Abstract Composite Games with Moves in All Components Compulsory, Part I

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Abstract

An impartial combinatorial game played under *normal* rules has two players who alternate moving. There is no infinite sequence of moves, both players have the same moves available, and the winner is the last player to make a move. In this paper the games can be both *transfinite* and *abstract*. By transfinite we mean that positions exist for which the number of moves (although finite) might be arbitrarily large. By abstract we mean that instead of defining what the moves are, we will simply list certain properties that a move must have. The basic idea here is to create abstract transfinite analogies for all the normal concrete games. In this paper we generalize a game of Hugo Steinhaus. See [5].

Suppose $G_1, G_2, \cdots, G_n$ are normal impartial games. There are two variations that can be played. Due to the complexity of the abstract games, we only study Game A in Part I. Game B is significantly harder than game A and will be studied in Part II.

In Game A, for **all** $i = 1, 2, \cdots, n$, the moving player must make a move in G_i if the position in G_i is not a terminal position. The loser is the first player to face a position in which for all $i = 1, 2, \cdots, n$, the position in G_i is terminal. Also, the moving player can move in any order, e.g., $G_1, G_2, G_3; G_2, G_3, G_1; G_1, G_3, G_2$. This becomes important in the abstract games.

In Game B, the moving player must make a move in each $G_i, i = 1, 2, \cdots, n$, with no exceptions. The loser is the first player to face a position in which he is unable to move in *some* $G_i, i = 1, 2, \cdots, n$. Also, the moving player can move in any order.

Concrete normal impartial Games 1. *A normal impartial game G_i is said to be concrete if the positions and moves of the game correspond to the vertices and directed edges of a digraph (V, E) with vertex set V and directed edge set E.*

At the end of the paper, we show that all concrete normal impartial games can be represented by the following model. Let $(P, \leq)$ be a well-ordered set of arbitrary ordinality with 0 being the first element and 1 being the second element. Also, $\forall a \in P$ we are given a set (possibly empty) $S(a) \subseteq [0, a)$, where $[0, a) = \{x : x \in P, 0 \leq x < a\}$. $[0, a)$ is called the initial segment of a and is usually denoted by $s(a)$.

At the beginning of the game, a position $p_0 \in P$ is designated the initial position. Two players alternate moving. The game is impartial which means that both players have the same moves available. If the moving player is facing a position $p \in P$, then he can move to any position $q \in S(p)$ if $S(p)$ is non-empty. Of course, $q < p$ since $S(p) \subseteq [0, p)$. If $S(p)$ is empty, we call p a terminal position and in particular 0 is a terminal position. The winner is the first player to land on a terminal position which means the winner is the last player to move. If p_0 is a terminal position, the first moving player loses automatically.

Since $(P, \leq)$ is well-ordered, there can be no infinite sequence of moves. To see this, suppose $p_0 \to p_1 \to p_2 \to \cdots$ is an infinite sequence of moves. Now $p_0 > p_1 > p_2 \cdots$. However, since $(P, \leq)$ is well-ordered, we know that $\{p_0, p_1, p_2, \cdots,\}$ has a first element. Call it p_k. But this is impossible if $\{p_0, p_1, p_2, \cdots,\}$ is infinite since $p_k > p_{k+1} > p_{k+2} > \cdots$.

Notation. The game that we have specified can be denoted $G = (P, \leq, S)$. Before defining abstract games, we deal with $(ord, +, \leq)$. $(ord, +, \leq)$ is the usual ordinal addition on the well-ordered proper class of all ordinal numbers. If a $\in ord$, then the immediate successor of a is $a + 1$. Also, if a has an immediate predecessor, let $a - 1$ denote the immediate predecessor of a.

Basic Set Theory. *For any well-ordered set $(P, \leq)$, there exists a unique $a \in ord$ and a unique $1 - 1$ onto function $f : P \to [0, a)$ such that f is a similarity mapping of $(P, \leq)$ onto $([0, a), \leq)$. That is, $\forall a, b \in P, a \leq b$ in $(P, \leq)$ if and only if $f(a) \leq f(b)$ in $([0, a), \leq)$.*

$\forall x \in P$ let $ord(x) = f(x)$ denote the ordinal equivalent of x.

Lemma 1. $\forall a_0 \in ord$, there is no infinite sequence $a_0, a_1, a_2, \cdots$ such that each a_{i+1} is the immediate predecessor of a_i.

Proof. Since $a_0 > a_1 > a_2 > \cdots$, the same proof used above shows that $\{a_0, a_1, a_2, \cdots\}$ cannot be infinite. □

Definition 1. *$\forall a \in ord, a$ is called a limit ordinal if a has no immediate predecessor. In* *particular, 0 is a limit ordinal.*

Definition 2. *Reversing the notation, $\forall a_k \in ord$, define $a_k, a_{k-1}, a_{k-2}, \cdots, a_1, a_0$ where a_0 is a limit ordinal and each a_{i-1} is the immediate predecessor of a_i. Then the degree of a_k is $d(a_k) = k$. Thus, $\forall a_0 \in ord$, a_0 is a limit ordinal if and only if $d(a_0) = 0$.*

Definition 3. *$\forall a_k \in ord$, we say that a_k is an odd ordinal if $d(a_k)$ is odd. Also, a_k is an even ordinal if $d(a_k)$ is even. Also, a_k is a very even ordinal if $d(a_k)$ is even and $d(a_k) \geq 2$.*

Lemma 2. $\forall a \in ord, a$ has an immediate predecessor, $a - 1$, if and only if a is odd or very even.

Lemma 3. $\forall a, b \in ord$, if a is odd and b is even, then $a \neq b$.

Lemma 4. $\forall a \in ord$, if a is odd, then $a + 1$ is very even and $a - 1$ is even. If a is even, then $a + 1$ is odd. If a is very even, then $a - 1$ is odd.

