On a Problem of Arthur Engel

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1 Introduction

Problem 21, page 10 of [1] states

> Three integers a, b, c are written on a blackboard. Then one of the integers is erased and replaced by the sum of the other two diminished by 1. This operation is repeated many times with the final result $17, 1967, 1983$. Could the initial numbers be (a) $2, 2, 2$ (b) $3, 3, 3$?

This paper develops a mathematical context for a class of problems that includes this one and solves them. We deal with Arthur Engel's problem in section 8.

A set F of triplets of integers is said to be a *Fibonacci set* if

1. each $t \in F$ is a triplet of the form $t = \{x, y, x+y\}$ where x and y are positive integers and $x = y$ is allowed and
2. if $t = \{x, y, x+y\} \in F$ then $\{x, x+y, 2x+y\} \in F$ and $\{y, x+y, x+2y\} \in F$.

The main purpose of this paper is to compute in a closed form the smallest Fibonacci set $F_t = g(t)$ that contains the single element $t = \{a, b, a+b\}$ where $a \le b$ and $a, b \in \{1, 2, 3, \ldots\}$. It is also possible to think of a, b as purely algebraic symbols. In the end, we devise two algorithms for determining if $\left\{\overline{a}, \overline{b}, \overline{a} + \overline{b}\right\} \in g(\{a, b, a+b\})$ when $\overline{a}, \overline{b}, a, b$ have specific numerical values.

Throughout this paper, we use the notation $\mathbb{N} = \{0, 1, 2, 3, \ldots\}, \mathbb{N}^+ = \{1, 2, 3, \ldots\}$. The main result is that if $a, b \in \mathbb{N}^+$ and $a \le b$, then

$$g(a, b, a+b) = \left\{ \{x, y, x+y\} : \begin{pmatrix} x \\ y \end{pmatrix} = M \cdot \begin{pmatrix} a \\ b \end{pmatrix}, M \in \overline{M} \right\}$$

where $M \cdot \begin{pmatrix} a \\ b \end{pmatrix}$ is matrix multiplication and

$$\overline{M} = \left\{ \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} : \theta, \phi, \psi, \pi \in \mathbb{N}, \begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1, \theta + \phi \le \psi + \pi \right\}.$$

2 Preliminary Concepts

For our purposes we will modify the definition of a Fibonacci set as follows:

Definition 1. *A set F is said to be a Fibonacci set if 1. and 2. are true.*

1. *Each $t \in F$ is an ordered triple of the form $t = (x, y, x+y)$ where $x, y \in \mathbb{N}^+$ and $x \leq y$.*
2. *If $(x, y, x+y) \in F$ then $(x, x+y, 2x+y) \in F$ and $(y, x+y, x+2y) \in F$.*

Notation 1. *If $(x, y, x+y)$, where $x \leq y, x, y \in \mathbb{N}^+$, is a member of a Fibonacci set F we will often use an abbreviated notation and write $(x, y, x+y) = (x, y)$.*

Definition 2. *Suppose $(x, y) = (x, y, x+y) \in F$ where F is a Fibonacci set and $x \leq y, x, y \in \mathbb{N}^+$.*

We define $(x, x+y) = (x, x+y, 2x+y)$ and $(y, x+y) = (y, x+y, x+2y)$ to be the *immediate successors* of $(x, y) = (x, y, x+y)$ in F. Of course, if $x = y$ then the two immediate successors of (x, y) are equal, and if $x < y$ then the two immediate successors of (x, y) are unequal. We denote them by $(x, y) \to (x, x+y)$ and $(x, y) \to (y, x+y)$. Of course, we could also denote them by $(x, y, x+y) \to (x, x+y, 2x+y)$ and $(x, y, x+y) \to (y, x+y, x+2y)$. Call $(x, y) = (x, y, x+y)$ the *immediate predecessor* of $(x, x+y) = (x, x+y, 2x+y)$ and call $(x, y) = (x, y, x+y)$ the immediate predecessor of $(y, x+y) = (y, x+y, x+2y)$.

Note that if $(\overline{x}, \overline{y})$ is an immediate successor of (x, y) in F then $\overline{x} < \overline{y}$.

Lemma 1. *Suppose F is a Fibonacci set and $(x, y, x+y) \in F$ where $x \leq y, x, y \in \mathbb{N}^+$. If $(x, y, x+y)$ has an immediate predecessor $(\theta, \phi, \theta+\phi)$ in F, where $\theta \leq \phi, \theta, \phi \in \mathbb{N}^+$, then $(\theta, \phi, \theta+\phi)$ is unique.*

Proof. Suppose $(\theta, \phi, \theta+\phi)$ is an immediate predecessor of $(x, y, x+y)$ in F where $\theta \leq \phi, \theta, \phi \in \mathbb{N}^+$. Since $(x, y, x+y)$ must be an immediate successor of $(\theta, \phi, \theta+\phi)$ in F, we must have (1): $(\theta, \theta+\phi, 2\theta+\phi) = (x, y, x+y)$ or (2): $(\phi, \theta+\phi, \theta+2\phi) = (x, y, x+y)$.

Suppose (1). Then $(\theta, \theta+\phi, 2\theta+\phi) = (x, y, x+y)$. Then $\theta = x, \theta+\phi = y, 2\theta+\phi = x+y$. Therefore, $\theta = x, \phi = y-x$ and we require $x \leq y-x$.

Next, suppose (2). Then $(\phi, \theta+\phi, \theta+2\phi) = (x, y, x+y)$. Then $\phi = x, \theta+\phi = y, \theta+2\phi = x+y$. Therefore, $\phi = x, \theta = y-x$ and we require $1 \leq y-x \leq x$.

Of course, if $x = y-x$ then $(\theta, \phi, \theta+\phi) = (x, y-x, y)$ and $(\theta, \phi, \theta+\phi) = (y-x, x, y)$ are the same for both (1) and (2) and this implies that $(\theta, \phi, \theta+\phi)$ is unique.

