be forward transformed into R in exactly one unit forward transfer. We show later that $\left(\underline{\overline{p}}, \leq, \underline{p}, \overline{p}\right)$ is a distributing lattice with least element $\underline{p}$ and greatest element $\overline{p}$.

Suppose $R \in \underline{\overline{p}}$ and R can be forward transformed or back transformed into $R \to S$. Then $S \in \underline{\overline{p}}$. For example, if R can be back transformed into S then since $\overline{p}$ can be back transformed into R we see that $\overline{p}$ can be back transformed into S. This implies $S \in \underline{\overline{p}}$.

Suppose $R, S \in \left(\underline{\overline{p}}, \leq, \underline{p}, \overline{p}\right), R \leq S$. Also, $R = \sum_{i=0}^{n} a_i R^i = \frac{a}{b}, S = \sum_{i=0}^{n} \overline{a}_i R^i = \frac{a}{b}, a_i, \overline{a}_i \in \{0, 1, 2, \ldots\}$. We now analyze in detail the action of forward transforming $R \to S$. Of course, since $R \leq S$ we know that R can be forward transformed into S. From what we already know about $\underline{p} \to R \to S \to \overline{p}$ we know the following. The total number of unit forward transfers in $R \to S$ is (level S)– (level R). Also, the total number of unit forward transfers $p_{i+1}R^i \to q_{i+1}R^{i+1}$ is always the same for each i no matter in what arbitrary order we forward transform $R \to S$.

As always the standard way to forward transform $R = (a_0, a_1, \ldots, a_n) \to S = (\overline{a}_0, \overline{a}_1, \ldots, \overline{a}_n)$ is to start with a_0, a_1, $\ldots$, a_n markers in containers R^0, R^1, R^2, $\ldots$. First we leave $\overline{a}_0$ markers in the $R^0 = 1$ container and forward transfer the rest to container R. This will give us $a_1' = a_1 + q_1 \left[\frac{a_0 - \overline{a}_0}{p_1}\right]$ markers, where $p_1 | (a_0 - \overline{a}_0)$, in the R container. Next we leave $\overline{a}_1$ markers in the R container and forward transfer the rest to the R^2 container. This will give us $a_2^{'} = a_2 + q_2 \left[\frac{a_1' - \overline{a}_1}{p_2}\right]$ markers, where $p_2 | a_1' - \overline{a}_1$, in the R^2 container. We now continue this same pattern until we forward transform $R \to S$. Analogous to the standard argument for $N \to \overline{p}$ used in Section 3 where N, N_1, $N_2 \ldots$ are analogous to a_0, a_1', a_2', $\ldots$ the above standard transformation of $R \to S$ tells us the number of unit forward transfers from each $p_{i+1}R^i \to q_{i+1}R^{i+1}$. As we already know from a few pages back, this number will also be the same for any arbitrary forward transformation of $R \to S$ since a_0, a_1', a_2', a_3' will always be the same. When $R \leq S$ we now show that we can forward transform $R \to S$ in any arbitrary order that we choose as long as we never exceed these known number of unit forward transfers $p_{i+1}R^i \to q_{i+1}R^{i+1}$, $i = 0, 1, 2, \ldots$, and as long as we always make valid unit forward transfers. Also, if we follow these two rules there will always be open to us such a valid forward unit move that we can make until we completely forward transform $R \to S$.

Suppose $R \to \overline{R}$ follows the above rules. Also, suppose

$$\overline{R} = a_0' + a_1' R + a_2' R^2 + \cdots + a_n' R^n$$

and

$$S = \overline{a}_0 + \overline{a}_1 R + \overline{a}_2 R^2 + \cdots + \overline{a}_n R^n.$$

Also, suppose $a_0' = \overline{a}_0$, $a_1' = \overline{a_1}$, $\ldots$, $a_{k-1}' = \overline{a}_{k-1}$, $a_k' \neq \overline{a}_k$.

Note that the action that has taken place in containers R^{k+1}, R^{k+2}, $\ldots$ is of no concern to us here and we know the complete action that has taken place in containers R^0, R^1, R^2, $\ldots$, R^{k-1}. We now focus on containing R^k.

Since we know by hypothesis that we have not exceeded the known number of unit forward transfers $p_{i+1}R^i \to q_{i+1}R^{i+1}$, $i = 0, 1, 2, \ldots$, and also $a_k' \neq \overline{a}_k$ we see that we must have $a_k' = \overline{a}_k + tp_{k+1}, t = 1, 2, 3, \ldots$. Therefore, there is a valid forward unit move that we can make namely $p_{k+1}R^k \to q_{k+1}R^{k+1}$ that is also compatible with the rules that we are using since we know by hypothesis that we could not have already make the known number of unit forward transfers $p_{k+1}R^k \to q_{k+1}R^{k+1}$ that must be made to forward transform R into S. Carefully note that we will always have a situation where $a_k' = \overline{a}_k + tp_{i+1}, t = 1, 2, 3, \ldots$ and where we have not reached the number of unit forward transfers $p_{k+1}R^k \to q_{k+1}R^{k+1}$ that must be made to forward transform $R \to S$. Therefore, there is always open to us at least one valid move that we can make if we follow the rules.

We now show that we can also forward transform

$$R = (a_0, a_1, \ldots, a_n) \to S = (\overline{a}_0, \overline{a}_1, \ldots \overline{a}_n)$$

in any order using the following rule. If at any time in the forward transformation of $R \to R' \to S$ where $R' = (a_0', a_1', a_2', \ldots, a_n',)$ we have $a_i' > \overline{a}_i$ and $a_i' > p_{i+1}$ we can forward unit transfer

$$R' = (a_0', a_1', \ldots, a_n') \to \left(a_0', a_1', \ldots, a_{i-1}', a_i' - p_{i+1}, a_{i+1}' + q_{i+1}, a_{i+2}', \ldots a_n'\right).$$

Also if we follow this rule, there will always be open to us such a move that we can make until we completely forward transform $R \to S$.

The proof of this is really just a corollary of the first proof. The following proof is optional.

To start the action suppose $R = (a_0, a_1, \dots, a_n) \le S = (\overline{a}_0, \overline{a}_1, \dots \overline{a}_n)$ and $a_i > \overline{a}_i$ and $a_i \ge p_{i+1}$. Since $R \le S$ from the first proof such a situation will always exist. Now a valid unit forward transfer is

$$R = (a_0, a_1, \dots, a_n) \to R_1 = (a_0, \dots, a_{i-1}, a_i - p_{i+1}, a_{i+1} + q_{i+1}, a_{i+2}, \dots, a_n).$$

Also, this unit forward transfer $p_{i+1}R^i \to q_{i+1}R^{i+1}$ must be made sometime in forward transforming $R \to S$. Otherwise we would end with too many markers in the R^k container to forward transform $R \to S$.

From the first proof we are justified to make this unit forward transfer at the very beginning and this is compatible with the first proof. Also, from the first proof we can now do the same thing with $R_1 \to S$ until we forward transform $R \to R_1 \to R_2 \to \cdots \to S$ by following the rules. In Section 11 we give a simple idea which greatly extends the structure $\left(\underline{\overline{p}}, \le, \underline{p}, \overline{p}\right)$ given in this section.

We need to point out that $\left(\underline{\overline{p}}, \le, \underline{p}, \overline{p}\right)$ has a fractal type property that each $Q \in \left(\underline{\overline{p}}, \le, \underline{p}, \overline{p}\right)$ has embedded in itself the property that the entire structure $\left(\underline{\overline{p}}, \le, \underline{p}, \overline{p}\right)$ can be generated from Q. Thus seems remarkable since $\left(\underline{\overline{p}}, \le, \underline{p}, \overline{p}\right)$ can be arbitrary large or even be infinite if we drop the covergence condition on the base.

