The Theory of Linear Fractional Transformations of Rational Quadratics

Arthur Holshouser
3600 Bullard St.
Charlotte, NC,
USA
Harold Reiter
Department of Mathematical Sciences
University of North Carolina Charlotte,
Charlotte, NC 28223, USA
hbreiter@uncc.edu

1 Abstract

A standard technique for solving the recursion $x_{n+1} = g(x_n)$ where $g : \mathbf{C} \to \mathbf{C}$ is a complex function is to first find a fairly simple function $\overline{g} : \mathbf{C} \to \mathbf{C}$ and a bijection $f : \mathbf{C} \to \mathbf{C}$ such that $g = f \circ \overline{g} \circ f^{-1}$ where $\circ$ is the composition of functions. Then $x_n = g^n(x_0) = (f \circ \overline{g}^n \circ f^{-1})(x_0)$ where g^n and $\overline{g}^n$ are the n-fold composition of functions and $\overline{g}^n$ is fairly easy to compute. With this motivation we find all pairs of rational quadratic functions $g, \overline{g}$ such that for some $a, b, c, d \in \mathbf{C}$, $g = \frac{ax+b}{cx+d} \circ \overline{g} \circ \left(\frac{ax+b}{cx+d}\right)^{-1}$ where a, b, c, d satisfy $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$. We denote such pairs by $g \sim \overline{g}$ and we show that $g \sim \overline{g}$ if and only if g and $\overline{g}$ have the same *signature*. The signature is defined as the ordered pair (θ, ϕ) of the two invariants that rational quadratic functions have under the above linear fractional transformation. These invariants were heuristically derived and computer proved. We will explain this heuristic thinking.

At the end, we apply this to study some specific examples, and we again mention the recursion $x_{n+1} = g(x_n)$. Also, we mention the possible extensions to higher degree rational functions.

2 Introduction

In this paper we use the following terminology. A rational quadratic function (called a *RQ*) is a complex function of the form $\frac{Ax^2+Bx+C}{Hx^2+Dx+E}$ where (1) $A, B, C, H, D, E \in \mathbf{C}$, (2) $A \neq 0$ or $H \neq 0$, (3) $(A, B, C) \neq (0,0,0)$ and $(H, D, E) \neq (0,0,0)$ and (4) $Ax^2 + Bx + C$ and $Hx^2 + Dx + E$ have no roots in common. Thus, $\frac{3x^2+2x+1}{x^2+x+1}, x^2 + 4x + 2, \frac{x+1}{x^2+2x+3}, \frac{1}{2x^2+3x+2}$ are *RQ*'s. It is a standard lemma that if g is a *RQ* and $f = \frac{ax+b}{cx+d}$ where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, then $f \circ g$ and $g \circ f$ are *RQ*'s and this fact is used often in the paper

In general two functions $g : \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}, \overline{g} : \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}$, where $\mathbf{C}$ is the complex numbers, are *similar* if there exists a bijection $f : \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}$ such that $g = f \circ \overline{g} \circ f^{-1}$ where $\circ$ is the composition of functions.

Two functions $g : \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}, \overline{g} : \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}$ are *linearly similar* (denoted by $g \sim \overline{g}$) if there exists a (complex) linear fractional transformation

$f = \frac{ax+b}{cx+d}, \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, such that $g = f \circ \overline{g} \circ f^{-1}$. Also, two functions $g : \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}, \overline{g} : \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}$ are *very linearly similar* (denoted by $g \approx \overline{g}$) if there exists a (complex) linear transformation $f = ax + b, a \neq 0$, such that $g = f \circ \overline{g} \circ f^{-1}$. Of course, both $\sim$ and $\approx$ are equivalence relations.

Theorem 3 states necessary and sufficient conditions so that two (normal) rational quadratics g and $\overline{g}$ are linearly similar.

The two invariants that are used in theorem 3 were heuristically derived and were proved by a computer. We know of no way to derive these invariants by mathematical logic, and we know of no way to prove them without using a computer. The same heuristic thinking generates at least a few analogous invariants for higher degree rational functions, and these invariants would soon be unprovable by any computer. So the unknown at least for us far exceeds what we know. The good news for the reader is that the invariants are powerful enough to make this paper a rather easy paper to read.

3 Computing the Linear Fractional Transformations of RQ's

It is convenient to consider two cases for the linear fractional transformation of rational quadratics (RQ's).