Before we define an abstract normal impartial game, we need to give some examples. A basic reason for defining abstract games is that the amount of information in concrete games is often far more than needed.

Example 1. Two players, Art and Beth, are facing a pile of counters, and the winner is the last player to move. The game is impartial so by symmetry suppose it is Art's move. Art's move consists of the following 3 steps.

1. Before Art moves, Beth must block exactly one of Art's options.
2. Art then removes x counters where $x \in \{1, 2, 3\}$ provided that x is not blocked and x counters remain in the pile.
3. Beth has the option of removing one additional counter if the pile still has counters in it at this point.

Of course, Beth's move consists of interchanging the words Art and Beth in the definition of Art's move. We say that a player has made a move if at the end of steps 1,2,3 the number of counters in the pile has been reduced.

Example 2. Example 2 is the same as example 1, except step 2 is changed to read as follows:

2′ If Art is facing an even pile of counters, he removes $x \in \{1,2,3,4\}$ provided x is not blocked and the pile has x counters. If Art is facing an odd pile of counters, he removes $x \in \{1,2,3\}$ provided x is not blocked and the pile has x counters.

Example 3. Two players, Art and Beth, are facing a pile of counters. Art and Beth alternate moving, and the winner is the last player to move. The game is impartial so by symmetry suppose it is Art's move. Art's move consists of one step.

1. On Art's move, Beth removes $1, 2$, or 3 counters as <u>she</u> chooses.

   Note that on Art's move, he does nothing at all and yet this is still considered to be his move. By symmetry on Beth's move, Art removes $1, 2$ or 3 counters as <u>he</u> chooses.

Example 4. Two players, Art and Beth, are facing a pile of counters. Art and Beth alternate moving, and the winner is the last player to move. The game is impartial so by symmetry suppose it is Art's move. Art's move consists of the following 3 steps.

1. Unknown to each other, Art writes down $x \in \{1,2,3\}$ on a card and Beth writes down $y \in \{0,1\}$ on a card.

2. Art and Beth simultaneously lay their numbers down on the table.

3. If the pile size n satisfies $n \geq x + y$, Art removes $x + y$ counters leaving $n - x - y$ counters. If the pile size n satisfies $n < x + y$, Art removes all of the remaining counters leaving an empty pile.

By symmetry on Beth's move, Beth writes down $x \in \{1,2,3\}$ and Art writes down $y \in \{0,1\}$.

Based on these examples, we now define an abstract impartial normal game and from that definition we define an abstract impartial normal *combinatorial* game. In [3] we give a rigorous derivation of this definition starting with a simple, natural definition in which the term 'well-ordered' is not mentioned.

Abstract impartial normal games 2. *Let $(P, \leq)$ be a well-ordered set of arbitrary ordinality with 0 being the first element and 1 the second element. A set $T \subseteq P$, where $0 \in T$, is given and each $p \in T$ is called a terminal position.*

$\forall x \in P \backslash T$, we are given sets $A(x), B(x), C(x)$ that satisfy the following conditions.

1. *$A(x), B(x), C(x)$ is a partition of $2^{[0,x)}$, where $2^{[0,x)}$, called the power set, is the family of all subsets of $[0,x)$ including ϕ. That is, $A(x) \cap B(x) = A(x) \cap C(x) = B(x) \cap C(x) = \phi$ and $A(x) \cup B(x) \cup C(x) = 2^{[0,x)}$.*

2. *$\forall \theta \subseteq [0,x), \forall \psi \subseteq [0,x)$, if $\theta \subseteq \psi$, then*

   (a) $\theta \in A(x) \Rightarrow \psi \in A(x)$, and

   (b) $\psi \in B(x) \Rightarrow \theta \in B(x)$.

3. $[0,x) \in A(x), \phi \in B(x)$.

At the beginning of the game, a position $p_0 \in P$ is designated the initial position. Two players, Art and Beth, alternate moving and the winner is the first player to land on a terminal position. If $p_0 \in T$, the first moving player loses automatically. The game is impartial or symmetric. However, a move is undefined, and we will only list the properties that a move must have.

By symmetry, suppose it is Art's move and Art is facing a position $p \in P \backslash T$. Art's move consists of some undefined encounter (whatever that is) between Art and Beth that has the following properties.

1. When Art's move is over, the new position will be $\bar{p} \in P$. (The move is denoted $p \rightarrow \bar{p}$).

2. The new position $\bar{p}$ will lie inside of $[0,p)$. That is, $\bar{p} \in [0,p)$.

3. $\forall \theta \in A(p)$, if Art uses perfect play (whatever that is), he can force $\bar{p} \in \theta$.

4. $\forall \theta \in B(p)$, if Beth uses perfect play, she can force $\bar{p} \in [0,p) \backslash \theta$. That is, she can force the end of Art's move, $\bar{p}$, to lie outside of θ.

5. $\forall \theta \in C(p)$, no information is given about θ.

By symmetry, if it is Beth's move and Beth is facing $p \in P \backslash T$, the words Art and Beth are interchanged in (1), (3), (4). For example, in (3), $\forall \theta \in A(p)$, if Beth uses perfect play, she can force $\bar{p} \in \theta$.

Note$\forall x \in P \backslash T$, *the three sets $A(x), B(x), C(x)$ is a structure.*

Notation. We will denote this abstract normal impartial game by $(P, \leq, T, A, B, C)$.

Definition 4. *The abstract normal impartial game* $(P, \leq, T, A, B, C)$ *is a combinatorial game if* $\forall x \in P \backslash T, C(x) = \phi$.

Notation. We will denote this abstract normal impartial combinatorial game by $(P, \leq, T, A, B)$.