5. Preliminary Results for Proving that $\left(\underline{\overline{p}}_N, \le\right), \left(\underline{\overline{p}}, \le\right)$ are Lattices.

Lemma 1. Let $\left(\underline{\overline{p}}, \le, \underline{p}, \overline{p}\right) = \left(\underline{\overline{p}}, \le\right)$ be the partial order defined in Section 4. The partial order $\left(\underline{\overline{p}}_N, \le\right)$ of Section 3 is a special case of this. Suppose $\underline{p} = \sum_{i=0}^{r} b_i R^i = \frac{a}{b}$, $\frac{a}{b} \in \mathbb{Q}^+$, is the least element of $(\underline{\overline{p}}, \le, \underline{p}, \overline{p})$. Suppose $p = \sum_{i=0}^{n} a_i R^i = \frac{a}{b}$, $Q = \sum_{i=0}^{m} \overline{a}_i R^i = \frac{a}{b}$ are any two members of $\left(\underline{\overline{p}}, \le\right)$. This means that $\underline{p}$ can be forward transformed into p and Q. Suppose $a_0 = \overline{a}_0, a_1 = \overline{a}_1, \dots, a_{k-1} = \overline{a}_{k-1}, a_k > \overline{a}_k$. Let us represent p, Q as markers $(a_0, a_1, a_2, \dots), (\overline{a}_0, \overline{a}_1, \overline{a}_2, \dots)$ in containers $R^0, R^1, R^2, R^3, \dots$. Then from container R^k for p that contains a_k markers we can remove tp_{k+1} markers, $t \in \{1, 2, 3, \dots\}$, and place tq_{k+1} markers in container R^{k+1} so that $a_k - tp_{k+1} = \overline{a}_k$.

Proof. We represent $\underline{p}$ as markers $(b_0, b_1, b_2, \dots)$ in containers $R^0 = 1$, R^1, R^2, R^3, … We now forward transform $\underline{p}$ into p, Q in the standard order $R^0 \to R^1$, $R_1 \to R^2$, $R^2 \to R^3$, …

In other words we first remove markers from the $R^0 = 1$ container and leave $a_0 = \overline{a}_0$ markers in the $R^0 = 1$ container and place markers in the R container. We continue this process until we have

$$a_0 = \overline{a}_0, a_1 = \overline{a}_1, a_2 = \overline{a}_2, \dots, a_{k-1} = \overline{a}_{k-1}, b_k^*$$

markers in containers

$$R^0 = 1, R^1, R^2, \dots, R^{k-1}, R^k.$$

Starting with the b_k^* markers in the R^k container, let us continue to remove markers in the standard order from containers $R^k, R^{k+1}, \dots$ until we have completely forward transformed $\underline{p}$ into p and $\underline{p}$ into Q. Since $a_k > \overline{a}_k$ we may suppose $b_k^* = kp_{k+1} + a$ where $k \in \{1, 2, 3, \dots\}$ and $a \in \{0, 1, 2, \dots, p_{k+1} - 1\}$.

Then, $a_k = a + k'p_{k+1} > a + \overline{k}p_{k+1} = \overline{a}_k$ where $k \geq k' > \overline{k}$. For polynomial p it is now obvious that we can remove $\left(k' - \overline{k}\right) p_{k+1}$ markers from the R^k container of p and place $\left(k' - \overline{k}\right) q_{k+1}$ markers in the R^{k+1} container of p so that

$$a_k - \left(k' - \overline{k}\right) p_{k+1} = (a + k'p_{k+1}) - \left(k' - \overline{k}\right) p_{k+1} = a + \overline{k}p_{k+1} = \overline{a}_k,$$

as required ▯

Lemma 2. *Suppose* $\left(\underline{\overline{p}}, \leq, \underline{p}, \overline{p}\right) = \left(\underline{\overline{p}}, \leq\right)$ *is the partial order of Section 4. Let* $p, Q \in \underline{\overline{p}}$ *be from Lemma 1 where*

$$p = \sum_{i=0}^{n} a_i R^i = \frac{a}{b}, \quad Q = \sum_{i=0}^{m} \overline{a}_i R^i = \frac{a}{b}$$

and where $a_0 = \overline{a}_0$, $a_1 = \overline{a}_1$, $\ldots$, $a_{k-1} = \overline{a}_{k-1}$, $a_k > \overline{a}_k$. *Since* $a_k = \overline{a}_k + tp_{k+1}$, $t \in \{1, 2, 3, \ldots\}$, *we can forward transform*

$$p = a_0 + a_1 R + \cdots + a_{k-1} R^{k-1} + a_k R^k + \cdots + a_n R^n$$

into

$$p^* = a_0 + a_1 R + \cdots + a_{k-1} R^k + \overline{a}_k R^k + (a_{k+1} + tq_{k+1}) R^{k+1} + a_{k+2} R^{k+2} + \cdots + a_n R^n.$$

Thus, p is forward transformed into p^* by using $tp_{k+1}R^k \to tq_{k+1}R^{k+1}$. Of course, $p < p^*$. Suppose $x = \sum\limits_{i=0}^{r} x_i R^i, x \in \underline{\overline{p}}$, is any upper bound of both p and Q. That is, $p \leq x, Q \leq x$ and this means that both p and Q can be forward transformed into x. Then x is an upper bound of both p^* and Q. That is $p < p^* \leq x, Q \leq x$. This means that both p^* and Q can be forward transformed into x.

Proof. We can forward transform both p and Q into x in the standard order as explained in Sections 3, 4. Since $a_0 = \overline{a}_0, a_1 = \overline{a}_1, \ldots, a_{k-1} = \overline{a}_{k-1}, a_k > \overline{a}_k$, when we forward transform p and Q into x and agree to use the standard order $R^0 \to R, R \to R^2, R^2 \to R^3, \ldots$, we see that the forward transfers $R^0 \to R, R \to R^2, \ldots R^{k-1} \to R^k$ must be identical for both p and Q.

Also, when we try to forward transform $p^* \to x$ the above transfers $R^0 \to R, R \to R^2, \ldots R^{k-1} \to R^k$ that were used for p and Q must also be used for $p^* \to x$ since p^*, p, Q all have the same coefficients $a_0 = \overline{a}_0, a_1 = \overline{a}_1, \ldots, a_{k-1} = \overline{a}_{k-1}$. When we reach the R^k container suppose when we forward transform $Q \to x$ that we take out $t'p_{k+1}$ markers from the R^k container and place $t'q_{k+1}$ markers in the R^{k+1} container.

Since by Lemma 1, $a_k = \overline{a}_k + tp_{k+1}, t \in \{1, 2, \ldots\}$, when we forward transform $p \to x$ we must take out an additional tp_{k+1} markers from the R^k container. Therefore, in forward transforming $p \to x$ we must take out a total of $t'p_{k+1} + tp_{k+1}$ markers from container R^k and place $t'q_{k+1} + tp_{k+1}$ markers in container R^{+1}.

From this it is very easy to see that we can forward transform both p and p^* into x since in the very beginning we made the forward transfer $p \to p^*$ by making the forward transfer $tp_{k+1}R^k \to tq_{k+1}R^{k+1}$ and also p, p^* are exactly the same except for this one forward transfer $tp_{k+1}R^k \to tq_{k+1}R^{k+1}$. ▯

6. Proving that $\left(\underline{\overline{p}}, \leq, \underline{p}, \overline{p}\right)$ is a Lattice.

Lemma 3. *The partial order* $\left(\underline{\overline{p}}, \leq, \underline{p}, \overline{p}\right) = \left(\underline{\overline{p}}, \leq\right)$ *of Section 4 is a lattice. The partial order* $\left(\underline{\overline{p}}_N, \leq\right)$ *is a special case.*

Proof. We must show for each $p, Q \in \underline{\overline{p}}$ that p and Q have a least upper bound denoted by $p \wedge Q$ and a greatest lower bound denoted by $p \vee Q$. Of course, $\overline{p}$ is one upper bound of p, Q and $\underline{p}$ is one lower bound of p, Q.