Case (a). $\frac{ax+b}{x+d} \circ \frac{Ax^2+Bx+C}{Hx^2+Dx+E} \circ \frac{b-dx}{x-a}$ where $\begin{vmatrix} a & b \\ 1 & d \end{vmatrix} \neq 0$ and $\left(\frac{ax+b}{x+d}\right)^{-1} = \frac{b-dx}{x-a}$.

Case (b). $\left(\frac{x}{a} - \frac{b}{a}\right) \circ \frac{Ax^2+Bx+C}{Hx^2+Dx+E} \circ (ax+b)$ where $a \neq 0$ and $(ax+b)^{-1} = \frac{x}{a} - \frac{b}{a}$.

We also observe that $ax + b = \lim_{t\to\infty} \frac{atx+bt}{x+t}$.

Case (a). By straight forward calculations we compute $\frac{ax+b}{x+d} \circ \frac{Ax^2+Bx+C}{Hx^2+Dx+E} \circ \frac{b-dx}{x-a} = \frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{H}x^2+\overline{D}x+\overline{E}}$ where $\overline{A}, \overline{B}, \overline{C}, \overline{H}, \overline{D}, \overline{E}$ are defined as follows. Also, from $\overline{A}$ and $\overline{D}$ we define the functions $F, G, \overline{F}, \overline{G}$.

$\overline{A} = a\left[Ad^2 - Bd + C\right] + b\left[Hd^2 - Dd + E\right] = a\overline{F}(d) + b\overline{G}(d)$.

$\overline{B} = a^2\left[Bd - 2C\right] + ab\left[(D - 2A)d + B - 2E\right] + b^2\left[-2Hd + D\right]$.

$\overline{C} = a^3C + a^2b\left[E - B\right] + ab^2\left[A - D\right] + b^3H$.

$\overline{H} = Hd^3 + \left[A - D\right]d^2 + \left[E - B\right]d + C$.

$\overline{D} = a\left[Dd^2 + (B - 2E)d - 2C\right] + b\left[-2Hd^2 + (D - 2A)d + B\right] = aF(d) + bG(d)$.

$\overline{E} = a^2\left[Ed + C\right] - ab\left[Dd + B\right] + b^2\left[Hd + A\right]$.

Observe that we are defining $\overline{A} = a\overline{F}(d) + b\overline{G}(d)$ and $\overline{D} = aF(d) + bG(d)$. Also, observe that $\overline{H}(d)$ is a function of d only.

The following lemma for case a is easy to prove.

Lemma 1. (1) $F(d) + dG(d) = -2\overline{H}(d)$, (2) $\overline{F}(d) + d\overline{G}(d) = \overline{H}(d)$.

Case (b). $\left(\frac{x}{a} - \frac{b}{a}\right) \circ \frac{Ax^2+Bx+C}{Hx^2+Dx+E} \circ (ax+b) = \frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{H}x^2+\overline{D}x+\overline{E}}$

$= \frac{\left[Aa^2-Ha^2b\right]x^2+\left[Ba+(2A-D)ab-2Hab^2\right]x+\left[-Hb^3+(A-D)b^2+(B-E)b+C\right]}{Ha^3x^2+\left[2Ha^2b+Da^2\right]x+\left[Hab^2+Dab+Ea\right]}$.

4 Two Invariants of RQ's Under Linear Fractional Transformations

We proceed by analogy with the following.

By calculation we see that

$$\begin{aligned}\frac{ax+b}{x+d}\circ\frac{Ax+B}{Cx+D}\circ\frac{b-dx}{x-a} &= \frac{\overline{A}x+\overline{B}}{\overline{C}x+\overline{D}} \\ &= \frac{[Ba-Aad+Db-Cbd]\,x+\left[Aab-Ba^2+Cb^2-Dab\right]}{\left[B-Ad+Dd-Cd^2\right]x+[Ab-Ba+Cbd-Dad]}.\end{aligned}$$

An invariant of $\frac{Ax+B}{Cx+D}$ under this linear fractional transformation is $\frac{A^2+2BC+D^2}{AD-BC}$ and this can be easily verified by hand and by showing that $\frac{A^2+2BC+D^2}{AD-BC}=\frac{\overline{A}^2+2\overline{B}\overline{C}+\overline{D}^2}{\overline{A}\overline{D}-\overline{B}\overline{C}}$. This invariant is also equivalent to the invariant $\frac{A^2+2BC+D^2}{AD-BC}-2=\frac{(A-D)^2+4BC}{AD-BC}$.