Note. $\forall x \in P \backslash T$, *the two sets* $A(x), B(x)$ *is a structure.*

Remark 1. Our example 4 is obviously not a combinatorial game. Even so, it is an amazing fact that it is possible to compute the generalized Sprague-Grundy values of this game. Then we can use these Sprague-Grundy values to play composite games in the usual way. See [3].

Safe (0) **and unsafe** (1) **positions.** $(P, \leq, T, A, B)$ *is given.* $\forall p_0 \in P$, *if the first moving player starting at* p_0 *can win with perfect play, we call* p_0 *an unsafe* (1) *position. If the second moving player can win with perfect play when the game starts at* p_0, *we call* p_0 *a safe* (0) *position. Of course, all terminal positions, i.e.,* $p_0 \in T$, *are safe* (0).

Safe (0) **and unsafe** (1) **Algorithm 1.** *In the game* $(P, \leq, T, A, B)$, *we assign to each* $x \in P$ *a* 0 *or* 1 *by transfinite induction on* $(P, \leq)$. *First, we assign a* 0 *to the first element* 0 *of* $(P, \leq)$ *since the first element* 0 *of* $(P, \leq)$ *is terminal and therefore safe. Suppose all* $y \in [0, x)$ *have been assigned* $0's$ *or* $1's$. *We then assign* 0 *or* 1 *to* x *by the following rules.*

1. *If* x *is a terminal position, we assign* x *a* 0 *since* x *is safe.*

   If x *is non-terminal, define* $0(x) = \{y : y \in [0, x), y \text{ is assigned a } 0\}$

2. *If* $0(x) \in A(x)$, *we assign* 1 *to* x.

3. *If* $0(x) \in B(x)$, *we assign* 0 *to* x.

Explanations. *In Step 2, if* $0(x) \in A(x)$, *the player facing* x *with perfect play can land inside of* $0(x)$. *But no matter where he lands in* $0(x)$, *he must land on a safe* (0) *position since* $0(x)$ *consists of all the safe positions in* $[0, x)$. *A similar explanation holds for Step 3.*

We now develop the machinery for studying game A in the abstract.

Notation. In the game $(P, \leq, T, A, B), \forall x \in P, 0(x) = \{y \in [0, x) : y \text{ is safe } (0)\}$, $1(x) = \{y \in [0, x) : y \text{ is unsafe } (1)\}$.

Remark 2. Since $(ord, \leq)$ is well-ordered, if $\exists t \in ord$ such that t has a certain property, then it makes sense to talk about the **smallest** $t \in ord$ that has this property.

The following definition is illustrated by a suggestive drawing in the appendix which we advise the reader to refer to.

The $M(x)$ function 2. *In $(P, \leq, T, A, B), \forall x \in P$ we assign to x an ordinal number $M(x) \in ord$ by transfinite induction on $(P, \leq)$. Also, by this transfinite induction, we show (a), (b),*

(a) $\forall x \in P, M(x)$ *is odd if x is unsafe (1), and $M(x)$ is even if x is safe* (0).

(b) $\forall x \in P, M(x) \leq ord(x) + 1$, *where $ord(x)$ is the ordinal equivalent of x specified in the basic set theory section.*
We provide some additional information as well.

(1) *First, $M(0) = 0$, since 0 is terminal.*

(2) *Suppose $\forall y \in [0, x), M(y) \in ord$ has been assigned. Also, suppose $\forall y \in [0, x)$, y satisfies (a), (b). Then $M(x)$ is computed as follows. Also, $M(x)$ satisfies (a), (b).*

A. *If x is terminal, $M(x) = 0$.*

B. *Suppose x is non-terminal and safe* (0). *Let $\underline{t} \in ord$ be the smallest member of $(ord, \leq)$ satisfying $0(x) \cup \{y \in 1(x) : M(y) \leq \underline{t}\} \in A(x)$. This implies $\forall t' \in ord$, if $t' < \underline{t}$, then $0(x) \cup \{y \in 1(x) : M(y) \leq t'\} \in B(x)$. Such a $\underline{t}$ exists and $\underline{t} \leq ord(x)$ since by induction $\forall y \in [0, x), M(y) \leq ord(y) + 1 \leq ord(x)$, which implies $0(x) \cup \{y \in 1(x) : M(y) \leq ord(x)\} = 0(x) \cup 1(x) = [0, x) \in A(x)$. Also, $1 \leq \underline{t}$ is easy to prove since $\underline{t} = 0$ leads to the contradiction $0(x)$ is a member of $A(x)$. $M(x)$ is now defined as follows.*

(a′) *If $\underline{t}$ is a limit ordinal, $M(x) = \underline{t}$.*

(b′) *If $\underline{t}$ is not a limit ordinal, $M(x) = \underline{t} + 1$.*

Still assuming (B) x is non-terminal and safe (0), *we have the following.*

i. *If $\underline{t}$ is a limit ordinal, then $\underline{t}$ is even which means $M(x) = \underline{t}$ is even. Suppose $y \in 1(x)$ and $M(y) \leq \underline{t}$. Now by induction, $M(y)$ is odd since $y \in 1(x)$. Therefore, $M(y) \neq \underline{t}$. Therefore, if $y \in 1(x)$, then $M(y) \leq \underline{t}$ implies $M(y) < \underline{t}$. Therefore, if $\underline{t}$ is a limit ordinal, $0(x) \cup \{y \in 1(x) : M(y) \leq \underline{t}\} = 0(x) \cup \{y \in 1(x) : M(y) < \underline{t}\} \in A(x)$.*