We show that p, Q have a least upper bound. The proof that p, Q have a greatest lower bound is true by the duality of the unit forward and unit backward transfers $p_{k+1}R^k \rightleftarrows q_{k+1}R^{k+1}$ and the duality of $\underline{p}$ and $\overline{p}$.

In other words we just turn the partial order upside down and call $\leq = \geq$.

From Lemma 1 if

$$\begin{aligned} p &= a_0 + a_1R + \cdots + a_{k-1}R^{k-1} + a_kR^k + \cdots + a_nR^n, \\ Q &= a_0 + a_1R + \cdots + a_{k-1}R^{k-1} + \overline{a}_kR^k + \cdots + \overline{a}_nR^n \end{aligned}$$

where $a_k > \overline{a}_k$ we can forward transform $p \to p^*$ where $p^* = a_0 + a_1R + \cdots + a_{k-1}R^{k-1} + \overline{a}_kR^k + [a_{k+1} + tq_{k+1}]\, R^{k+1} + a_{k+2}R^{k+2} + \cdots a_nR^n$ and $t \in \{1, 2, 3, \ldots\}$. Of course, $p < p^*$. Note we use the same n for p, Q. Also, if $p \leq x, Q \leq x$ then from Lemma 2 $p < p^* \leq x, Q \leq x$. That is, x is an upper bound of both p, Q iff x is also an upperbound of both p^*, Q.

Note that p, Q have the same coefficients $a_0,\ a_1,\ a_2,\ \ldots,\ a_{k-1}$ but $a_k > \overline{a}_k$. However, p^*, Q have the same coefficients $a_0,\ a_1,\ a_2,\ \ldots,\ a_{k-1},\ \overline{a}_k$. Also, if $p \leq x$, $Q \leq x$ then $p < p^* \leq x$, $Q \leq x$.

We just continue this exact same pattern over and over by defining

$$p < p^* < p^{**} < \cdots < p^{****}$$

and

$$Q < Q^* < Q^{**} < \cdots < Q^{***}$$

where eventually (as we show below) $p^{****} = Q^{***}$ so that if $p \leq x$ and $Q \leq x$ then

$$p < p^* < p^{**} < \cdots < p^{****} \leq x$$

and

$$Q < Q^* < Q^{**} < \cdots < Q^{***} \leq x.$$

Since $p^{****} = Q^{***}$ and $p \leq p^{****} \leq x, Q \leq Q^{***} \leq x$ we see that $p \wedge Q = p^{****} = Q^{***}$.

Thus we have constructed the least upper bound $p \wedge Q$ of p and Q.

The reason that eventually we must have $p^{****} = Q^{***}$ is that by the convergence condition placed on our generalized base $R^0 = 1,\ R,\ R^2,\ R^3,\ \ldots$ we cannot keep forward transforming forever. In other words in our finite partial order $\left(\underline{\overline{p}}, \leq\right)$ we cannot construct an infinite sequence of different levels $A < B < C < D < \cdots$. ∎

The last statement in the proof is also true for another more basic reason which is true even when our generalized base does not satisfy the convergence condition.

Suppose $p = \sum_{i=0}^{n} a_iR^i = \frac{a}{b}$, $Q = \sum_{i=0}^{n} \overline{a}_iR^i = \frac{a}{b}$ where we sum from 1 to n in both sums. Since n is the same in both sums eventually in the proof we will reach a point where $p \to p^{****}$, $Q \to Q^{***}$ and where

$$p^{****} = \left(\sum_{i=0}^{n-1} a_i^* R^i\right) + a_n' R^n = \frac{a}{b}$$

and

$$Q^{***} = \left(\sum_{i=0}^{n-1} a_i^* R^i\right) + a_n^{''} R^n = \frac{a}{b}.$$

Therefore, $a_n' = a_n^{''}$ and $p^{****} = Q^{***}$.

We soon show that the lattice $\left(\underline{\overline{p}}, \leq\right)$ is distributive. Of course, the lattice $\left(\underline{\overline{p}}_N, \leq\right)$ of Section 3 is a special case of $\left(\underline{\overline{p}}, \leq\right)$.

7. Transforming p **into** $Q, p, Q \in \left(\underline{\overline{p}}, \leq\right)$**.** Suppose $p, Q \in \left(\underline{\overline{p}}, \leq\right)$. Then we can always transform $p \to Q$ by using a sequence of unit forward and unit backward transfers. A simple way to do this is to first forward transform p into $p \wedge Q$ and then backward transform $p \wedge Q$ into q. Or we can first backward transform p into $p \vee Q$ and then forward transform $p \vee Q$ into Q.

We show later that both of these two methods take the same number of unit transfers and also these two methods use the smallest possible total number of unit transfers. Lemma 4 is very optional.

LEMMA 4.*Suppose* $p, Q \in \left(\underline{\overline{p}}, \leq\right)$. *Also suppose*

$$\begin{aligned} p &= a_0 + a_1 R + \cdots + a_{k-1} R^{k-1} + a_k R^k + a_{k+1} R^{k+1} + \cdots + a_n R^n, \\ Q &= a_0 + a_1 R + \cdots + a_{k-1} R^{k-1} + \overline{a}_k R^k + \overline{a}_{k+1} R^{k+1} + \cdots + \overline{a}_m R^m. \end{aligned}$$

If $a_k > \overline{a}_k$ *then from Lemma 1 we can forward transform* $p \to p^*$ *such that*

$$p^* = a_0 + a_1 R + \cdots + a_{k-1} R^{k-1} + \overline{a}_k R^k + a_{k+1}^* R^{k+1} + a_{k+2} R^{k+2} + \cdots + a_n R^n.$$

We now show that if $a_k < \overline{a}_k$ *then we can backward transform* $p \to p^*$ *such that*

$$p^* = a_0 + a_1 R + \cdots a_{k-1} R^{k-1} + \overline{a}_k R^k + a_{k+1}^* R^{k+1} + \cdots + a_r^* R^r,$$

$r \leq n$.

Proof. We know that $\underline{p} = b_0 + b_1 R + \cdots + b_t R^t$ can be forward transformed into both p and Q in the standard order in which the unit forward transfers $p_{k+1} R^k \to q_{k+1} R^{k+1}$ are made in the order $R^0 \to R$, $R \to R^2$, $R^2 \to R^3, \ldots$.

The first k steps $R^0 \to R$, $R \to R^2$, $\ldots R^{k-1} \to R^k$ are identical for both $\underline{p} \to p$ and $\underline{p} \to Q$. In these first k steps $R^0 \to R^1$, $R^1 \to R^2$, $\ldots$, $R^{k-1} \to R^k$ we leave a_0 markers in container R^0, we leave a_1 markers in container R, we leave a_2 markers in container $R^2, \ldots$, we leave a_{k-1} markers in container R^{k-1} and place some additional markers in the R^k container.

At this point in the forward transfers of both $\underline{p} \to p$, $\underline{p} \to Q$ we have a_0 markers in R^0, a_1 markers in R, a_2 markers in $R^2, \ldots, a_{k-1}$ markers in R^{k-1} and b_k^* markers in R^k, b_{k+1} markers in R^{k+1}, b_{k+2} markers in $R^{k+2}, \ldots$ and b_t markers in R^t. In $\underline{p} \to p$ the next move $R^k \to R^{k+1}$ takes out $\overline{t} p_{k+1}$ markers from R^k and in $\underline{p} \to Q$ the next move $R^k \to R^{k+1}$ takes out $t p_{k+1}$ markers from R^k where $\overline{t} > t$ must be true since $a_k < \overline{a}_k$. Thus, $a_k = b_k^* - \overline{t} p_{k+1}$ and $\overline{a}_k = b_k^* - t p_{k+1}$ where $\overline{t} > t$.