The last invariant equals the discriminant of $\overline{C}\,(d)=-Cd^2+(D-A)\,d+B$ divided by $\rho\,(Ax+B,Cx+D)=\begin{vmatrix} A & B \\ C & D\end{vmatrix}$ where $\rho\,(Ax+B,Cx+D)$ is the *resultant* of the two polynomials $Ax+B,Cx+D$. We will use the analogy of this observation to deal with rational quadratics. In general the resultant $\rho\,(P,Q)$ of two polynomials P,Q is a standard determinant, p.99, [1], which gives by its zero or non-zero value the necessary and sufficient conditions so that P and Q have no common roots.

Also, the *discriminant* of a polynomial P is $\rho\,(P,P')$ and it is the determinant that likewise gives the necessary and sufficient conditions so that P has no repeated roots.

We now proceed totally by analogy to heuristically find two invariants for rational quadratics under the linear fractional transformation $g=\frac{ax+b}{cx+d}\circ\overline{g}\circ\left(\frac{ax+b}{cx+d}\right)^{-1}$.

It is possible to use any of the six case a, section 3 definitions $\overline{A},\overline{B},\overline{C},\overline{H},\overline{D},\overline{E}$ in an analogous way to derive these two invariants.

However, we will use only $\overline{H}\,(d)$ and $\overline{D}=aF\,(d)+bG\,(d)$ in heuristically deriving these two invariants since it is easy for us to explain what they mean. Using $\overline{A},\overline{B},\overline{C},\overline{E}$ will likewise lead to these same two invariants.

Suppose that $H\neq 0$. We observe that a root d of $\overline{H}\,(d)$ must be used in $\frac{ax+b}{x+d}$ if we wish to transform $\frac{Ax^2+Bx+C}{Hx^2+Dx+E}\sim\frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{D}x+\overline{E}}$ by case a. This is because a root d of $\overline{H}\,(d)$ will force $\overline{H}\,(d)=0$. Thus, the information on whether any of the roots of $\overline{H}\,(d)$ repeat is probably invariant information under linear fractional transformation and we need to look at this. Also, the fact that Ax^2+Bx+C and Hx^2+Dx+E have no roots in common is probably another piece of invariant information that we need to look at.

Finally, if $F\,(d)$ and $G\,(d)$ have any root d in common, then since $-2\overline{H}\,(d)=F\,(d)+dG\,(d)$ we know that $\overline{H}\,(d)=0$. Thus, we can use this common root of $F\,(d)\,,G\,(d)\,,H\,(d)$ to transform $\frac{Ax^2+Bx+C}{Hx^2+Dx+E}\sim\frac{\overline{A}}{\overline{E}}x^2+\frac{\overline{B}}{\overline{E}}x+\frac{\overline{C}}{\overline{E}}$ which is a polynomial.

Therefore, we also need to look at whether $F\,(d)$ and $G\,(d)$ have any root d in common. We heuristically believe that the above information is the glue that holds the transformation invariants together.

We now use this information to conjecture two invariants for rational quadratics under linear fractional transformation.

We define invariant 1: $\theta = \frac{\rho\left(\overline{H}(d), \overline{H}'(d)\right)}{\rho(Ax^2+Bx+C, Hx^2+Dx+E)}$ and invariant 2: $\phi = \frac{\rho(F(d), G(d))}{\rho(Ax^2+Bx+C, Hx^2+Dx+E)}$.

Now $\rho\left(Ax^2+Bx+C, Hx^2+Dx+E\right) = \begin{vmatrix} A & B & C & 0 \\ 0 & A & B & C \\ H & D & E & 0 \\ 0 & H & D & E \end{vmatrix}$ $= (AE - CH)^2 + (BE - CD)(BH - AD)$. See p.99, [1]. When $(A, H) \neq (0,0)$ this resultant will always give the necessary and sufficient conditions so that Ax^2+Bx+C and Hx^2+Dx+E have no common roots. This includes all degenerate cases including the extreme case where one of $(A, B) = (0,0)$, or $(H, D) = (0,0)$.