ii. *If $\underline{t}$ is not a limit ordinal, then $\underline{t}$ has an immediate predecessor,* $\underline{t}-1$. *Suppose* $\forall y \in [0,x), [y \in 1(x)$ *and* $M(y) \le \underline{t}]$ *implies* $M(y) < \underline{t}$. *This would mean that* $\{y \in 1(x) : M(y) \le \underline{t}\} = \{y \in 1(x) : M(y) < \underline{t}\} = \{y \in 1(x) : M(y) \le \underline{t}-1\}$, *which contradicts the fact that $\underline{t}$ is the smallest ordinal number satisfying* $0(x) \cup \{y \in 1(x) : M(y) \le \underline{t}\} \in A(x)$.
Therefore, if $\underline{t}$ is not a limit ordinal, $\exists \bar{y} \in 1(x)$ *such that* $M(\bar{y}) = \underline{t}$. *Since by induction $M(\bar{y})$ is odd, this means $\underline{t}$ is odd and* $M(x) = \underline{t}+1$ *is even when $\underline{t}$ is not a limit ordinal.*
From i and *ii it follows that in both* (a'), (b'), $M(x)$ *is even. Also,* $M(x)$ *is very even if $\underline{t}$ is not a limit ordinal.*
Also, since $\underline{t} \le ord(x)$, *in both* (a'), (b'), $M(x) \le \underline{t}+1 \le ord(x)+1$.

C. *Suppose x is unsafe (1). Of course, x is automatically non-terminal.*
Let $\underline{r} \in ord$ *be the smallest member of* $(ord, \le)$ *satisfying*
$\{y \in 0(x) : M(y) > \underline{r}\} \in B(x)$.

This implies $\forall r' \in ord$, *if* $r' < \underline{r}$ *then* $\{y \in 0(x) : M(y) > r'\} \in A(x)$. *Such an* $\underline{r} \in ord$ *will exist and* $\underline{r} \le ord(x)$ *since by induction,* $\forall y \in 0(x), M(y) \le ord(y)+1 \le ord(x)$ *which implies* $\{y \in 0(x) : M(y) > ord(x)\} = \phi \in B(x)$.

(c′) *We now define* $M(x) = \underline{r}+1$. *Since* $\underline{r} \le ord(x)$, *this implies* $M(x) \le ord(x)+1$.

Still assuming (C) x is unsafe (1), *we have the following:*

i' *Suppose $\underline{r}$ is a limit ordinal. Then $\underline{r}$ is even and* $M(x) = \underline{r}+1$ *is odd.*

ii' *Suppose $\underline{r}$ is not a limit ordinal. Then $\underline{r}-1$ is the immediate predecessor of $\underline{r}$. By the definition of $\underline{r}$,* $\{y \in 0(x) : M(y) > \underline{r}-1\} = \{y \in 0(x) : M(y) \ge \underline{r}\} \in A(x)$. *But since* $\{y \in ord(x) : M(y) > \underline{r}\} \in B(x)$, *this implies* $\exists \bar{y} \in 0(x)$ *such that* $M(\bar{y}) = \underline{r}$. *Now by induction,* $\forall \bar{y} \in 0(x), M(\bar{y})$ *is even. Therefore,* $\underline{r} = M(\bar{y})$ *is even and* $M(x) = \underline{r}+1$ *is odd.*

Notation. Given the abstract normal impartial games $G_i = (P_i, \le, T_i, A_i, B_i), i = 1, 2, \cdots, n$, let two players play Game A specified in the abstract. The positions in Game A are denoted $(p_1, p_2, \cdots, p_n)$ where $\forall i = 1, 2, \cdots, n, p_i \in P_i$.

Convention. *Suppose* $(p_1, p_2, \cdots, p_n)$ *is a position in Game A. Also, suppose some of the p_i's are safe* (0) *and some are unsafe* (1). *By symmetry we may suppose that* $p_1, p_2, \cdots, p_k$ *are unsafe and* $p_{k+1}, \cdots, p_n$ *are safe where* $1 \le k < n$. *Of course,* $\forall i = 1, 2, \cdots, n$ *we say that p_i is safe\unsafe if and only if p_i is safe\unsafe in G_i. Also,* $\forall i = 1, 2, \cdots, n$, *when we are dealing with* $G_i = (P_i, \le, T_i, A_i, B_i)$ *we will sometimes just call* $G_i = (P, \le, T, A, B)$ *since it will be obvious which game G_i we are dealing with.*

For the same reason, $\forall i = 1, 2, \cdots, n$*, we will call* $M_i(x) = M(x), 0_i(x) = 0(x)$ *and* $1_i(x) = 1(x)$.

Definition 5. Suppose $(p_1, p_2, \cdots, p_n)$ is a position in Game A. We say that the unsafe positions are dominant if (a) all p_i's are unsafe or (b) some p_i's are safe and some are unsafe and $\max\{M(p_1), M(p_2), \cdots, M(p_k)\} > \max\{M(p_{k+1}), \cdots, M(p_n)\}$. Note that $\forall i = 1, 2, \cdots, k, M(p_i)$ is odd and $\forall j = k+1, \cdots, n, M(p_j)$ is even. Therefore, from Lemma 3, $\forall i = 1, 2, \cdots, k, \forall j = k+1, \cdots n, M(p_i) \neq M(p_j)$. Also, we say that the safe positions are *dominant* if (a') all p_i's are safe or (b') some p_i's are safe and some are unsafe and $\max\{M(p_1), M(p_2), \cdots, M(p_k)\} < \max\{M(p_{k+1}), \cdots, M(p_n)\}$.

Lemma 5. Suppose $(p_1, p_2, \cdots, p_n)$ is a position in Game A and the unsafe positions are dominant. Then the moving player can move from $(p_1, p_2, \cdots, p_n) \to (\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_n)$ such that the safe positions of $(\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_n)$ are dominant.