At this point in the forward transformation $\underline{p} \to p' \to p$ we see that

$$p' = a_0 + a_1 R + \cdots + a_{k-1} R^{k-1} + a_k R^k + \left(b_{k+1} + \overline{t} q_{k+1}\right) R^{k+1} + b_{k+2} R^{k+2} + \cdots + b_t R^t$$

where $a_k = b_k^* - \overline{t} p_{k+1}$.

We now note that $a_k + (\bar{t} - t)\, p_{k+1} = b_k^* - tp_{k+1} = \bar{a}_k$. Therefore, by making the back transfers on p' of $(\bar{t} - t)\, q_{k+1}R^{k+1} \to (\bar{t} - t)\, p_{k+1}R^k$ we see that p' is back transformed into $p' \to p^*$ where p^* satisfies the conditions of the lemma.

Now p' can be forward transformed into $p' \to p$. Therefore, the back transforms $p \to p' \to p^*$ will back transform $p \to p^*$ where p^* satisfies the conditions of the lemma. $\square$

The reader might like to think about backward transforming $p \to p*$ in the smallest number of unit backward transfers.

Observation 2: For $p, Q \in (\underline{\overline{p}}, \leq)$ suppose $p = a_0 + a_1R + \cdots + a_nR^n + a_{n+1}R^{n+1} + \cdots + a_mR^m = \frac{a}{b}$, $Q = a_0 + a_1R + \cdots + a_nR^n = \frac{a}{b}$ where $m \geq n, a_i \geq 0$.

Since p and Q have the same numerical value $p = \frac{a}{b}, Q = \frac{a}{b}$ and since $a_i \geq 0, R^i > 0$ we see that $a_{n+1} = a_{n+2} = \cdots = a_m = 0$. Therefore, lemmas 1, 4 show another way to transform p into Q in a sequence of unit forward and unit backward transfers.

8. Efficiently Transforming p into $Q, p, Q \in (\underline{\overline{p}}, \leq)$. If $p, Q \in (\underline{\overline{p}}, \leq)$ we know that we can transform $p \to Q$ by building a sequence of unit forward and unit backward transfers. In the transforming $p \to Q$ for each R^i, R^{i+1}, suppose we make x_i unit forward transfers $p_{i+1}R^i \to q_{i+1}R^{i+1}$ and y_i unit backward transfers $p_{i+1}R^i \leftarrow q_{i+1}R^{i+1}$. Let $t_i = x_i - y_i$. Then in $p \to Q$ we are making the equivalent of t_i unit transfers $p_{i+1}R^i \rightleftarrows q_{i+1}R^{i+1}$. If $t_i > 0$ this means we make the equivalent of t_i unit forward transfers $p_{i+1}R^i \to q_{i+1}R^{i+1}$. If $t_i < 0$ this means that we make the equivalent of $|t_i|$ unit backward transfers $p_{i+1}R^i \leftarrow q_{i+1}R^{i+1}$. If $t_i = 0$, we make the equivalent of no transfers at all.

In any transformation of $p \to Q$ we soon show that the t_i-values $t_i = x_i - y_i$, $i = 0, 1, 2, \ldots$ are always the same. As the example below makes clear, this means that if we know the t_i-value t_i, $i = 0, 1, 2, \ldots$ of $p \to Q$ then the t_i unit transfers $p_{i+1}R^i \rightleftarrows q_{i+1}R^{i+1}$, $i = 0, 1, 2, \ldots$ in any arbitrary order will transform $p \to Q$.

For example if $t_0 = 2, t_1 = -1, t_2 = 1$ then we can make the four unit transfers $p_1R^0 \to q_1R^1$, $p_1R^0 \to q_1R^1$, $p_2R^1 \leftarrow q_2R^2$, $p_3R^2 \to q_3R^3$ in any arbitrary order. Of course, most of these unit transfers will not be compatible with the fact that $p_{i+1}R^i \to q_{i+1}R^{i+1}$ is possible iff $a_i \geq p_{i+1}$ and $p_{i+1}R^i \leftarrow q_{i+1}R^{i+1}$ is possible iff $a_{i+1} \geq q_{i+1}$.

Let us call $|t_0| + |t_1| + \cdots + |t_{n-1}| = t$ the total t-value of $p \to Q$ and say that $p \to Q$ is an efficient transformation if it uses exactly t unit transfers. Also, in $p \to Q$, a unit transfer $p \to p^*$ is an efficient unit transfer if it reduces the total t-values of $p \to Q$ by 1.

If $p \to p_1 \to p_2 \to \cdots \to p_{t-1} \to Q$ is an efficient transformation of $p \to Q$ then $Q \to p_{t-1} \to \cdots \to p_2 \to p_1 \to p$ is an efficient transformation $Q \to p$ having the same total t-value.

If the t_i-values of $p \to Q$ are $(t_0, t_1, t_2, \ldots, t_{n-1})$ then the t_i-values of $Q \to p$ are $(-t_0, -t_1, -t_2, \ldots, -t_{n-1})$.

For $p, Q \in (\underline{\overline{p}}, \leq)$ we now show that the t_i-values of $p \to Q$ are the same for any arbitrary transformation of $p \to Q$.

Let $p = a_0 + a_1R + a_2R^2 + \cdots + a_nR^n = \frac{a}{b}, Q = \overline{a}_0 + \overline{a}_1R + \overline{a}_2R^2 + \cdots + \overline{a}_nR^n = \frac{a}{b}$. Note that we are using the same exponent R^n for both p and Q.

As always we use the standard order $R^0 \to R^1 \to R^2 \to \cdots$.

For $p \to Q$ and for all case of $a_0, \overline{a}_0$ we know the t_i-value t_0 of the number of unit transfers $p_1R^0 \rightleftarrows q_1R$ that must be made so that p has the correct $\overline{a}_0$ value. After these t_0 unit transfers $p_1R^0 \rightleftarrows q_1R$ are made we now know the t_i-value t_1 of the number of unit transfers $p_2\ R \rightleftarrows q_2R^2$ that we must make so that p has the correct $\overline{a}_1$ value. After these t_1 unit transfers $p_2R \rightleftarrows q_2R^2$ are made we now know the t_i-value t_2 of the number of unit transfers $p_3R^2 \rightleftarrows q_3R^3$ that we must make so that p has the correct $\overline{a}_2$ value.

This pattern continues until p has been transformed into

$$p \to Q = \overline{a}_0 + \overline{a}_1 R + \cdots + \overline{a}_{n-1} R^{n-1} + a_n^* R^n.$$

Since $p = Q = \frac{a}{b}$ is true numerically we see that $a_n^* R^n = a_n R^n$ must be true. Thus, we know all of $t_0,\ t_1,\ t_2,\ \ldots t_{n-1}$.

We now show by induction on $t = |t_0| + |t_1| + \cdots$ that for any $p, Q \in \left(\underline{\overline{p}}, \leq\right)$ an efficient transformation of $p \to Q$ is possible. Of course, since $p, Q \in (\underline{\overline{p}}, \leq)$, at least an inefficient transformation of p into Q is possible and the t_i values actually exist.

If $t = 0$ then $p = Q$ and there is nothing to prove. Therefore for $p \to Q$ suppose $|t_0| + |t_1| + \cdots = t \geq 1$.

Of course, an efficient transformation of $p \to Q$ is possible iff an efficient transformation of $Q \to p$ is possible.