Using this formula we can now easily compute

$$\begin{aligned} \rho\left(F(d), G(d)\right) &= \rho\left(Dd^2 + (B - 2E)d - 2C, -2Hd^2 + (D - 2A)d + B\right) \\ &= (BD - 4CH)^2 \\ &\quad + \left[B^2 - 2BE + 2CD - 4AC\right]\left[-D^2 + 2AD - 2BH + 4EH\right]. \end{aligned}$$

Now the standard discriminant $\rho\left(P(x), P'(x)\right)$ of the cubic $P(x) = a_0x^3 + a_1x^2 + a_2x + a_3$ is $\rho(P, P') = -27a_0^2a_3^2 + 18a_0a_1a_2a_3 - 4a_1^3a_3 - 4a_0a_2^3 + a_1^2a_2^2$. See p.117, [1].

Since $\overline{H}(d) = Hd^3 + (A - D)d^2 + (E - B)d + C$ we have $\rho\left(\overline{H}(d), \overline{H}'(d)\right) = -27H^2C^2 + 18H(A - D)(E - B)C - 4(A - D)^3C - 4H(E - B)^3 + (A - D)^2(E - B)^2$.

Therefore, we are conjecturing that the following are two invariants of the given RQ under linear fractional transformation.

Invariant 1:

$$\theta = \frac{-27H^2C^2 + 18H(A - D)(E - B)C - 4(A - D)^3C - 4H(E - B)^3 + (A - D)^2(E - B)^2}{(AE - CH)^2 + (BE - CD)(BH - AD)}.$$

Invariant 2:

$$\phi = \frac{(BD - 4CH)^2 + [B^2 - 2BE + 2CD - 4AC][-D^2 + 2AD - 2BH + 4EH]}{(AE - CD)^2 + (BE - CD)(BH - AD)}.$$

Therefore, θ and ϕ must remain unchanged when we substitute $\overline{A}, \overline{B}, \ldots, \overline{H}$ for $A, B, \ldots, H$ using either case a or case b of section 3.

Prof. Ben Klein, Davidson College, Davidson, N.C., has verified both of these invariants using the Mathematica software.

Also, he has verified that in case a the common factor involving a, b, d that appears in both the numerator and denominator of each of the two invariants is $\begin{vmatrix} a & b \\ 1 & d \end{vmatrix}^6$. Also, we observe that case b of section 3 can be considered a special case of case a since $ax + b = \lim_{t\to\infty} \frac{atx+bt}{x+t}$.

Since $\begin{vmatrix} a & b \\ 1 & d \end{vmatrix} \neq 0$ and $\begin{vmatrix} a & b \\ 1 & d \end{vmatrix}^6$ is cancelled out of both the numerator and denominator of θ and ϕ, we see that it is impossible for a degeneracy $\frac{0}{0}$ to ever appear in θ or ϕ as we vary a, b, d in case a subject to $\begin{vmatrix} a & b \\ 1 & d \end{vmatrix} \neq 0$. It is also true in case b.

(Note also the following information).

Since we assume that Ax^2+Bx+C and Hx^2+Dx+E have no roots in common, it will always be true that $\overline{A}x^2+\overline{B}x+\overline{C}$ and $\overline{H}x^2+\overline{D}x+\overline{E}$ have no roots in common when $\frac{Ax^2+Bx+C}{Hx^2+Dx+E} \sim \frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{H}x^2+\overline{D}x+\overline{E}}$.

Therefore, since $(A,H) \neq (0,0)\left(\overline{A},\overline{H}\right) \neq (0,0)$, we know that $\rho\left(Ax^2+Bx+C, Hx^2+Dx+E\right) \neq 0$ and $\rho\left(\overline{A}x^2+\overline{B}x+\overline{C}, \overline{H}x^2+\overline{D}x+\overline{E}\right) \neq 0$. Therefore, the denominator of both θ and ϕ is never 0 as we vary a,b,d in case a subject to $\begin{vmatrix} a & b \\ 1 & d \end{vmatrix} \neq 0$, and this fact will include all degenerate cases. It is also true in case b. For simplicity in this paper we will often assume that $\theta \neq 0$ and we call the RQ normal.

But $\theta \neq 0$ implies in case (a) that $\rho\left(\overline{H}(d), \overline{H}'(d)\right) \neq 0$. Also, from the definition of θ this implies that it is impossible for both $H=0$ and $A-D=0$. Thus, in case (a), $\overline{H}(d) = Hd^3+(A-D)d^2+(E-B)d+C$ is a polynomial of at least degree 2.