Also, if $x < y-x$ we have $(\theta, \phi, \theta+\phi) = (x, y-x, y)$ where $\theta < \phi, \theta, \phi \in \mathbb{N}^+$ and if $1 \leq y-x < x$, we have $(\theta, \phi, \theta+\phi) = (y-x, x, y)$ where $\theta < \phi, \theta, \phi \in \mathbb{N}^+$. Therefore, $(\theta, \phi, \theta+\phi)$ is uniquely determined from $(x, y, x+y)$ if $(x, y, x+y)$ has an immediate predecessor $(\theta, \phi, \theta+\phi)$ in F. ■

Definition 3. *Suppose A is any set such that for every $t \in A, t$ satisfies the condition $t = (x, y, x+y) = (x, y)$ where $x \leq y, x, y \in \mathbb{N}^+$. Then $F_A = g(A)$ is the smallest Fibonacci set such that $A \subseteq F$. We say that $F_A = g(A)$ is generated by A and we generate $F_A = g(A)$ in a standard way by first insuring that $A \subseteq F$ and then insuring that for all t in F_A the two immediate successors of t are also in F_A. Also, if $t = (x, y, x+y) = (x, y)$, where $x \leq y, x, y \in \mathbb{N}^+$, we define $F_t = F_{\{t\}} = g(\{t\}) = g(t)$, and we say that F_t is the Fibonacci set generated by the single element t.*

It is fairly easy to convince yourself that

$$F_{\{t_1,t_2,\cdots,t_k\}} = \bigcup_{i=1}^{k} F_{t_i}.$$

We do not give a formal proof of this since it is not used in this paper. Also, see Section 8 for another property.

In Fig. 1 we illustrate $F_{(1,1)} = F_{(1,1,2)}$. Again note that if $(\overline{x}, \overline{y})$ is an immediate successor of (x, y), then we must have $\overline{x} < \overline{y}$. Therefore, from Definition 2 and Lemma 1, we easily see that $F_{(1,1)}$ is a binary tree since each vertex in $F_{(1,1)}$, (except the initial vertex $(1, 1, 2)$), has two immediate successors and one immediate predecessor in $F_{(1,1)}$. We later explain why $F_{(1,1)}$ is the basic or universal Fibonacci set.

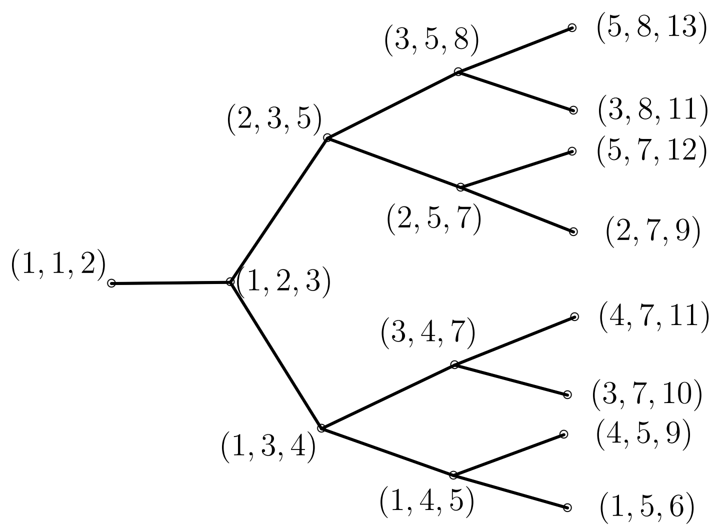


Fig. 1. The binary tree $F_{(1,1)} = g\,(1, 1)$.

Lemma 2. *Suppose* $a \in \mathbb{N}^+$. *Then* $F\,(a, a) = g\,(a, a) = \{(ax, ay) : (x, y) \in g\,(1, 1)\}$. *Also, suppose* t *is the greatest common divisor of* a, b *where* $a < b$, *and* $a, b \in \mathbb{N}^+$. *Then* $F_{(a,b)} = g\,(a, b) = \left\{(tx, ty) : (x, y) \in g\left(\frac{a}{t}, \frac{b}{t}\right)\right\}$.

Proof. This is obvious. ■

3 Statement of the Two Problems

Main Problem. Suppose that a, b are algebraic literal numbers where we agree that $a < b, a, b \in \mathbb{N}^+$. The secondary problem will take care of the case where $a = b$. We wish to compute in a closed form the Fibonacci set $F_{(a,b)} = F_{(a,b,a+b)} = g\,(a, b)$. We will use the following easy secondary problem to help us solve the main problem.

Secondary Problem. Suppose that a is an algebraic literal number in $\mathbb{N}^+$. We wish to compute in a closed form the Fibonacci set $F_{(a,a)} = F_{(a,a,2a)}$.

4 The Solution to the Secondary Problem

If $a, b \in \mathbb{N}^+$, the notation $(a, b) = 1$ means that a and b are relatively prime.

Solution of the Secondary Problem. If $a \in \mathbb{N}^+$ is arbitrary but fixed, then $F_{(a,a)} = g(a, a) = \{(\theta a, \phi a) : \theta, \phi \in \mathbb{N}^+, \theta \leq \phi, (\theta, \phi) = 1\}$.

Note 1. Of course, this solution implies that

$$F_{(1,1)} = F_{(1,1,2)} = \left\{(\theta, \phi) : \theta, \phi \in \mathbb{N}^+, \theta \leq \phi, (\theta, \phi) = 1\right\}.$$

Proof of the Solution. Of course, $(a, a) \in F_{(a,a)}, (a, a) = (1 \cdot a, 1 \cdot a)$ and $(1, 1) = 1$. Now $F_{(a,a)}$ is the Fibonacci set generated by (a, a) and we observe that if $(\theta a, \phi a) \in F_{(a,a)}$, where $\theta \leq \phi, \theta, \phi \in \mathbb{N}^+, (\theta, \phi) = 1$, then the two immediate successors of $(\theta a, \phi a)$ are $(\theta a, (\theta + \phi) a)$ and $(\phi a, (\theta + \phi) a)$. We see that $\theta < \theta + \phi, \theta, \theta + \phi \in \mathbb{N}^+$ and $(\theta, \theta + \phi) = 1$ since $(\theta, \phi) = 1$. Also, $\phi < \theta + \phi, \phi, \theta + \phi \in \mathbb{N}^+$ and $(\phi, \theta + \phi) = 1$ since $(\theta, \phi) = 1$.