By symmetry suppose that $p = a_0 + a_1R + \cdots + a_{k-1}R^{k-1} + a_kR^k + a_{k+1}R^{k+1} + \cdots$, $Q = a_0 + a_1R + \cdots + a_{k-1}R^{k-1} + \overline{a}_kR^k + \overline{a}_{k+1}R^{k+1} + \cdots$ where $a_k > \overline{a}_k$. From Lemma 1, we know that $a_k = \overline{a}_k + r_kp_{k+1}$, $r_k \in \{1, 2, 3, \ldots\}$.

Now obviously $t_0 = t_1 = \cdots t_{k-1} = 0$ and t_k for $p \to Q$ is $t_k = r_k \geq 1$. Therefore, the unit forward transfer $p_{k+1}R^k \to q_{k+1}R^{k+1}$ is a possible move and it transforms $p \to p^*$. Of course, $p^* \in \left(\underline{\overline{p}}, \leq\right)$.

If $(t_0, t_1, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots)$ are the t-values for $p \to Q$ then $t_k \geq 1$ and the t-values for $p^* \to Q$ are $(t_0, t_1, \ldots, t_{k-1}, t_k - 1, t_{k+1}, \ldots)$ and $|t_0| + |t_1| + \cdots + |t_k - 1| + |t_{k+1}| + \cdots = t - 1$.

Thus, $p \to p^*$ is an efficient unit transfer of $p \to Q$. Since $p^*, Q \in \left(\underline{\overline{p}}, \leq\right)$, by induction on t we know that p^* can be efficiently transformed into Q in exactly $t - 1$ unit transfers. Also, p was transformed into p^* in exactly one unit transfer. Therefore, p can be transformed into Q in exactly $(t - 1) + 1 = t$ unit transfers.

Also, p can be efficiently transformed into Q iff Q can be efficiently transformed into p. Note that we assumed that $a_k > \overline{a}_k$. If $a_k < \overline{a}_k$ we just reverse the roles of p and Q. Let $(t_0, t_1, \ldots)$ be the t_i-values of $p \to Q$. Since p can be efficiently transformed into Q for any $p, Q \in \left(\underline{\overline{p}}, \leq\right)$, we know that there is always open to p at least one efficient unit transfer either a unit forward transfer or a unit backward transfer.

Of course, an efficient unit transfer reduces the total t-value by 1.

In a moment we show that for any $p, Q \in \left(\underline{\overline{p}}, \leq\right)$ there is always open to p in the transformation $p \to Q$ both an efficient unit forward transfer and an efficient unit backwards transfer if some t_i are positive and some t_i are negative. We now carry the preceding ideas much further.

Suppose $p, Q \in \left(\underline{\overline{p}}, \leq\right)$. We now show that $p \to p \wedge Q \to Q$ is an efficient transformation of $p \to Q$ if we first transform $p \to p \wedge Q$ by a sequence of unit forward transfers and then transform $p \wedge Q \to Q$ by a sequence of unit backward transfers. The same is true for $p \to p \vee Q \to Q$.

We note that $p \to p \wedge Q \to Q$ is an efficient transformation iff $Q \to p \wedge Q \to p$ is an efficient transformation. Therefore, we have the licence to prove $Q \to p \wedge Q \to p$ is an efficient transformation if this is convenient for us.

We first use induction on $|t_0| + |t_1| + |t_2| + \cdots = t$ to show that there exists at least one efficient transformation of $p \to Q$ that transforms $p \to p \wedge Q \to p$ where we first transform $p \to p \wedge Q$ by some sequence of unit forward transfers and then transform $p \wedge Q \to Q$ by some sequence of unit back transfers.

If $t = 0$ then $p = Q$ and there is nothing to prove. Therefore, suppose $t \geq 1$.

By symmetry we can suppose that $p = a_0 + a_1 R + \cdots + a_{k-1} R^{k-1} + a_k R^k + a_{k+1} R^{k+1} + \cdots + a_n R^n$, $Q = a_0 + a_1 R + \cdots + a_{k-1} R^{k-1} + \overline{a}_k R^k + \overline{a}_{k+1} R^{k+1} + \cdots + \overline{a}_n R^n$ where $a_k > \overline{a}_k$. If $a_k < \overline{a}_k$ we just deal with $Q \to p \wedge Q \to p$.

As always $a_k = \overline{a}_k + r_k p_{k+1}$, $r_k \in \{1, 2, 3, \ldots\}$.

Define $p^* = a_0 + a_1 R + \cdots + (a_k - p_{k+1}) R^k + (a_{k+1} + q_{k+1}) R^{k+1} + a_{k+2} R^{k+2} + \cdots + a_n R^n$. Lemma 2 uses a slightly different p^* but we can reason with Lemma 2 to see that $p < p^* \leq p^* \wedge Q = p \wedge Q$.

Also, $p \to p^*$ uses one unit forward transfer.

Also, the total t-value of $p^* \to Q$ is $t - 1$.

By induction on $|t_0| + |t_1| + \cdots = t - 1, p^* \to p^* \wedge Q = p \wedge Q \to Q$ is an efficient transformation of $p^* \to Q$ if we transform $p^* \to p^* \wedge Q = p \wedge Q$ by a sequence of unit forward transfers and then transform $p^* \wedge Q = p \wedge Q \to Q$ by a sequence of unit backward transfers. By induction on $|t_0| + |t_1| + \cdots = t - 1$, $p^* \to p^* \wedge Q = p \wedge Q \to Q$ can transform $p^* \to Q$ in exactly $t - 1$ unit transfers.

Since $p \to p^*$ uses one unit forward transfer we see that $p \to p^* \to p^* \wedge Q = p \wedge Q \to Q$ transforms $p \to Q$ in exactly $1 + (t - 1) = t$ unit transfers.

Therefore, $p \to p^* \to p^* \wedge Q = p \wedge Q \to Q$ is a least one efficient transformation of $p \to Q$.

However, much more general than this, since $p \leq p \wedge Q$ we know from Section 4 that any forward transformation of $p \to p \wedge Q$ uses exactly the same number of unit forward transfers. Also, since $p \wedge Q \geq Q$ any back transformation of $p \wedge Q \to Q$ uses exactly the same number of unit back transfers. Therefore, any transformation $p \to p \wedge Q \to Q$ that first forward transforms $p \to p \wedge Q$ in a sequence of unit forward transfers and then back transforms $p \wedge Q \to Q$ in a sequence of unit back transfers is an efficient transformation of $p \to Q$ that uses $|t_0| + |t_1| + \cdots = t$ unit transfers.

We also need to note that since $p \to p \wedge Q \to Q$ is an efficient transformation of $p \to Q$ and since $p \to p \wedge Q$ only uses unit forward transfers on p and $p \wedge Q \to Q$ only uses unit backward transfers on p then there could not be any interaction between these two types of transfers on p since that would not be efficient. This means that the forward unit transfers on p in $p \to p \wedge Q$ and the back unit transfers on p in $p \wedge Q \to Q$ must be completely independent in order to efficiently transfer $p \to Q$ in exactly $|t_0| + |t_1| + \cdots + |t_{n-1}| = t$ unit transfers.

Therefore, in efficiently transforming $p \to p \wedge Q$ all of the positive t_i-values of $p \to Q$ are used and in efficiently transforming $p \wedge Q \to Q$ all of the negative t_i-values of $p \to Q$ are used. Also, when all of the positive t_i-values of $p \to Q$ are used and no negative t_i-values are used we will always end in $p \wedge Q$ no matter in what order we make the unit forward transfers. Of course, we must always make valid moves along the way.

The proof for $p \to p \vee Q \to Q$ follows by the duality between $p_{k+1} R^k \to q_{k+1} R^{k+1}$ and $p_{k+1} R^k \leftarrow q_{k+1} R^{k+1}$ and the duality of $\underline{p}$ and $\overline{p}$. Let $(t_0, t_1, \cdots)$ be the t_i-values of $p \to Q$.