Also, if $\overline{H}(d)$ is of degree 2 and $\theta \neq 0$, then $\rho\left(\overline{H}(d), \overline{H}'(d)\right) = -4(A-D)^3C + (A-D)^2(E-B)^2 = (A-D)^2\left[(E-B)^2 - 4(A-D)C\right] \neq 0$ and $(E-B)^2-4(A-D)C \neq 0$. This implies that the two roots of $\overline{H}(d)$ are unequal. In general the two invariants θ,ϕ provide the same information for all degenerate cases.

Definition 1. If $\frac{Ax^2+Bx+C}{Hx^2+Dx+E}$ is a rational quadratic, then the *signature* of this RQ is (θ,ϕ).

Definition 2. A rational quadratic $Q(x)$ is said to be normal if the signature (θ,ϕ) of $Q(x)$ satisfies $\theta \neq 0$.

The rest of this paper is devoted mainly to proving theorems 1,2,3.

Theorem 1. A rational quadratic Q having a signature (θ,ϕ) is linearly similar to a rational quadratic polynomial P if and only if $(\theta,\phi) = (\theta,0)$.

Theorem 2. Any normal rational quadratic Q is linearly similar to a normal rational quadratic $\overline{Q}$ of the form $\overline{Q}(x) = \overline{A}x + \overline{B} + \frac{1}{x}$ where $\overline{A} \neq 0$.

Theorem 3. Two normal rational quadratics Q, Q' are linearly similar if and only if Q and Q' have the same signature $(\theta,\phi), \theta \neq 0$.

5 Proving Theorem 1 and Related Lemmas

Lemma 2. The signature of the rational quadratic polynomial $Ax^2+Bx+C, A \neq 0$, is $(\theta,\phi) = \left((B-1)^2 - 4AC, 0\right)$.

Proof. Using $H=D=0, E=1$ we have

$$\begin{aligned} \theta &= \frac{-4A^3C + A^2(1-B)^2}{A^2} \\ &= -4AC + (1-B)^2. \end{aligned}$$

Also, $\phi = \frac{0}{A^2} = 0$. ■

Theorem 1. Suppose $\frac{Ax^2+Bx+C}{Hx^2+Dx+E}$ is a RQ having a signature (θ,ϕ).

Then $\exists$ a rational quadratic polynomial $\overline{A}x^2+\overline{B}x+\overline{C}, \overline{A} \neq 0$, such that $\frac{Ax^2+Bx+C}{Hx^2+Dx+E} \sim \overline{A}x^2+\overline{B}x+\overline{C}$ if and only if $(\theta,\phi) = (\theta,0)$.

Proof. Since the signature (θ, ϕ) of a RQ is invariant under linear fractional transformation, from Lemma 2, $\phi = 0$ is necessary for a RQ to be linearly similar to a rational quadratic polynomial. Therefore, suppose $\phi = 0$ for $\frac{Ax^2+Bx+C}{Hx^2+Dx+E}$.

We may assume $(H, D) \neq (0, 0)$. Since the denominator of ϕ is never 0 and since the numerator of ϕ is $\rho(F(d), G(d))$, we see that $\phi = 0$ if and only if $\rho(F(d), G(d)) = 0$ where $F(d), G(d)$ are defined in case a, section 3, as $\overline{D} = aF(d) + bG(d)$. Since $(H, D) \neq (0, 0)$ we know that $\rho(F(d), G(d)) = 0$ implies that $F(d), G(d)$ have a common root d. From Lemma 1, $F(d) + dG(d) = -2\overline{H}(d)$. Therefore, $F(d) = G(d) = \overline{H}(d) = 0$. Using this d in case a of section 3 with $\frac{ax+b}{x+d}$ where $\begin{vmatrix} a & b \\ 1 & d \end{vmatrix} \neq 0$, we have $\frac{Ax^2+Bx+C}{Hx^2+Dx+E} \sim \frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{H}x^2+\overline{D}x+\overline{E}} = \frac{\overline{A}x^2+\overline{B}x+\overline{C}}{0x^2+0x+\overline{E}} = \frac{\overline{A}}{\overline{E}}x^2 + \frac{\overline{B}}{\overline{E}}x + \frac{\overline{C}}{\overline{E}}$.