From this it follows that each $(x, y) \in F_{(a,a)}$ must be of the form $(x, y) = (\theta a, \phi a)$ where $\theta \leq \phi, \theta, \phi \in \mathbb{N}^+$ and $(\theta, \phi) = 1$.

We now reverse directions and show that any arbitrary (x, y) that satisfies $(x, y) = (\theta a, \phi a), \theta \leq \phi, \theta, \phi \in \mathbb{N}^+, (\theta, \phi) = 1$ must be in $F_{(a,a)} = g(a, a)$. We do this by mathematical induction on $\theta + \phi = n$.

Now if $n = 2$ then $\theta = \phi = 1$ and $(\theta a, \phi a) = (1 \cdot a, 1 \cdot a) = (a, a) \in F_{(a,a)}$. So we have started the induction on n, and we now suppose that the conclusion is true for each $\theta + \phi = \overline{n}$ where $\overline{n} \in \{1, 2, 3, \ldots, n - 1\}$ and $n \geq 3$. We now show that the conclusion is true for any (θ, ϕ) when $\theta + \phi = n, \theta \leq \phi, \theta, \phi \in \mathbb{N}^+, (\theta, \phi) = 1$.

We consider three case.

Case (1). $\theta = \phi - \theta$.

Case (2). $\theta < \phi - \theta$.

Case (3). $\phi - \theta < \theta$.

We first observe that the conditions $\theta, \phi \in N^+, \theta \leq \phi, (\theta, \phi) = 1, \theta + \phi \geq 3$ together imply that $\theta < \phi$. Thus $1 \leq \phi - \theta$.

Case (1). Now $\theta = \phi - \theta$ implies $2\theta = \phi$ which implies $\theta = 1, \phi = 2$ since $(\theta, \phi) = 1$. Therefore, $(\theta a, \phi a) = (a, 2a)$. Also $(a, a) \in F_{(a,a)}$ implies $(a, 2a) \in F_{(a,a)}$.

Case (2). By induction $(\theta a, (\phi - \theta) a) \in F_{(a,a)}$ since $\theta, \phi - \theta \in \mathbb{N}^+, \theta < \phi - \theta, (\theta, \phi - \theta) = 1$ and $\theta + (\phi - \theta) < \theta + \phi = n$.

Also, $(\theta a, (\phi - \theta) a,) = (\theta a, (\phi - \theta) a, \phi a) \in F_{(a,a)}$ implies $(\theta a, \phi a) \in F_{(a,a)}$.

Case(3). Now by induction $((\phi - \theta)a, \theta a) \in F_{(a,a)}$ since (1) $\theta < \phi$ implies $\phi - \theta, \theta \in N^+$, (2) $\phi - \theta < \theta$, (3) $(\phi - \theta, \theta) = 1$, and (4) $(\phi - \theta) + \theta < \theta + \phi = n$. Also, $((\phi - \theta)a, \theta a) = ((\phi - \theta)a, \theta a, \phi a) \in F_{(a,a)}$ implies $(\theta a, \phi a) \in F_{(a,a)}$. ■

Observation 1. From Lemma 2, we know that if t is the greatest common divisor of a, b where $a \leq b, a, b \in \mathbb{N}^+$, then $F_{(a,b)} = g(a, b) = \left\{(tx, ty) : (x, y) \in g\left(\frac{a}{t}, \frac{b}{t}\right)\right\}$.

Also, $\left(\frac{a}{t}, \frac{b}{t}\right) = 1$ and from Note 1 this implies that $\left(\frac{a}{t}, \frac{b}{t}\right) \in F_{(1,1)}$. Thus, $\left(\frac{a}{t}, \frac{b}{t}\right)$ is a member of the binary tree $F_{(1,1)}$ which is shown in Fig. 1. Therefore, $F_{\left(\frac{a}{t}, \frac{b}{t}\right)}$ consists of all of those vertices on the binary tree $F_{(1,1)}$ that are generated by the single vertex $\left(\frac{a}{t}, \frac{b}{t}\right)$. Thus, the binary tree $F_{(1,1)}$ contains embedded in itself sufficient information to compute all $F_{(a,b)}$ where $a, b \in \mathbb{N}^+, a \leq b$. This is why we call $F_{(1,1)}$ the basic or universal Fibonacci set. Before we solve the Main Problem, we will first develop the very basic matrix machinery that we will need.

5 Basic Matrix Machinery

Lemma 3. *Suppose* $\theta, \phi, \psi, \pi \in \mathbb{N}$ *and* $\det \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} = \begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1.$

Then $(\theta, \psi) = 1, (\phi, \pi) = 1, (\theta, \phi) = 1, (\psi, \pi) = 1, (\theta + \phi, \psi + \pi) = 1$ and $(\theta + \psi, \phi + \pi) = 1$.

Proof. $(\theta, \psi) = (\phi, \pi) = (\theta, \phi) = (\psi, \pi) = 1$ is obvious. We show that $(\theta + \phi, \psi + \pi) = 1$. The proof of $(\theta + \psi, \phi + \pi) = 1$ is the same. Suppose p is a prime such that $p|m, p|n$ where $\theta + \phi = m, \psi + \pi = n$.

Now $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \begin{vmatrix} \theta & m - \theta \\ \psi & n - \psi \end{vmatrix} = \begin{vmatrix} \theta & m \\ \psi & n \end{vmatrix}$.