Proof. If each $p_i, i = 1, 2, \cdots, n$, is unsafe, then $\forall i = 1, 2, \cdots, n$, the moving player can move from $p_i \to \bar{p}_i$ such that $\bar{p}_i$ is safe. Therefore, the safe positions of $(\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_n)$ are dominant. Remember, if p_i is unsafe in $G_i = (P_i, \leq, T_i, A_i, B_i)$, then $0_i(p_i) \in A_i(p_i)$, which implies the moving player can land inside of $0_i(p_i)$. Since we know that we are dealing with G_i, we can shorten this as $0(p_i) \in A(p_i)$ which implies the moving player can land inside of $0(p_i)$

Therefore, suppose $p_1, p_2, \cdots, p_k$ are unsafe, $p_{k+1}, \cdots, p_n$ are safe, where $1 \leq k < n$, and $\max\{M(p_1), \cdots, M(p_k)\} > \max\{M(p_{k+1}), \cdots, M(p_n)\}$. Now if some of the p_i's,$i = k+1, \cdots, n$, are terminal positions, these positions are already out of the game and also $M(p_i) = 0$ for these terminal positions. Therefore, the terminal positions can be ignored and there is no loss of generality in assuming that each $p_i, i = k+1, \cdots, n$, is both safe and non-terminal.

By symmetry let us further assume that $M(p_1) = \max\{M(p_1), \cdots, M(p_k)\}$ which implies $\forall i = k+1, \cdots, n, M(p_1) > M(p_i)$. First, $\forall i = 2, 3, \cdots, k$, the moving player moves from $p_i \to \bar{p}_i$ such that $\bar{p}_i$ is safe, and that is all he has to do with these $\bar{p}_i$'s.

Now $\forall i = k+1, \cdots, n$, if the opposing player allows the moving player to move from $p_i \to \bar{p}_i$ such that $\bar{p}_i$ is safe, this can only have the effect of possibly further increasing the dominance of the safe positions in $(\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_n)$. Therefore, we can assume that $\forall i = k+1, \cdots, n$, the moving player moves from $p_i \to \bar{p}_i$ such that $\bar{p}_i$ is unsafe. Using this assumption, we will now show that the moving player can move from $p_1 \to \bar{p}_1$, and $\forall i = k+1, \cdots, n$, move from $p_i \to \bar{p}_i$ such that (1) $\bar{p}_1$ is safe, (2) $\forall i = k+1, \cdots, n$, $\bar{p}_i$ is unsafe and (3) $\forall i = k+1, \cdots, n$, $M(\bar{p}_1) > M(\bar{p}_i)$. This means that the safe positions are dominant in the new position $(\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_n)$. Let us now focus on a fixed $p_i \in P_i, G_i = (P_i, \leq, T_i, A_i, B_i)$, where $i = k+1, \cdots, n$. Now $\forall i = k+1, \cdots, n$, the position p_i is safe. Therefore, $\forall i = k+1, \cdots, n$, $\exists \underline{t}_i \in ord$ such that (1) and (2) are true.

(1) If $\underline{t}_i$ is a limit ordinal, $M(p_i) = \underline{t}_i$. Also, from i. in the $M(x)$ definition, $0(p_i) \cup \{y \in 1(p_i) : M(y) < \underline{t}_i\} \in A(p_i)$.

(2) If $\underline{t}_i$ is not a limit ordinal, $M(p_i) = \underline{t}_i + 1$ and $0(p_i) \cup \{y \in 1(p_i) : M(y) \leq \underline{t}_i\} \in A(p_i)$.

Also, (1′) and (2′) are true.

(1′) If $\underline{t}_i$ is limit ordinal, the moving player can move from $p_i \to \bar{p}_i$ such that (by assumption) $\bar{p}_i$ is unsafe and $M(\bar{p}_i) < \underline{t}_i = M(p_i)$.

(2′) If $\underline{t}_i$ is not a limit ordinal, the moving player can move from $p_i \to \bar{p}_i$ such that (by assumption) $\bar{p}_i$ is unsafe and $M(\bar{p}_i) \leq \underline{t}_i < \underline{t}_i + 1 = M(p_i)$.

In both (1′) and (2′), $\forall i = k+1, \cdots, n$, the moving player can move from $p_i \to \bar{p}_i$ such that (by assumption) $\bar{p}_i$ is unsafe and $M(\bar{p}_i) < M(p_i)$.

We now consider p_1. However, we need to point out that since the moving player can move in any order, he will move in $G_{k+1}, G_{k+2}, \cdots, G_n$ before he moves in G_1.

Since p_1 is unsafe, $\exists\ \underline{r}_1 \in ord$ such that $M(p_1) = \underline{r}_1 + 1$. Now $M(p_1) = \underline{r}_1 + 1 > M(p_i), i = k+1, \cdots, n$. Therefore, $\underline{r}_1 \geq M(p_i) > M(\bar{p}_i), i = k+1, \cdots, n$.

Since $\underline{r}_1 > M(\bar{p}_i), i = k+1, \cdots, n$, we know that $\underline{r}_1 > r_1' = \max\{M(\bar{p}_i) : i = k+1, \cdots, n\}$. Therefore, since $r_1' < \underline{r}_1$, we know from the properties of $\underline{r}_1$ that $\{y \in 0(p_1) : M(y) > r_1'\} \in A(p_1)$. Therefore, the moving player can move from $p_1 \to \bar{p}_1$, such that $\bar{p}_1$ is safe and $M(\overline{p}_1) > r_1' \geq M(\overline{p}_i), i = k+1, \cdots, n$. That is, $\bar{p}_1$ is safe and $\forall i = k+1, \cdots, n$, $\overline{p}_i$ is unsafe and $M(\overline{p}_1) > M(\overline{p}_i)$. Therefore, the safe positions of $\overline{p}_1, \overline{p}_2, \cdots, \overline{p}_n$ are dominant. □

Lemma 6. Suppose $(p_1, p_2, \cdots, p_n)$ is a position in Game A and the safe positions are dominant. Then the opposing player can force the moving player to move from $(p_1, p_2, \cdots, p_n) \to (\overline{p}_1, \overline{p}_2, \cdots, \overline{p}_n)$ such that the unsafe positions of $(\overline{p}_1, \overline{p}_2, \cdots, \overline{p}_n)$ will be dominant.