From the above we also see that for any $p, Q \in \left(\underline{\overline{p}}, \leq\right)$ by using $p \to p \wedge Q \to Q, p \to p \vee Q \to Q$ there is always open to p in transforming $p \to Q$ and efficient unit forward transfer it any of the t_i-values t_i are positive and there is always open to p an efficient unit backward transfer if any of the t_i-values t_i are negative.

Let $(t_0, t_1, \cdots)$ be the t_i-values of $p \to Q$ with total t_i-values of t. To efficiently transform $p \to Q$ we can in general proceed as follows. We use the above list of t_i-values to keep track of what we are doing.

We know there is open to p at least one valid efficient unit transfer either one unit forward or one unit backwards or both if some $t_i > 0$ and some $t_i < 0$.

The first move can be to arbitrarily choose any valid efficient unit transfer $p \to p^*$ that is available to p. This reduces the total t-value of $p \to Q$ by 1 and also reduces the magnitude of one of the t_i-values by 1. Also, $p^* \in (\underline{\overline{p}}, \leq)$. Since the total t-value of $p^* \to Q$ is $t-1$ we then do the same thing to p^* to transfer $p^* \to p^{**}$. We then do the same thing to $p^{**} \to p^{***}$. This creates a sequence of exactly t unit transfers $p \to p^* \to p^{**} \to p^{***} \to \cdots \to Q$.

Suppose $p, Q \in (\underline{\overline{p}}, \leq)$ and we want to construct $p \wedge Q$. First we compute the t_i-values of $p \to Q$ which are $(t_0, t_1, t_2, \ldots)$. We now efficiently forward transform $p \to p \wedge Q$ by pretending to efficiently transform $p \to Q$. In efficiently transforming $p \to Q$ we first make all of the unit forward transfers which correspond to the positive t_i-values. Such forward unit transfers will always be available to us. This can be done in any arbitrary order that is available to us As long as all of the positive t_i-values are used and no negative t_i-values are used we end in the same polynomial. When we have finished this we will have forward transformed $p = p \wedge Q$. Each of these efficient unit forward transfers of $p \to Q$ will reduce one of the positive t_i-values of $p \to Q$ by 1. We use the list $(t_0, t_1, t_2, \ldots)$ to keep track of our unit forward transfers of $p \to Q$. By duality in efficiently transforming $p \to p \vee Q$ we efficiently transform $p \to Q$ and first make all of the unit back transfers which correspond to the negative t_i-values. We keep track of this by using our list $(t_0, t_1, t_2, \ldots)$. Let θ be the total number of unit forward transfers of $p \to Q$ and ϕ be the total number of unit back transfers of $p \to Q$. Then level $p \wedge Q-$ level $p =$ level $Q-$ level $p \vee Q = \theta$. Also, level $(p \wedge Q)-$ level $Q =$ level $p-$ level $p \vee Q = \phi$. Let $(t_0, t_1, \ldots)$ be the t_i-values of $p \to Q$. Then

$$\text{level } Q - \text{level } p = \sum_{i=0}^{n} t_i.$$

Also, by arranging $\theta+$'s and $\phi-$'s in any arbitrary sequence we see that in efficiently transforming $p \to Q$ we can create a path that oscillates up and down between adjacent levels of $(\underline{\overline{p}}, \leq)$ according to this arbitrary sequence of $+$'s and $-$'s. We conclude this section by noting that if $p \leq \overline{p} \leq p \wedge Q$ and $Q \leq \overline{Q} \leq p \wedge Q$, then $P \wedge Q = \overline{P} \wedge \overline{Q}$.

9. Proving that $(\underline{\overline{p}}, \leq)$ is Distributive. In this proof we use the obvious fact that if the t_i-values of $p \to Q$ are $(t_0, t_1, \ldots)$ and the t_i-values of $Q \to R$ are $(\bar{t}_0, \bar{t}_1, \ldots)$ then the t_i-values of $p \to R$ are $(t_0 + \bar{t}_0, t_1 + \bar{t}_1, \ldots)$.

Suppose that $p \in (\underline{\overline{p}}, \leq)$ is fixed. Let $Q, R \in (\underline{\overline{p}}, \leq)$ be arbitrary. Let $(t_0, t_1, t_2, \ldots)$ be the t_i-values of $p \to Q$ and let $(\bar{t}_0, \bar{t}_1, \bar{t}_2, \ldots)$ be the t_i-values of $p \to R$. From the above statement it is obvious that $(\bar{t}_0 - t_0, \bar{t}_1 - t_1, \bar{t}_2 - t_2, \ldots)$ must be the t_i-values of $Q \to R$.

From the end of Section 8 we know how to compute the t_i-values of $Q \to Q \wedge R$ when we know the t_i-values $(\bar{t}_0 - t_0, \bar{t}_1 - t_1, \ldots)$ of $Q \to R$. The t_i-values of $Q \to Q \wedge R$ called $(t_0^*, t_1^*, t_2^*, \ldots)$ are computed as follows. If $\bar{t}_i - t_i \geq 0$ then $t_i^* = \bar{t}_i - t_i$. If $\bar{t}_i - t_i \leq 0$ then $t_i^* = 0$.

Since the t_i-values of $p \to Q$ are $(t_0, t_1, t_2, \ldots)$, the t_i-values of $p \to Q \wedge R$ must be $(t_0, t_1, t_2, \ldots) + (t_0^*, t_1^*, t_2^*, \ldots) = (t_0 + t_0^*, t_1 + t_1^*, t_2 + t_2^*, \ldots)$.

If $\bar{t}_i - t_i \geq 0$ then $t_i + t_i^* = t_i + (\bar{t}_i - t_i) = \bar{t}_i$.

If $\bar{t}_i - t_i \le 0$ then $t_i + t_i^* = t_i + 0 = t_i$. In both cases the t_i-values of $p \to Q \wedge R$ are $\left(\max\left(t_0, \bar{t}_0\right), \max\left(t_1 \bar{t}_1\right), \ldots\right)$.

The proof that the t_i-values of $p \to Q \vee R$ must be $\left(\min\left(t_0, \bar{t}_0\right), \min\left(t_1 \bar{t}_1\right), \ldots\right)$ is almost exactly the same.

It is well known that the $\max(x, y)$ and $\min(x, y)$ operators distribute over one another. Therefore $\left(\underline{\overline{p}}, \le\right)$ is a distribute lattice.

In summary suppose $\left(\underline{\overline{p}}, \le, \underline{p}, \overline{p}\right)$ is generated by $p \in \underline{\overline{p}}$. Then as always each $Q \in \underline{\overline{p}}$ can be represented by the t_i-values of $p \to Q$. That is, each $Q \in \underline{\overline{p}}$ is represented by $Q = (t_0, t_1, t_2, \ldots)$, $t_i \in \mathbb{Z}$. Suppose $\overline{Q} \in \underline{\overline{p}}$ is represented by $\overline{Q} = \left(\bar{t}_0, \bar{t}_1, \bar{t}_2, \ldots\right)$. Then $Q \le \overline{Q}$ iff $t_i \le \bar{t}_i$, $i = 0, 1, 2, \ldots$.

If $Q = (t_0, t_1, t_2, \ldots) \in \underline{\overline{p}}, \overline{Q} = \left(\bar{t}_0, \bar{t}_1, \bar{t}_2, \ldots\right) \in \underline{\overline{p}}$ then $Q \wedge \overline{Q} = lub\left(Q, \overline{Q}\right) = \left(\max\left(t_0, \bar{t}_0\right), \ldots \max\left(t_n, \bar{t}_n\right)\right) \in \underline{\overline{p}}$ and $Q \vee \overline{Q} = glb\left(Q, \overline{Q}\right) = \left(\min\left(t_0, \bar{t}_0\right), \ldots \min\left(t_n \bar{t}_n\right)\right) \in \underline{\overline{p}}$. Thus $\left(\underline{\overline{p}}, \le \underline{p}, \overline{p}\right)$ is closed under the two $\max, \min$ operators, lub and glb.