Now $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1$ is impossible since $p|m, p|n$. ■

Lemma 4. *Suppose* $1 \leq m \leq n$ *are relatively prime positive integers. Then there exists a unique* 2×2 *matrix* $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ *that satisfies the following conditions.*

(1). θ, ϕ, ψ, π are non-negative integers.

(2). $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = 1.$

(3). $\theta + \phi = m, \psi + \pi = n.$

Proof. First suppose that $1 = m \leq n$.

Now $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \theta\pi - \psi\phi = 1$ implies $\theta \neq 0, \pi \neq 0$. Therefore, $\theta + \phi = m = 1$ implies $\theta = 1, \phi = 0$. Therefore, $\theta = 1, \phi = 0, \theta\pi - \psi\phi = 1$ implies $\pi = 1$. Therefore, $\psi + \pi = n, \pi = 1$ implies $\psi = n - 1$. Therefore, $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ n - 1 & 1 \end{bmatrix}$.

Second, suppose that $2 \leq m \leq n$. From $\theta + \phi = m, \psi + \pi = n, \begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = 1$ we have the following:

$$\phi = m - \theta, \pi = n - \psi$$

and $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \begin{vmatrix} \theta & m - \theta \\ \psi & n - \psi \end{vmatrix} = \begin{vmatrix} \theta & m \\ \psi & n \end{vmatrix} = n\theta - m\psi = 1.$

Now obviously, $\theta \neq 0$. Also, $2 \leq n$ implies $\psi \neq 0$ since $\psi = 0$ would imply $n|1$. Suppose $\phi = 0$. Then $\theta + \phi = m$ implies $\theta = m$ and $n\theta - m\psi = nm - m\psi = 1$ is impossible since $m \geq 2$.

Therefore, $\phi \neq 0$.

Suppose $\pi = 0$. Then $\psi + \pi = n$ implies $\psi = n$ and $n\theta - m\psi = n\theta - mn = 1$ is impossible since $2 \leq n$.

Therefore, $\pi \neq 0$. Therefore, $\theta + \phi = m, \psi + \pi = n, \theta \neq 0, \phi \neq 0, \psi \neq 0, \pi \neq 0$ imply $1 \leq \theta \leq m - 1, 1 \leq \phi \leq m - 1, 1 \leq \psi \leq n - 1$ and $1 \leq \pi \leq n - 1$.

Since $2 \leq n, 2 \leq m$ and n, m are relatively prime we know from number theory (the Euclidean algorithm) that there exists a unique (θ, ψ) with $1 \leq \theta \leq m - 1, 1 \leq \psi \leq n - 1$

that satisfies $n\theta - m\psi = 1$. From this unique (θ, ψ) and from $\theta + \phi = m, \psi + \pi = n$ we see that (ϕ, π) is also unique. Therefore $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ is unique. ■

Corollary 1. *Suppose $1 \leq m \leq n$ are relatively prime positive integers. Then, there exists a unique 2×2 matrix $\begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix}$ that satisfies the following conditions.*

(1). $\overline{\theta}, \overline{\phi}, \overline{\psi}, \overline{\pi} \in \mathbb{N}$.
(2). $\begin{vmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{vmatrix} = -1$.
(3). $\overline{\theta} + \overline{\phi} = m, \overline{\psi} + \overline{\pi} = n$.

Also, the unique matrix $\begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix}$ of Corollary 1 and the unique matrix $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ of Lemma 4 are related by $\begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix} = \begin{bmatrix} \phi & \theta \\ \pi & \psi \end{bmatrix}$.

Proof. We use Lemma 4 with the matrix $\begin{bmatrix} \phi & \theta \\ \pi & \psi \end{bmatrix}$. ■

Corollary 2. *The conclusion of Lemma 4 remains true if we drop $1 \leq m \leq n$ and simply assume that m, n are any relatively prime positive integers.*

Proof. If $1 \leq n < m$ we use Corollary 1 with the matrix $\begin{bmatrix} \psi & \pi \\ \theta & \phi \end{bmatrix}$. ■

Corollary 3. *The conclusion of Corollary 1 remains true if we drop $1 \leq m \leq n$ and simply assume that m, n are any relatively prime positive integers.*

6 Solving the Main Problem

In the main problem we consider a, b to be algebraic literal numbers and we assume that $a < b, a, b \in \mathbb{N}^+$. We can assume that $a < b$ since the case where $a = b, a \in \mathbb{N}^+$ was solved in the secondary problem.

Starting with $(a, b) = (a, b, a + b)$, in Fig. 2 we show a few of the branches in the binary tree that represents the Fibonacci set $F(a, b) = g(a, b)$.