Of course, $\overline{A} \neq 0, \overline{E} \neq 0$ since $\frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{E}}$ must be a RQ. Also, note that $\overline{H} = \overline{D} = 0, \overline{A} \cdot \overline{E} = 0$ implies $\left(\overline{AE} - \overline{CH}\right)^2 + \left(\overline{BE} - \overline{CD}\right)\left(\overline{BH} - \overline{AD}\right) = 0$, which is impossible. ■

Lemma 3. Two rational quadratic polynomials $Ax^2 + Bx + C$ and $\overline{A}x^2 + \overline{B}x + \overline{C}$ are very linearly similar (i.e. $Ax^2 + Bx + C \approx \overline{A}x^2 + \overline{B}x + \overline{C}$) if and only if they have the same signature $\left((B-1)^2 - 4AC, 0\right) = \left(\left(\overline{B}-1\right)^2 - 4\overline{AC}, 0\right)$.

Note 1. Of course, this implies that two RQ polynomials $P, \overline{P}$ are linearly similar if and only if they have the same signature. Thus, two RQ polynomials $P, \overline{P}$ are linearly similar if and only if they are very linearly similar.

Proof of Lemma 3. Let us assume that $Ax^2+Bx+C, A \neq 0$, and $\overline{A}x^2+\overline{B}x+\overline{C}, \overline{A} \neq 0$, have the same signature $(\theta, \phi) = (\theta, 0)$.

Using case b, section 3 with $ax + b, a \neq 0$, we have

$$\begin{aligned} Ax^2 + Bx + C &= \frac{Ax^2 + Bx + C}{0x^2 + 0x + 1} \sim \frac{Aa^2x^2 + (Ba + 2Aab)x + Ab^2 + (B-1)b + C}{0 \cdot x^2 + 0 \cdot x + 1 \cdot a} \\ &= Aax^2 + (B + 2Ab)x + \frac{Ab^2 + (B-1)b + C}{a}. \end{aligned}$$

Now, $Aa = \overline{A}, B+2Ab = \overline{B}$ defines $a \neq 0$ and b. Since the signature $(\theta, \phi) = (\theta, 0)$ is invariant under linear fractional transformation and since $A \neq 0$, $\overline{A} \neq 0$ we know that it is automatically true that $\frac{Ab^2+(B-1)b+C}{a} = \overline{C}$ since $\overline{A}, \overline{B}$ and the signature $(\theta, \phi) = \left(\left(\overline{B}-1\right)^2 - 4\overline{AC}, 0\right)$ of $\overline{A}x^2 + \overline{B}x + \overline{C}$ will uniquely determine $\overline{C}$. ■

Corollary 1. Suppose $\frac{Ax^2+Bx+C}{Hx^2+Dx+E}$ and $\frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{H}x^2+\overline{D}x+\overline{E}}$ are two RQ's having the same signature $(\theta, \phi) = (\theta, 0)$. Then $\frac{Ax^2+Bx+C}{Hx^2+Dx+E} \sim \frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{H}x^2+\overline{D}x+\overline{E}}$.

Proof. Follows from Theorem 1, Lemma 3, Note 1 and the fact that $\sim$ is an equivalence relation. ■

6 Computing the signature of $\frac{\overline{A}x^2+\overline{B}x+1}{x}, \overline{A} \neq 0$.

We first compute the signature (θ, ϕ) of the RQ $\frac{\overline{A}x^2+\overline{B}x+1}{x}, \overline{A} \neq 0$.

Then we compute all of the normal RQ's $\frac{\overline{A}x^2+\overline{B}x+1}{x}, \overline{A} \neq 0$, that have a given signature (θ, ϕ), (where normality means that $\theta \neq 0$).

To compute the signature of $\frac{\overline{A}x^2+\overline{B}x+1}{x}$ we use $\overline{H}=\overline{E}=0, \overline{C}=\overline{D}=1$. Therefore, $\theta=\frac{-4\left(\overline{A}-1\right)^3+\left(\overline{A}-1\right)^2\overline{B}^2}{\overline{A}}$.

Also, $\phi=\frac{\overline{B}^2+\left[\overline{B}^2+2-4\overline{A}\right]\left[-1+2\overline{A}\right]}{\overline{A}}=\frac{2\overline{A}\overline{B}^2-2\left(2\overline{A}-1\right)^2}{\overline{A}}$.