Proof. If some of the p_i's are terminal, then these p_i's are already out of the game and $M(p_i) = 0$ for terminal positions. Therefore, there is no loss of generality in supposing that no p_i is both safe and terminal. Now if all p_i's are safe, then $\forall i = 1, 2, \cdots, n, 0_i(p_i) \in B_i(p_i)$, (which we write as $0(p_i) \in B(p_i)$), and the opposing player can force the moving player to move from $p_i \rightarrow \overline{p}_i$ such that $\overline{p}_i$ is unsafe. Therefore, the unsafe positions of $(\overline{p}_1, \overline{p}_2, \cdots, \overline{p}_n)$ are dominant.

Therefore, we suppose $p_1, p_2, \cdots, p_k$ are unsafe and $p_{k+1}, p_{k+1}, \cdots, p_n, 1 \leq k < n$, are safe and non-terminal.

Since the safe positions are dominant, we know that $\max\{M(p_1), \cdots, M(p_k)\}$ $< \max\{M(p_{k+1}), \cdots M(p_n)\}$. Now $\forall i = k+1, \cdots, n$, if $p_i \rightarrow \overline{p}_i$ is a move, the opposing player can force $\overline{p}_i$ to be unsafe since $0(p_i) \in B(p_i)$. Now $\forall i = 1, 2, \cdots, k$, if the moving player moves from $p_i \rightarrow \overline{p}_i$ such that $\overline{p}_i$ is unsafe, this can only have the effect of possibly further increasing the dominance of the unsafe positions of $(\overline{p}_1, \overline{p}_2, \cdots, \overline{p}_n)$. Therefore, we may assume that $\forall i = 1, 2, \cdots, k$, the moving player moves from $p_i \rightarrow \overline{p}_i$ such that $\overline{p}_i$ is safe.

By symmetry, we may suppose that $M(p_n) = \max\{M(p_{k+1}), \cdots, M(p_n)\}$.

Therefore, $\forall i = 1, 2, \cdots, k$, $M(p_i) < M(p_n)$. Since p_n is safe and non-terminal, (from the definition of the $M(x)$ function), there exists a unique $\underline{t}_n \in ord$ such that $(1')$ and $(2')$ are satisfied as well as the other properties specified in the definition of the $M(x)$ function.

$(1')$ If $\underline{t}_n$ is a limit ordinal, $M(p_n) = \underline{t}_n$, which means $M(p_n)$ is even.

$(2')$ If $\underline{t}_n$ is not a limit ordinal, then $\underline{t}_n$ will be odd and $M(p_n) = \underline{t}_n + 1$, which means $M(p_n)$ is very even.

We now prove $(**)$.

$(**)$ $\forall t'_n \in ord$, if $t'_n < M(p_n)$, then the opposing player can force a move $p_n \rightarrow \overline{p}_n$ such that $\overline{p}_n$ is unsafe and $M(\overline{p}_n) \geq t'_n$.

(a) First, suppose $\underline{t}_n$ is not a limit ordinal. Then $M(p_n) = \underline{t}_n + 1$ and $0(p_n) \cup \{y \in 1(p_n) : M(y) < \underline{t}_n\} = 0(p_n) \cup \{y \in 1(p_n) : M(y) \leq \underline{t}_n - 1\} \in B(p_n)$, since $\underline{t}_n - 1 < \underline{t}_n$. Therefore, the opposing player can force a move $p_n \rightarrow \overline{p}_n$ such that $\overline{p}_n$ is unsafe and $M(\overline{p}_n) \geq \underline{t}_n = M(p_n) - 1$, which proves $(**)$ since $M(p_n) - 1$ is the immediate predecessor of $M(p_n)$.

(b) If $\underline{t}_n$ is a limit ordinal, then $\forall t'_n \in ord$, if $t'_n < \underline{t}_n = M(p_n)$, then $0(p_n) \cup \{y \in 1(p_n) : M(y) \leq t'_n\} \in B(p_n)$. Therefore, if t'_n is an ordinal such that $t'_n < M(p_n) = \underline{t}_n$, then the opposing player can force a move $p_n \rightarrow \overline{p}_n$ such that $\overline{p}_n$ is unsafe and $M(\overline{p}_n) > t'_n$. Therefore, in both (a), (b), (**) is true.

$\forall i = 1, 2, \cdots, k$, let the moving player move from $p_i \rightarrow \overline{p}_i$ and suppose $\overline{p}_i$ is safe. Since $\forall i = 1, 2, \cdots, k$, p_i is unsafe, $M(p_i) = \underline{r}_i + 1$, where $\underline{r}_i \in ord$. Now $\forall i = 1, 2, \cdots, k, \{y \in 0(p_i) : M(y) > \underline{r}_i\} \in B(p_i)$. Therefore since we are assuming that $\forall i = 1, 2, \cdots, k$, $\overline{p_i}$ is safe, we know that $\forall i = 1, 2, \cdots, k$, the opposing player can force $M(\overline{p_i}) \leq \underline{r}_i = M(p_i) - 1$. That is, $\forall i = 1, 2, \cdots, k$, (assuming $\overline{p}_i$ is safe), the opposing player can force $M(\overline{p}_i) \leq M(p_i) - 1$. Now $\forall i = 1, 2, \cdots, k, M(p_i) < M(p_n)$. Therefore, $\forall i = 1, 2, \cdots, k$, the moving player can be forced to move $p_i \rightarrow \overline{p}_i$ such that $\overline{p}_i$ is safe (by assumption) and $M(\overline{p}_i) \leq M(p_i) - 1 < M(p_i) < M(p_n)$.