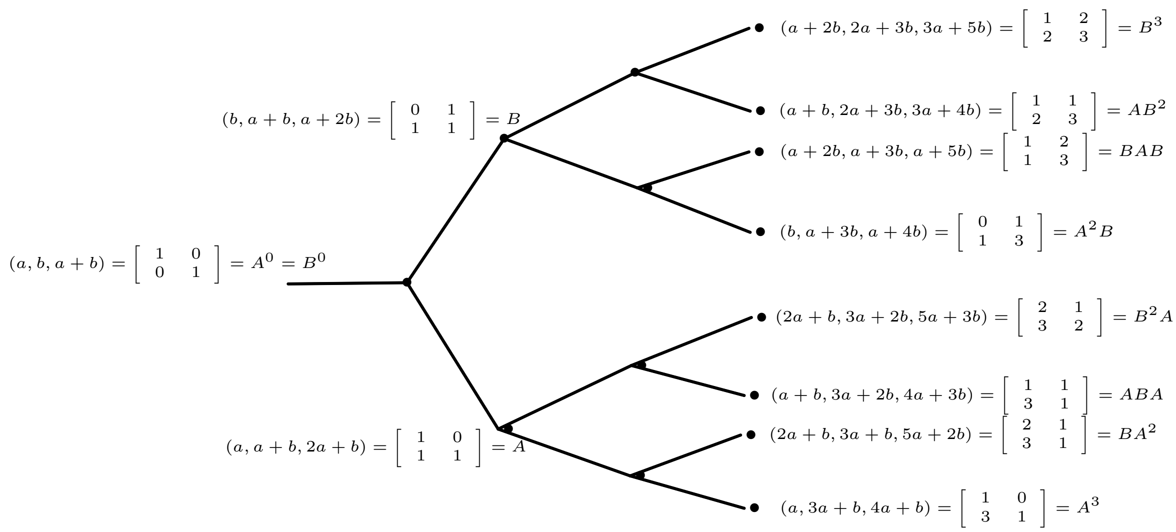


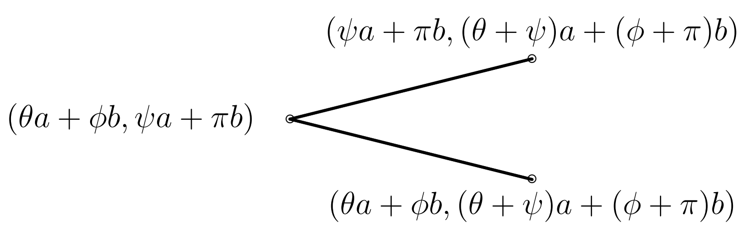
Fig. 2. The binary tree $F_{(a,b)} = g(a,b)$.

Of course, $F_{(a,b)}$ must be a binary tree since it is a binary tree for specific values of a, b.

As always, each vertex on the binary tree $F_{(a,b)}$ has exactly two immediate successors on the tree and each vertex except (a,b) has exactly one immediate predecessor on the tree. As always, from this it follows that all of the vertices shown on the binary tree $F_{(a,b)}$ must be distinct. Also, the successive levels of the tree have $1, 2, 4, 8, 16, \cdots$ vertices respectively.

The following statement $(*)$ is easy to prove by mathematical induction.

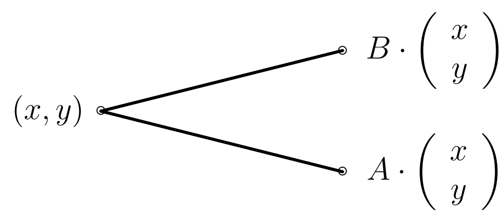
$(*)$ If $(\theta a + \phi b, \psi a + \pi b) = (\theta a + \phi b, \psi a + \pi b, (\theta + \psi) a + (\phi + \pi) b)$ is any vertex on the binary tree $F_{(a,b)}$ except (a,b), then $\theta, \psi, \phi, \pi \in \mathbb{N}, \theta + \phi \in \mathbb{N}^+, \psi + \pi \in \mathbb{N}^+, \theta \le \psi, \phi \le \pi$ and at least one of $\theta < \psi, \phi < \pi$. Therefore, $\theta + \phi < \psi + \pi$. $(*)$ follows by induction because each vertex $(\theta a + \phi b, \psi a + \pi b)$ of the binary tree $F_{(a,b)}$ has two immediate successors namely



Suppose $(\theta a + \phi b, \psi a + \pi b)$ satisfies the above conditions $(*)$. Also, suppose we wish to decide whether $(\theta a + \phi b, \psi a + \pi b) = (\theta a + \phi b, \psi a + \pi b, (\theta + \psi) a + (\phi + \pi) b)$ lies on the binary tree $F_{(a,b)}$. To do this, let us first define $ka + hb < \overline{k}a + \overline{h}b$ if $k, h, \overline{k}, \overline{h} \in \mathbb{N}, k + h \in \mathbb{N}^+, \overline{k} + \overline{h} \in \mathbb{N}^+, k \le \overline{k}, h \le \overline{h}$ and at least one of $k < \overline{k}, h < \overline{h}$. We next assume that $(\theta a + \phi b, \psi a + \pi b)$ lies on $F_{(a,b)}$. Then since each vertex of $F_{(a,b)}$ except (a,b) has exactly one immediate predecessor in $F_{(a,b)}$, we work backwards from $(\theta a + \phi b, \psi a + \pi b)$ one step at a time, using the above definition of $<$, until we either reach (a,b) or else reach a point where an immediate predecessor does not exist. By using the above definition of $<$, these immediate predecessors can be computed exactly as we did in the proof of Lemma 1. We now derive a lemma that will tell us directly whether $(\overline{a}, \overline{b}) = (\theta a + \phi b, \psi a + \pi b)$ lies on $F_{(a,b)}$ or not.

Notation 2. *Let* $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. *Then* $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ *and* $B^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$.

Lemma 5. *Suppose* $(x, y) = (x, y, x + y)$, *where* $x, y \in \mathbb{N}^+$, $x \leq y$, *is an element in a Fibonacci set* F. *Then the two immediate successors of* $(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}$ *in* F *are the following.*



Proof. This is obvious. ■

Observations 2. It follows from Lemma 5 that each element $(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}$ of the binary tree $F_{(a,b)} = g(a, b)$ can be written $(x, y) = \begin{pmatrix} x \\ y \end{pmatrix} = T \cdot \begin{pmatrix} a \\ b \end{pmatrix}$ where $T = C_1 \cdot C_2 \cdots C_t$ with each $C_i \in \{A, B\}$ and where we also include $T = A^\circ = B^\circ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Since $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and since we are also including $A^\circ = B^\circ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ we immediately see that $T = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ satisfies the following conditions which are slightly weaker than the conditions $(*)$ mentioned earlier.

(1). $\det T = \pm 1$.

(2). $\theta, \phi, \psi, \pi \in \mathbb{N}$.

(3) $1 \leq \theta + \phi \leq \psi + \pi$.

From Lemma 3, conditions (1), (2) also imply that $(\theta + \phi, \psi + \pi) = 1$.