We now compute all normal $\frac{\overline{A}x^2+\overline{B}x+1}{x}, \overline{A}\neq 0$, that have a given $(\theta,\phi), \theta\neq 0$. Since $\theta\neq 0$, we see that $\overline{A}\neq 1$. Therefore, using the above formulas for θ and ϕ we have

$$(*) \qquad \overline{B}^2=\frac{\theta\overline{A}+4\left(\overline{A}-1\right)^3}{\left(\overline{A}-1\right)^2}=\frac{\phi\overline{A}+2\left(2\overline{A}-1\right)^2}{2\overline{A}}.$$

Also,

$$(**) \qquad \phi\overline{A}^3-2\left(\theta+\phi-1\right)\overline{A}^2+\left(\phi-4\right)\overline{A}+2=0.$$

If $(\theta,\phi), \theta\neq 0$, is given them $\overline{A}, \overline{B}$ is uniquely computed from $(*)$ and $(**)$. Note that $(**)$ follows from $(*)$ since $\overline{A}\neq 0, \overline{A}\neq 1$.

Also, $\overline{A}=0$ and $\overline{A}=1$ do not solve $(**)$ since $\theta\neq 0$,

Lemma 4. If the signature (θ,ϕ) of the normal RQ $\frac{Ax^2+Bx+C}{Hx^2+Dx+E}$ satisfies $\theta\neq 0, \phi=0$, then $\frac{Ax^2+Bx+C}{Hx^2+Dx+E}\sim\frac{\overline{A}x^2+\overline{B}x+1}{x}$ where $\overline{A}$, satisfies $\left(\theta-1\right)\overline{A}^2+2\overline{A}-1=0$ and $\overline{B}=\pm\frac{\left(2\overline{A}-1\right)}{\sqrt{\overline{A}}}$.

Proof. From $(*)$, $(**)$ we know that $\frac{\overline{A}x^2+\overline{B}x+1}{x}$ has the same signature $(\theta,\phi), \theta\neq 0, \phi=0$ as the given RQ $\frac{Ax^2+Bx+C}{Hx^2+Dx+E}$.

Therefore, from Corollary 1, $\frac{Ax^2+Bx+C}{Hx^2+Dx+E}\sim\frac{\overline{A}x^2+\overline{B}x+1}{x}$. ■

7 Proving Theorem 2

Since Lemma 4 took care of Theorem 2 for the easy case where (θ,ϕ) satisfies $\theta\neq 0, \phi=0$, we now assume that $\theta\neq 0, \phi\neq 0$.

Lemma 5. If the signature (θ,ϕ) of the normal RQ $\frac{Ax^2+Bx+C}{Dx+E}$, where $A\neq 0$, satisfies $\theta\neq 0, \phi\neq 0$ then $D\neq 0$ and $\exists B^*\in C$ such that $\frac{Ax^2+Bx+C}{Dx+E}\sim\frac{A}{D}x+B^*+\frac{1}{x}$.

Proof. Obviously $D\neq 0$ since the signature (θ,ϕ) of $\frac{Ax^2+Bx+C}{E}$ would be $(\theta,\phi)=(\theta,0)$.

We now use case b of section 3 (where $a\neq 0$) with $H=0$. Therefore, $\frac{Ax^2+Bx+C}{Dx+E}\sim\frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{D}x+\overline{E}}$ where $\overline{A}=Aa^2, \overline{D}=Da^2, \overline{E}=a\left(Db+E\right)$. Since $D\neq 0$ we can define b so that $Db+E=0$ which gives $\frac{Ax^2+Bx+C}{Dx+E}\sim\frac{Aa^2x^2+\overline{B}x+\overline{C}}{Da^2x}$.

Since $\frac{Aa^2x^2+\overline{B}x+\overline{C}}{Da^2x}$ is a RQ we see that $\overline{C}\neq 0$. Therefore, since $D\neq 0, \overline{C}\neq 0$ we can define a so that $Da^2=\overline{C}$ where $a\neq 0$. Therefore, $\frac{Ax^2+Bx+C}{Dx+E}\sim\frac{Aa^2x^2+\overline{B}x+\overline{C}}{Da^2x}=\frac{A}{D}x+\frac{\overline{B}}{Da^2}+\frac{1}{x}$.
■

Lemma 6. If the signature (θ,ϕ) of the normal RQ $\frac{Ax^2+Bx+C}{Hx^2+Dx+E}$ satisfies $\theta\neq 0, \phi=0$, then $\exists$ a normal RQ $\frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{D}x+\overline{E}}, \overline{A}\neq 0, \overline{D}\neq 0$, such that $\frac{Ax^2+Bx+C}{Hx^2+Dx+E}\sim\frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{D}x+\overline{E}}$.