Any collection of real n-tuples with the above closure property on max, min defines a distributive lattice. Finally we note that (level Q) $-$ (level p) $= t_0 + t_1 + \cdots + t_n$.

10. A Simplified Construction. This section gives a simple but massive extension of our methods.

In Section 4 we used a polynomial $p = a_0 + a_1 R + \cdots + a_n R^n$, $a_i \in \{0, 1, 2, \ldots\}$ to generate a distributive lattice $\left(\underline{\overline{p}}, \le, \underline{p}, \overline{p}\right)$.

We used both unit forward and unit backward transfers from p.

Suppose now that we allow only unit forward transfers to be made from p.

This will generate a subpartial order $\left(\underline{\overline{p}}_{\in}, \le, p, \overline{p}\right) \subseteq \left(\underline{\overline{p}}, \le, \underline{p}, \overline{p}\right)$ with least element p and greatest element $\overline{p}$, where $\overline{p}$ is the same in both $(\underline{\overline{p}}_{\in}, \le, p, \overline{p})$ and $(\underline{\overline{p}} \le, \underline{p}, \overline{p})$.

Suppose $Q, R \in \left(\underline{\overline{p}}_{\in}, \le\right)$. Then $Q, R \in \left(\underline{\overline{p}}, \le, \underline{p}, \overline{p}\right)$ and $Q \wedge R$ and $Q \vee R$ exist in $\left(\underline{\overline{p}}, \le, \underline{p}, \overline{p}\right)$. Now $p \le Q \le Q \wedge R, p \le R \le Q \wedge R$ in $\left(\underline{\overline{p}}, \le, \underline{p}, \overline{p}\right)$ and this implies that p can be forward transformed into $Q \wedge R$. Therefore, $Q \wedge R \in \left(\underline{\overline{p}}_{\in}, \le\right)$ which means that a least upper bound $Q \wedge R$ exists for Q, R in $\left(\underline{\overline{p}}_{\in}, \le\right)$.

Likewise $Q \vee R \in \left(\underline{\overline{p}}, \le, \underline{p}, \overline{p}\right)$. Now $p \le Q, p \le R$ implies $p \le Q \vee R$ in $\left(\underline{\overline{p}}, \le, \underline{p}, \overline{p}\right)$. Therefore, p can be forward transformed into $Q \vee R$. Therefore, $Q \vee R \in \left(\underline{\overline{p}}_{\in}, \le\right)$ which means that a greatest lower bound $Q \vee R$ exists for Q, R in $\left(\underline{\overline{p}}_{\in}, \le\right)$. Also, $Q \wedge R$ and $Q \vee R$ are the same in both $\left(\underline{\overline{p}}, \le\right)$ and $\left(\underline{\overline{p}}_{\in}, \le\right)$ Also, p is the least element of $\left(\underline{\overline{p}}_{\in}, \le\right)$ and the $\overline{p}$ of $\left(\underline{\overline{p}}, \le, \underline{p}, \overline{p}\right)$ is the greatest element of $\left(\underline{\overline{p}}_{\in}, \le\right)$. Also, $\left(\underline{\overline{p}}_{\in}, \le\right)$ is a distributive lattice.

Of course p will generate $\left(\underline{\overline{p}}_{\in}, \le\right)$. We can also generate $\left(\underline{\overline{p}}_{\in}, \le\right)$ from every element $Q \in \left(\underline{\overline{p}}_{\in}, \le\right)$ if we use the following algorithm. Let $\overline{p} = \bar{a}_0 + \bar{a}_1 R + \cdots + \bar{a}_m R^m$. This defines m for us. Of course, $n \le m$

Also, let $p \to \overline{p}$ have t_i-values of $\left(\bar{t}_0, \bar{t}_1, \ldots, \bar{t}_{m-1}\right)$ where each $\bar{t}_i \ge 0$.

We now write $\overline{p}$ as a matrix,

$$\overline{p} = \begin{bmatrix} \bar{a}_0 & \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_{m-1} & \bar{a}_m \\ \bar{t}_0 & \bar{t}_1 & \bar{t}_2 & \cdots & \bar{t}_{m-1} & 0 \end{bmatrix}.$$

For any $Q \in \left(\underline{\overline{p}}_{\in}, \leq\right)$, let us first forward transform Q as far as we can go. This will transform Q into $\overline{p}$. Starting with $\overline{p}$, we now make unit backward transfers $p_i R^{i-1} \leftarrow q_i R^i$ where we can make any valid unit backward transfers $\overline{p} \to Q_1 \to Q_2 \to \cdots Q_t$ in the usual way except that

$$\overline{p} = \begin{bmatrix} \overline{a}_0 & \overline{a}_1 & \overline{a}_2 & \cdots & \overline{a}_{m-1} & \overline{a}_m \\ \overline{t}_0 & \overline{t}_1 & \overline{t}_2 & \cdots & \overline{t}_{m-1} & 0 \end{bmatrix} \to Q_t = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{m-1} & b_m \\ t_0 & t_1 & t_2 & \cdots & t_{m-1} & 0 \end{bmatrix}$$

must satisfy the added restriction that $t_i \leq 0$ and $t_i + \overline{t}_i \geq 0$, where $t_0, t_1, t_2, \ldots t_{m-1}$ are the t_i-values of Q_t with respect to $\overline{p}$. The material in Section 8 can be rigorously used to show that these unit back transfers define the Hasse diagram of the partial order $\left(\underline{\overline{p}}_{\in}, \leq\right)$.

Since $\left(\underline{\overline{p}}_{\in}, \leq\right)$ is a finite distributive lattice we will see from the finite topological t_1-space of Section 11 that $\left(\underline{\overline{p}}_{\in}, \leq\right)$ has all or most of the other properties mentioned in this paper. Finally we mention that we can probably create infinite distributive lattices by starting with an infinite symbolic polynomial $p = \sum_{i=0}^{\infty} a_i R^i$, $a_i \in \{0, 1, 2, \ldots\}$ and using only forward unit transfers $p_{k+1} R^k \to q_{k+1} R^{k+1}$.

11. Topological Spaces. In 1997 we discovered that all finite distributive lattices are equivalent to the collection of all finite topological t_1-spaces. A finite topological t_1-space(u, t) has the property that (u, t) is a topological space on the finite set u. Also, of course $\phi \in t$, $u \in t$. Also, if $A, B \in t$ then $A, B \subseteq u$, $A \cup B \in t$, $A \cap B \in t$.

The t_1-property states that for all $a, b \in u$ if $a \neq b$ there exists $A \in t$ such that $a \in A, b \notin A$ or $a \notin A, b \in A$. This is called a *separation* axiom.

The finite set u can be represented by the n-tuple $u = \{u_1, \ldots, u_n\} = (1, \ldots, 1)$. Also, all $A \in t$ can be represented by the n-tuple $A = (a_1, a_2, \ldots, a_n)$ where each $a_i = 1$ if $u_i \in A$ and $a_i = 0$ if $u_i \notin A$.

If $A = (a_1, a_2, \ldots, a_n) \in t$, $B = (b_1, b_2, \ldots, b_n) \in t$ then $A \leq B$ iff $A \subseteq B$ iff $a_i \leq b_i$ for all i and $lub\,(A, B) = A \cup B = A \wedge B = (\max(a_1, b_1), \max(a_2, b_2) \ldots, \max(a_n, b_n))$. Also, $glb\,(A, B) = A \cap B = A \vee B = (\min(a_1, a_2), \min(a_2, b_2), \ldots, \min(a_n, b_n)$.