If we include $T = A^\circ = B^\circ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then we will soon show that these three properties (1), (2), (3) exactly determine all of the 2×2 matrices T such that $T = C_1 \cdot C_2 \cdots C_t$ with each $C_i \in \{A, B\}$. This will be the complete solution to the Main Problem.

Also, suppose $T = C_1 \cdot C_2 \cdots C_r, \overline{T} = \overline{C}_1 \cdot \overline{C}_2 \cdots \overline{C}_s$ where each $C_i \in \{A, B\}$ and each $\overline{C}_i \in \{A, B\}$.

Starting at $(a,b) = \begin{pmatrix} a \\ b \end{pmatrix}$ on the binary tree. $F_{(a,b)} = g(a,b)$, where $a < b, a, b \in \mathbb{N}^+$, we see that $T \cdot \begin{pmatrix} a \\ b \end{pmatrix}$ and $\overline{T} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$ will be the same vertex on the tree $F_{(a,b)}$ if and only if $r = s$ and for every $i \in \{1, 2, \cdots, r = s\}, C_i = \overline{C}_i$.

Therefore, it follows that $T = \overline{T}$ if and only if $r = s$ and for every $i \in \{1, 2, \cdots, r = s\}, C_i = \overline{C}_i$.

Lemma 6. *Suppose $(x,y) = (x, y, x+y), x, y \in \mathbb{N}^+, x < y$, is an element in a Fibonacci set F.*

If $(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}$ has an immediate predecessor $(\overline{x}, \overline{y})$ in F, then either

$$(A) \qquad (\overline{x}, \overline{y}) = \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x+y \end{pmatrix}$$

or

$$(B) \qquad (\overline{x}, \overline{y}) = \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = B^{-1} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x+y \\ x \end{pmatrix}.$$

Proof. The proof is obvious. ■

Definition 4. *Suppose $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}, \begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix}$ are 2×2 matrices. We say that $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} \sim \begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix}$ if $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} = \begin{bmatrix} \overline{\phi} & \overline{\theta} \\ \overline{\pi} & \overline{\psi} \end{bmatrix}$.*

Lemma 7. *Suppose R, S are 2×2 matrices, and $R \sim S$. Then $AR \sim AS$ and $BR \sim BS$.*

Proof. Let $R = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}, S = \begin{bmatrix} \phi & \theta \\ \pi & \psi \end{bmatrix}$.

Then $AR = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} = \begin{bmatrix} \theta & \phi \\ \theta+\psi & \phi+\pi \end{bmatrix}$.

Also, $AS = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \phi & \theta \\ \pi & \psi \end{bmatrix} = \begin{bmatrix} \phi & \theta \\ \phi+\pi & \theta+\psi \end{bmatrix}$

Therefore, $AR \sim AS$. Likewise $BR \sim BS$. ■

Lemma 8. *Let T be the matrix product $T = C_1 \cdot C_2 \cdots C_t$ where each $C_i \in \{A, B\}$. Then* $\det T = \pm 1$.

Also, $TA \sim TB$.

Proof. Since $\det A = 1, \det B = -1$ it follows that $\det T = \pm 1$. Also, since $A \sim B$ it follows from repeated use of Lemma 7 that $TA \sim TB$. ■

Observations 3. In the Fig. 2 Fibonacci tree $F_{(a,b)}$, we observe that $A^3 \sim A^2B, BA^2 \sim BAB, ABA \sim AB^2, B^2A \sim B^3$. From Lemma 8, we note in general that the 2^n elements in level $n+1$ of the Fibonacci tree $F_{(a,b)}$ must occur in symmetric pairs $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \theta a + \phi b \\ \psi a + \pi b \end{pmatrix}$ and $\begin{bmatrix} \phi & \theta \\ \pi & \psi \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \phi a + \theta b \\ \pi a + \psi b \end{pmatrix}$. For example, in the 4th level of the Fig.

2 Fibonacci tree, we note that $(a+b, 2a+3b) = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = AB^2 \begin{pmatrix} a \\ b \end{pmatrix} \in F_{(a,b)}$ and $(a+b, 3a+2b) = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = ABA \begin{pmatrix} a \\ b \end{pmatrix} \in F_{(a,b)}$.

This is because $AB \cdot B \sim AB \cdot A$.

Lemma 9. *Suppose $m, n \in \mathbb{N}^+$ are any arbitrary members of $\mathbb{N}^+$ that satisfy $m \leq n$ and $(m,n) = 1$. Then there exists at least one matrix T of the form $T = C_1 \cdot C_2 \cdots C_t = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$, where each $C_i \in \{A, B\}$, such that $\theta + \phi = m, \psi + \pi = n$. This includes $T = A^\circ = B^\circ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Also, $\theta, \phi, \psi, \pi \in \mathbb{N}$ and $\det T = \pm 1$.*

Proof. From the solution of the Secondary Problem, we know that

$$g(a,a) = \left\{ \left(\overline{\theta} a, \overline{\phi} a\right) : \overline{\theta}, \overline{\phi} \in \mathbb{N}^+, \overline{\theta} \leq \overline{\phi}, \left(\overline{\theta}, \overline{\phi}\right) = 1 \right\}.$$

Let $\overline{\theta} = m, \overline{\phi} = n$. By letting $a = b$ and using the properties of the binary tree $F_{(a,b)}$ it follows that there exists $T = C_1 \cdot C_2 \cdots C_t = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ with each $C_i \in \{A, B\}$ such that

$$\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} \begin{pmatrix} a \\ a \end{pmatrix} = \begin{pmatrix} (\theta + \phi) a \\ (\psi + \pi) a \end{pmatrix} = \begin{pmatrix} ma \\ na \end{pmatrix}. \quad \blacksquare$$

Corollary 4. *Suppose $m, n \in \mathbb{N}^+, m \leq n$ and $(m,n) = 1$. Then from Lemma 8 and Observation 3 we know that there exists at least two distinct matrices $T, \overline{T}$ that satisfy the conclusion of Lemma 9.*

Also, by Lemma 8 and Observation 3 we can call $T = C_1 \cdot C_2 \cdots C_t \cdot A$ and call $\overline{T} = C_1 \cdot C_2 \cdots C_t \cdot B$. Since $\det T = -\det \overline{T}$ we also conclude that $\left\{\det T, \det \overline{T}\right\} = \{-1, 1\}$.