Now $\forall i = 1, 2, \cdots, k$, it is true that $M(\overline{p}_i) \leq M(p_i) - 1 < M(p_i) \leq \max\{M(p_1), \cdots, M(p_k)\} < M(p_n)$. From (**) the opposing player can force a move $p_n \rightarrow \overline{p}_n$ such that $\overline{p}_n$ is unsafe and $\max\{M(p_1), \cdots, M(p_k)\} \leq M(\overline{p}_n)$.

Therefore, $\forall i = 1, 2, \cdots, k, M(\overline{p}_i) < M(\overline{p}_n)$.

□

Main Theorem 1. *Suppose* $(p_1, p_2, \cdots, p_n)$ *is the initial position in Game A, and* $(p_1, p_2, \cdots, p_n)$ *is non-terminal. Then (1) the first moving player can win with perfect play if the unsafe positions of* $(p_1, p_2, \cdots, p_n)$ *dominate, and (2) the second moving player can win with perfect play if the safe positions of* $(p_1, p_2, \cdots, p_n)$ *dominate.*

Proof. Of course, the game can have only a finite number of moves. Also, observe that for any terminal position $(p_1, p_2, \cdots, p_n)$, the safe positions dominate since each p_i is safe in $G_i, i = 1, 2, \cdots, n$, when $(p_1, p_2, \cdots, p_n)$ is terminal. The player who is destined to win simply uses Lemmas 5, 6 over and over until he wins. □

Finding all concrete normal impartial games. As defined previously, a concrete normal impartial game can be thought of as a directed graph (V, E) with vertex set V and directed edge set E. Each vertex corresponds to a position in the game, and the directed edges correspond to the moves in the game. The followers of a vertex are those positions joined by an outgoing edge, and $\forall p, q \in V$, we denote $p \rightarrow q$ if there is a directed edge from p to q. A move in the game consists of going from p to any q such that $p \rightarrow q$.

The reader will note that (V, E) must be acyclic (i.e., have no directed cycles) since there can be no infinite sequence of moves.

We define a terminal position of (V, E) to be a position that has no followers. We must now find all (V, E) such that $\forall p_0 \in V$, there exist no infinite sequence $p_0 \to p_1 \to p_2 \to \cdots$, where each $p_i \in V$. Of course, as this paper demonstrated, this amount of information is often far more than is needed. See [3] for a derivation of the abstract version starting with very primitive assumptions.

Lemma 7. Suppose (V, E) represents a concrete normal impartial game. Suppose $\overline{V} \subseteq V$ and $\left(\overline{V}, E\right)$ denotes the graph $(V\ E)$ that is restricted to the vertex set $\overline{V}$. That is, $\forall p, q \in \overline{V}, p \to q$ in $\left(\overline{V}, E\right)$ if and only if $p \to q$ in (V, E). Then $\left(\overline{V}, E\right)$ has at least one terminal position.

Proof. Suppose $\left(\overline{V}, E\right)$ has no terminal positions. Then $\forall p_0 \in \overline{V}$, there exists an infinite sequence of moves $p_0 \to p_1 \to p_2 \to \cdots$ where each $p_i \in \overline{V} \subseteq V$, a contradiction. □

Theorem 2. Suppose (V, E) represents a concrete normal impartial game. Then $\exists$ a well-ordering on V, call it $(V, \leq)$, such that $\forall p, q \in V$, if $p \to q$ in (V, E) then $p > q$ in $(V, \leq)$, where $p > q$ means $p \geq q$ and $p \neq q$.

Remark From Theorem 2, we see that the model given in Concrete Normal Impartial Games 1 specifies all concrete normal impartial games.

Proof of Theorem 2. We first use transfinite induction on $(ord, \leq)$ to begin our well-ordering on V. Since ord is a proper class, it is bigger than any set since any set can be well-ordered and ord contains all ordinal numbers. Let us define the sets $T_x, x \in ord$, as follows.

(1) T_0 is the set of all terminal positions of (V, E). T_0 is not empty.

(2) Suppose we have specified T_y for all $y \in [0, x)$, and suppose $\forall y \in [0, x), T_y$ is non-empty. Then T_x is specified as follows.

   (a) If $V = \bigcup_{y \in [0,x)} T_y$, the transfinite induction on $(ord, \leq)$ stops at x, and we do not define T_x.

   (b) If $V \backslash \bigcup_{y \in [0,x)} T_y$ is non-empty, we define T_x to be the set of all terminal positions of $\left(V \backslash \bigcup_{y \in [0,x)} T_y, E\right)$, the restriction of (V, E) to the vertex set $\overline{V} =$

$V \backslash \bigcup_{y \in [0,x)} T_y$. By Lemma 7, T_x is non-empty.

Since each T_x that we specify will be non-empty, since $T_x \cap T_y = \phi$ when $x > y$, and since ord is bigger than any set V, eventually the transfinite induction on $(ord, \leq)$ must come to an end. This means that $\exists a \in ord$ such that $\forall x \in [0,a)\,, T_x$ is non-empty and $\bigcup_{y \in [0,a)} T_x = V$.

Of course, the sets $T_x, x \in [0,a)$, are well-ordered by $(ord, \leq)$. That is, $\forall x, y \in [0,a)$, we say that $T_x \geq T_y$ if and only if $x \geq y$ in $(ord, \leq)\,$.

Now $\forall x \in [0,a)\,, \forall p, q \in T_x$, it is obvious that $p \nrightarrow q, q \nrightarrow p$ in (V,E) since T_x is the set of terminal positions of (V,E) restricted to $V \backslash \bigcup_{y \in [0,x)} T_y$.