All or most of the properties in this paper are satisfied by this finite t_1-space. However, we do not know how to generate the finite topological t_1-space from every member set of this space.

12. Further Generalizations Including Infinite Distributive Lattices. We can see no reason why every idea in this paper cannot be generalized. This leads to much more mathematics than what we have in this paper.

If we allow our polynomial coefficients to satisfy $a_i \in \mathbb{Z}$ instead of $a_i \in \{0, 1, 2, \ldots\}$ it appears that we get infinite distributive lattices.

We can generalize our base further. Fig. 1 will make this base easier to understand.

For each $i \in \{1, 2, 3, \ldots\}$ let $0 < r_i = \frac{p_i}{q_i}$. For each $i \in \{0, 1, 2, \ldots\}$, $0 < r_{-i} = \frac{q_{-i}}{p_{-i}}$. We can let $p_i, q_i \in \{1, 2, 3, \ldots\}$ or $p_i, q_i \in \mathbb{Q}^+$ or $p_i, q_i \in \mathbb{R}^+$.

Also, for $n \geq 1$,

$$\begin{aligned} R^0 &= 1, \\ R^n &= r_1 r_2 \ldots r_n = \frac{p_1}{q_1} \frac{p_2}{q_2} \cdots \frac{p_n}{q_n}, \\ R^{-n} &= r_0 r_{-1} r_{-2} \cdots r_{-n+1} = \frac{q_0}{p_0} \frac{q_{-1}}{p_{-1}} \cdots \frac{q_{-n+1}}{p_{-n+1}}. \end{aligned}$$

Also, $p_{i+1}R^i = q_{i+1}R^{i+1}$, $i \in \mathbb{Z}$,

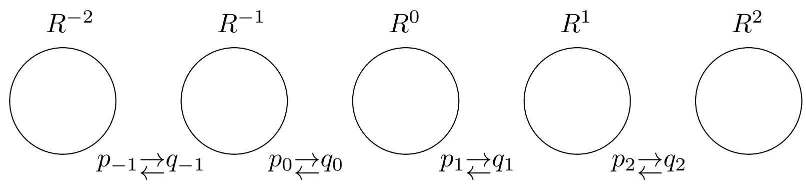

FIG. 1. *An extended generalized base.*

We call this base an extended generalized base.

We deal with polynomial $p = \sum\limits_{i=m}^{n} a_i R^i$, where $m \leq n, m, n \in \mathbb{Z}$.

We can restrict $a_i \in \{0, 1, 2, 3, \ldots\}$ or $a_i \in \mathbb{Q}^+ \cup \{0\}$ or $a_i \in \mathbb{R}^+ \cup \{0\}$ or $a_i \in \mathbb{Z}$ or $a_i \in \mathbb{Q}$ or $a_i \in \mathbb{R}$.

We use the identity $p_{i+1}R^i = q_{i+1}R^{i+1}$ to transform a polynomial $p \to Q$.

As always we can represent a polynomial p as markers a_m, a_{m+1}, $\ldots a_n$ in containers R^i, $m \leq i \leq n$, $m, n \in \mathbb{Z}$.

Starting with a given polynomial $p = \sum_{i=m}^{n} a_i R^i$, $m \leq n, m,$ $n \in \mathbb{Z}$, we create the set of all polynomial $\underline{\overline{p}}$ that p can be transformed into.

In the substitution $p_{i+1}R^i = q_{i+1}R^{i+1}$ we call this substitution a forward unit transfer if we replace $p_{i+1}R^i \to q_{i+1}R^{i+1}$ and call this substitution a backward unit transfer if we replace $p_{i+1}R^i \leftarrow q_{i+1}R^{i+1}$. We say that $p \leq Q$ if we can transform $p \to Q$ by a sequence of forward unit transfers. We can use a convergence condition in one or both directions, or use no convergence condition at all.

Although we have not done so we can see no reason why we cannot show that with or without convergence conditions that each $p, Q \in \underline{\overline{p}}$ has a least upper bound $p \wedge Q$ and a greatest lower bound $p \vee Q$. Also, we can see no reason why $\wedge$ and $\vee$ do not distribute over each other. Everything in this paper especially the material in Section 8 probably generalizes for $\ldots R^{-3}$, R^{-2}, R^{-1}, R^0, R, R^2, R^3, $\cdots$. Suppose we start with $p = \sum_{i=m}^{n} a_i R^i$, $m \leq n, m,$ $n \in \mathbb{Z}$ and use $p_{k+1}R^k = q_{k+1}R^{k+1}$ to generate a set $\underline{\overline{p}}$. Suppose the convergence condition does not hold. In this case if we agree to put an upper bound on the highest exponent R^k that we can allow in $\underline{\overline{p}}$ and also put a lower bound on the lowest exponent R^t that we can allow in $\underline{\overline{p}}$ then p will generate a finite distributive lattice $\left(\underline{\overline{p}}, \leq\right)$ exactly the same way that p generates a finite distributive lattice when we do have convergence conditions in both directions. If we do not put bounds on R^k, R^t and the upper and/or lower convergence conditions do not hold for p then p will probably generate on infinite distributive lattice in one or both directions. If we use $r_i = \frac{1}{0}$, $i = 1, 2, 3, \ldots$ we can use symbolic polynomials $p = a_0 + a_1 R + \cdots + a_n R^n$, $a_i \in \{0, 1, 2, \ldots\}$ and we can use the unit forward transfer $1 \cdot R^i \to 0 \cdot R^{i+1}$. Thus starting with $p = 2 + 2R$ we can use unit forward transfers to generate the following simple 2×2 tile lattice.

Also we can create an infinite tile lattice by embedding the lattices

$$g\left(1 + R\right) \subseteq g\left(2 + 2R\right) \subseteq g\left(3 + 3R\right) \subseteq g\left(4 + 4R\right) \subseteq \cdots$$

where $g(n + nR)$ is the lattice generated by $n + nR$.

In general we can embed lattices

$$g\left(p_1\right) \subseteq g\left(p_1 + p_2\right) \subseteq g\left(p_1 + p_2 + p_3\right) \subseteq g\left(p_1 + p_2 + p_3 + p_4\right) \subseteq \cdots$$

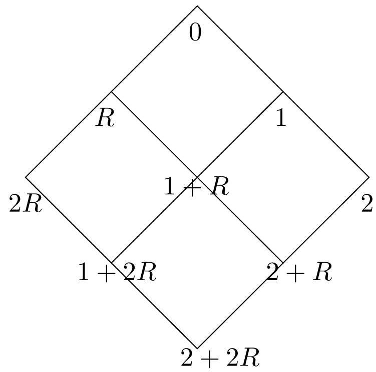

FIG. 2.

to create infinite lattices when $g(p_1 + p_2 + \cdots + p_n)$ is a superset of $g(p_1 + p + 2 + \cdots + p_{n-1})$.

The reader might like to experiment with $r_i = \frac{0}{1}$, $i = 0, 1, 2, \ldots$ and the unit forward transfers $0 \cdot R^i \to 1 \cdot R^{i+1}$ and the initial polynomial $p = 1$.

13. A Concluding Remark. We are not even remotely close to proving that our methods will generate all finite distributive lattices. However, the generalized construction of Section 10 with adjustments like Fig. 2 will generate all of the finite distributive lattices that we ourselves have recently studied.

One difficulty in solving this problem is to find a way to generate all finite distributive lattices from every single member of the lattice which is equivalent to generating all finite topological t_1-spaces from every single member of the space.

Section 10 generates such a massive number of finite distributive lattices that it would almost seem surprising if our methods could not generate at least all finite distributive lattices.

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Biosketch author 2