Proof. The proof is obvious. $\blacksquare$

Lemma 10. *Define $\overline{T} = \{C_1 \cdot C_2 \cdots C_t : t \in \mathbb{N},$ each $C_i \in \{A, B\}\}$ where we agree that $C_1 \cdot C_2 \cdots C_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ when $t = 0$.*

Also, $\overline{M} = \left\{ \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} : \theta, \phi, \psi, \pi \in \mathbb{N}, \begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1, \theta + \phi \leq \psi + \pi \right\}$.

Then $\overline{T} = \overline{M}$.

Erratum. Technically, $\overline{T} = \overline{M} \setminus \{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \}$. We patch this up by agreeing that $\overline{M} = \overline{M} \setminus \{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \}$. **Proof.** First we show that $\overline{T} \subseteq \overline{M}$.

Let $T = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} = C_1 \cdot C_2 \cdots C_t$ where each $C_i \in \{A, B\}$. We show $T \in \overline{M}$. Now obviously $\theta, \phi, \psi, \pi \in \mathbb{N}$ and $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1$. Also, since $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ it is obvious by induction that $\theta + \phi \leq \psi + \pi$. Also, when $t = 0$, we define $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Therefore, $T \in \overline{M}$.

Next we show that $\overline{M} \subseteq \overline{T}$, where we consider $\overline{M} = \overline{M} \setminus \{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$.

Therefore, suppose $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} \in \overline{M}$ is any fixed member of $\overline{M}$.

Now since $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1$, we know from Lemma 3 that $(\theta + \phi, \psi + \pi) = 1$.

Let us call $\theta + \phi = m, \psi + \pi = n$ where m, n are fixed, $m, n \in \mathbb{N}^+, m \leq n$ and $(m, n) = 1$. Since the case $m = n = 1$ is trivial, we suppose that $1 \leq m < n$.

From Lemma 4 and Corollary 1, we now know the following.

(a). If $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = 1$, then there is only one possible matrix $\begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix}$ that satisfies $\overline{\theta}, \overline{\phi}, \overline{\psi}, \overline{\pi} \in \mathbb{N}, \begin{vmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{vmatrix} = 1$ and $\overline{\theta} + \overline{\phi} = m, \overline{\psi} + \overline{\pi} = n$, namely $\begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix} = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$.

(b). If $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = -1$, then there is only one possible matrix $\begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix}$ that satisfies $\overline{\theta}, \overline{\phi}, \overline{\psi}, \overline{\pi} \in \mathbb{N}, \begin{vmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{vmatrix} = -1$ and $\overline{\theta} + \overline{\phi} = m, \overline{\psi} + \overline{\pi} = n$ namely, $\begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix} = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$.

Also, from Lemma 9 and Corollary 4, we know that this unique matrix $\begin{bmatrix} \overline{\theta} & \overline{\phi} \\ \overline{\psi} & \overline{\pi} \end{bmatrix} = \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ of cases (a), (b) must lie in $\overline{T}$ and this completes the proof. ■

Solution to the Main Problem.

We are required to compute $F_{(a,b)} = g(a, b)$ in a closed form.

Calling $(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}$ we know that $F_{(a,b)} = \left\{ T \cdot \begin{pmatrix} a \\ b \end{pmatrix} : T \in \overline{T} \right\}$ where $\overline{T}$ is defined in Lemma 10.

Now $\overline{T} = \overline{M}$, where $\overline{M}$ is also defined in Lemma 10. Therefore $F_{(a,b)} =$

$$\left\{ M \cdot \begin{pmatrix} a \\ b \end{pmatrix} : M \in \overline{M} \right\} = \left\{ \begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} : \theta, \phi, \psi, \pi \in \mathbb{N}, \begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1, \theta + \phi \leq \psi + \pi \right\}. \blacksquare$$

Note 2. It is easy to compute members $M \in \overline{M}$. Suppose, for example, that $\theta, \phi, \psi, \pi \in \mathbb{N}, \begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1$ and $\theta + \phi > \psi + \pi$. We just reverse the two rows and we have $\begin{vmatrix} \psi & \pi \\ \theta & \phi \end{vmatrix} = \pm 1$ with $\psi + \pi < \theta + \phi$

7 Solving Specific Numerical Problems

Suppose $a, b \in \mathbb{N}^+, a < b, \overline{a}, \overline{b} \in \mathbb{N}^+, \overline{a} < \overline{b}$ are specific positive integers and we wish to decide whether $(\overline{a}, \overline{b}) \in g(a, b)$ when $(\overline{a}, \overline{b}) \neq (a, b)$.

Define $t = \gcd(a,b)$, $\bar{t} = \gcd(\bar{a},\bar{b})$ where gcd denotes the greatest common divisor. From Lemma 2 (or by induction) it is easy to prove that if $(\bar{a},\bar{b}) \in g(a,b)$ then $t = \bar{t}$ is a necessary condition.

Also, if $a < b$ and $(\bar{a},\bar{b}) \in g(a,b)$, then the inequalities of Fig. 3 are also easily proved necessary conditions.

$$\begin{array}{ccc} a \bullet & \leq & \bullet \bar{a} \\ \bigwedge & & \bigwedge \\ b \bullet & < & \bullet \bar{b} \end{array}$$

Fig. 3. Inequalities when $(\bar{a},\bar{b}) \in g(a,b)$ and $a < b$.