Suppose $x, y \in [0,a)$ and $x > y$. Also, suppose $p \in T_x, q \in T_y$. Since q is a terminal position of $V \backslash \bigcup_{i \in [0,y)} T_i$ and since $p \in T_x \subseteq V \backslash \bigcup_{i \in [0,y)} T_i$, it is obvious that $q \rightarrow p$ is impossible in $(V,E)\,$.

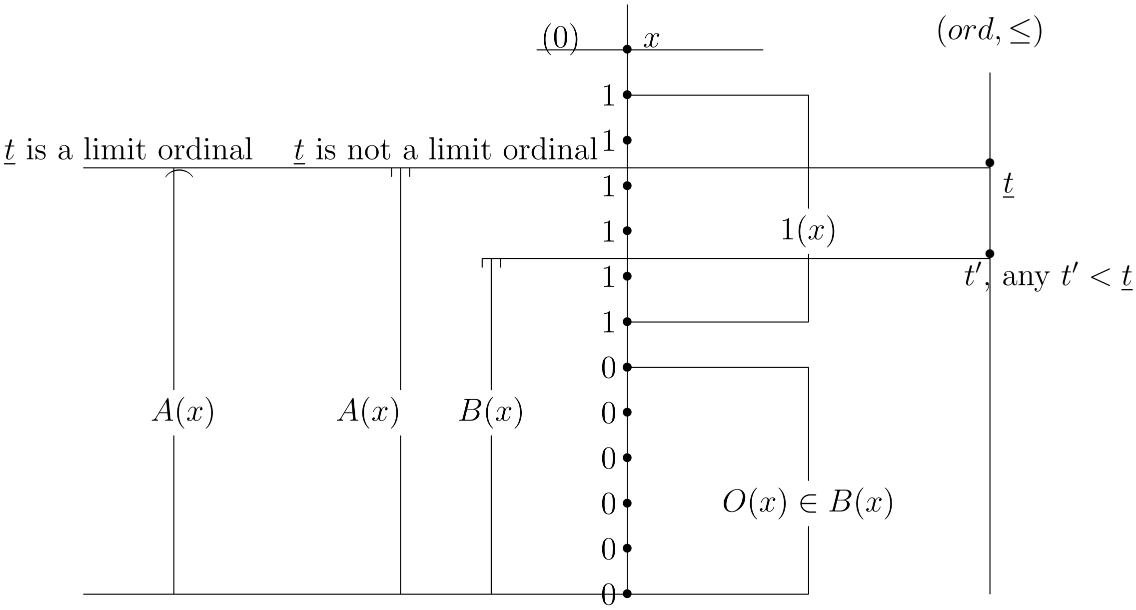
Let us now well-order each T_x, where $x \in [0,a)$, in any arbitrary way. Let us now well-order $V = \bigcup_{i \in [0,a)} T_i$ in the following standard lexicographical way, and let us call this lexicographical ordering on $V, (V, \leq)\,$.

(a) $\forall p, q \in V$, if p, q are members of the same T_x, we order p, q in $(V, \leq)$ by the well ordering that we have defined on T_x.

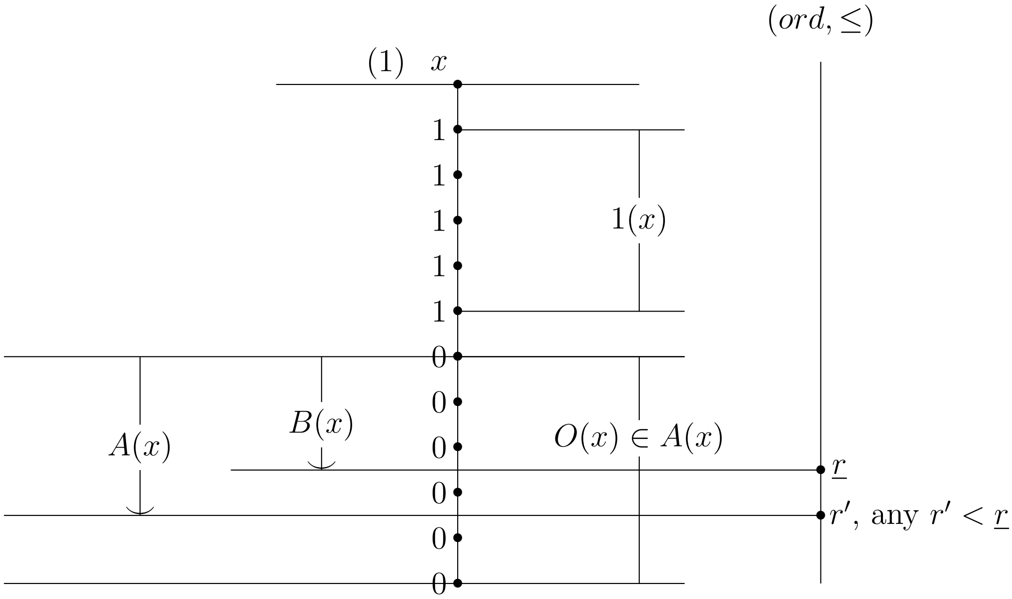
(b) $\forall p, q \in V$, if $p \in T_x, q \in T_y$ and $x > y$ in $(ord, \leq)$, then $p > q$ in $(V, \leq)\,$.

It is easy to see that this $(V, \leq)$ is a well-ordering. Also, it is easy to see that if $p \rightarrow q$ in (V,E), then $p > q$ in $(V, \leq)$. This shows that the model given in Concrete Normal Impartial Games 1 classifies all concrete normal impartial games. □

Appendix.



If $\underline{t}$ is not a limit ordinal, $M(x) = \underline{t} + 1$ (very even). If $\underline{t}$ is a limit ordinal, $M(x) = \underline{t}$ (even).



$M(x) = \underline{r} + 1$ (odd)

Fig 1 A suggestive drawing of $M(x)$

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