From Lemma 2, it is also easy to show that $(\bar{a},\bar{b}) \in g(a,b)$ if and only if $\left(\frac{\bar{a}}{t},\frac{\bar{b}}{t}\right) \in g\left(\frac{a}{t},\frac{b}{t}\right)$ where $t = \gcd(\bar{a},\bar{b}) = \gcd(a,b)$.

We will now develop two algorithms for deciding if $(\bar{a},\bar{b}) \in g(a,b)$ when $(\bar{a},\bar{b}) \neq (a,b)$, $\gcd(a,b) = \gcd(\bar{a},\bar{b}) = 1$ and the necessary inequalities of Fig. 3 are met. We will suppose $a < b$ since the secondary problem has already taken care of the easy case where $a = b$. Since $\gcd(a,b) = \gcd(\bar{a},\bar{b}) = 1$, we know that (a,b) and $(\bar{a},\bar{b})$ are vertices on the basic Fibonacci tree $F_{(1,1)}$. Therefore, it makes sense to talk about immediate predecessors on the tree.

(A). One way to numerically decide if $(\bar{a},\bar{b}) \in g(a,b)$ is to work backwards from $(\bar{a},\bar{b})$ by finding consecutive immediate predecessors until we either arrive at (a,b) or else arrive at a contradiction to the necessary inequalities of Fig. 3.

(B) We will now develop a matrix solution that uses the solution to the Main Problem. We know that $(\bar{a},\bar{b}) \in g(a,b)$ is true if and only if there exists a matrix $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ that satisfies the following conditions.

(1) $\theta, \phi, \psi, \pi \in \mathbb{N}$.

(2) $\begin{vmatrix} \theta & \phi \\ \psi & \pi \end{vmatrix} = \pm 1$.

(3) $\theta + \phi \leq \psi + \pi$.

(4) $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix}$.

Each matrix $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ that satisfies conditions (1), (2), (3) can be written as $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix} = C_1 \cdot C_2 \cdots C_t$ with each $C_i \in \{A, B\}$.

Also, each distinct $C_1 \cdot C_2 \cdots C_t$ places $(C_1 \cdot C_2 \cdots C_t) \begin{pmatrix} a \\ b \end{pmatrix}$ at a different vertex on the Fibonacci tree $F_{(a,b)}$. Thus, if $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ exists that satisfies conditions (1), (2), (3), (4) then it is unique.

From (1), (2), (3), (4) we have the following.

$(1') \ \theta\pi - \psi\phi = \pm 1.$
$(2') \ \theta a + \phi b = \overline{a}.$
$(3') \ \psi a + \pi b = \overline{b}.$
Therefore, $(2'') \ \theta = \frac{\overline{a} - \phi b}{a}.$
$(3'') \ \psi = \frac{\overline{b} - \pi b}{a}.$
Therefore, $(1'') \ \left[\frac{\overline{a} - \phi b}{a}\right]\pi - \left[\frac{\overline{b} - \pi b}{a}\right]\phi = \pm 1.$
Therefore, $\overline{a}\pi - b\pi\phi - \overline{b}\phi + b\pi\phi = \pm a.$

Thus $(**)$ $\overline{a}\pi - \overline{b}\phi = \pm a$. From $(3')$, we see that $0 \leq \pi < \overline{b}$ since $b \geq 2$. Also, from $(2')$ we see that $0 \leq \phi < \overline{a}$ since $b \geq 2$. From $(**)$ we have $(***) \, \pi = \frac{\pm a + \overline{b}\phi}{\overline{a}}$ subject to $0 \leq \phi < \overline{a}, 0 \leq \pi < \overline{b}$.

Since $\left(\overline{a}, \overline{b}\right) = 1$, it is easy to see that if solutions (ϕ, π) exist for $(***)$ then they must be unique for each $\pm a$.

Therefore, the matrix solution requires us to first find these unique solutions (ϕ, π) to $(***)$ subject to the side conditions $0 \leq \pi < \overline{b}, 0 \leq \phi < \overline{a}$ if such solutions exist.

If a solution (ϕ, π) to $(***)$ exists, for either $\pm a$, then (θ, ψ) can be uniquely computed from (ϕ, π). We then check to see if the matrix $\begin{bmatrix} \theta & \phi \\ \psi & \pi \end{bmatrix}$ satisfies the conditions $\theta, \psi \in \mathbb{N}$ and $\theta + \phi \leq \psi + \pi$. The other conditions in (1), (2), (3), (4) are automatically satisfied.

8 Some Concluding Remarks

It is possible to prove more properties of Fibonacci sets than we have proved in this paper.

As an example, suppose $a, b, \overline{a}, \overline{b} \in \mathbb{N}^+$ are $a < b, \overline{a} < \overline{b}$. We say that (a, b) and $\left(\overline{a}, \overline{b}\right)$ are independent if $(a, b) \notin g\left(\overline{a}, \overline{b}\right)$ and $\left(\overline{a}, \overline{b}\right) \notin g(a, b)$. If (a, b) and $\left(\overline{a}, \overline{b}\right)$ are independent, then we can show that $g(a, b) \cap g\left(\overline{a}, \overline{b}\right) = \phi$, the empty set.

As a further extension, the reader might like to use the isomorphism $f : (\mathbb{Z}, 0, +) \rightarrow (\mathbb{Z}, 1, *)$, where $\mathbb{Z}$ is the set of all integers, $f(x) = x + 1, a * b = a + b - 1$, and then substitute this operator $*$ for $+$ and study para-Fibonacci sets $\overline{F}$ that satisfy both (1) for all $t \in \overline{F}$, $t = \{x, y, x * y\}$ where $x, y, x * y \in \mathbb{N}^+$ and (2) if $t = \{a, b, a * b\} \in \overline{F}$, then $\{a, a * b, a * (a * b)\} \in \overline{F}$ and $\{b, a * b, b * (a * b)\} \in \overline{F}$.

References

[1] Engel, Arthur, Problem-Solving Strategies, Springer, 1998.