Note 2. Of course, $\overline{A}\neq 0$ and from Lemma 5, $\overline{D}\neq 0$.

Proof. Since (θ,ϕ) for $\frac{Ax^2+Bx+C}{Hx^2+Dx+E}$ satisfies $\phi\neq 0$, we know from Lemma 2 that it is impossible for both $H=0$ and $D=0$ to be true. Therefore, we can assume that $H\neq 0$ since if $H=0$ there is nothing to prove.

Using case a of section 3 with $\frac{ax+b}{x+d}$ where $\overline{H}(d)=0$ and $\begin{vmatrix} a & b \\ 1 & d \end{vmatrix} \neq 0$, we have $\frac{Ax^2+Bx+C}{Hx^2+Dx+E} \sim \frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{D}x+\overline{E}}$ where $\overline{A},\overline{B},\overline{C},\overline{H}(d)=0,\overline{D},\overline{E}$ are defined in case a.

Now if $\overline{D}=0$ we would have the signature of $\frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{D}x+\overline{E}}$ satisfying $(\theta,\phi)=(\theta,0)$ which is a contradiction. Therefore, $\overline{D}\neq 0$ and also $\overline{A}\neq 0$ since $\frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{D}x+\overline{E}}$ is a rational quadratic. ■

Theorem 2. Any normal rational quadratic Q is linearly similar to a normal RQ $\overline{Q}$ of the form $\overline{Q}=\overline{A}x+\overline{B}+\frac{1}{x}$ where $\overline{A}\neq 0$.

Proof. This follows from Lemmas 4, 5, 6. ■

8 Solving $(**)$ of Section 6 when all of the Roots are Equal

Lemma 7. If (θ,ϕ) satisfies $\theta\neq 0,\phi\neq 0$, then the three roots of $(**)$, section 6, are all equal if and only if $(\theta,\phi)=(-27,16)$ and when $(\theta,\phi)=(-27,16)$ the three equal roots of $(**)$ are $\left\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\}$.

Proof. Using x for $\overline{A}$, the 3 roots of $(**)$ are equal if and only if $\exists r\in C$ such that $x^3-\frac{2(\theta+\phi-1)}{\phi}x^2+\left(\frac{\phi-4}{\phi}\right)x+\frac{2}{\phi}=(x-r)^3=x^3-3rx^2+3r^2x-r^3$.

Of course, $r\neq 0$. Therefore, $\phi\neq 4$.

Now $r^3=\frac{-2}{\phi}, r^2=\frac{\phi-4}{3\phi}$.

Therefore, $r=\frac{r^3}{r^2}=\frac{-2}{\phi}\left(\frac{3\phi}{\phi-4}\right)=\frac{-6}{\phi-4}$.

Therefore, $\left(\frac{-6}{\phi-4}\right)^2=\frac{\phi-4}{3\phi}$.

Therefore, $\phi^3-12\phi^2-60\phi-64=(\phi+2)^2(\phi-16)=0$.

Therefore, $\phi=-2,\phi=16$.

(a) $\phi=16$ implies $r=\frac{-6}{\phi-4}=-\frac{1}{2}$.

Also, $-3r=\frac{-2(\theta+\phi-1)}{\phi}$ implies $\theta=-27$.

(b) $\phi=-2$ implies $r=\frac{-6}{\phi-4}=1$.

Also, $-3r=\frac{-2(\theta+\phi-1)}{\phi}$ implies $\theta=0$ which contradicts the hypothesis. ■

Note 3. When $\theta=-27,\phi=16$, the solution $\left(\overline{A},\overline{B}\right)$ to $(*),(**)$ of Section 6 is $\left(\overline{A},\overline{B}\right)=\left(-\frac{1}{2},0\right)$.

9 Proving Theorem 3

Lemma 8. If $A\neq 0$, then $\frac{Ax^2+Bx+1}{x}\sim\frac{Ax^2-Bx+1}{x}$.

Proof. Using $ax+b=-x+0$ with case b of Section 3 we have $\frac{Ax^2+Bx+1}{x}\sim\frac{A(-1)^2x^2+B(-1)x+1}{(-1)^2x}=\frac{Ax^2-Bx+1}{x}$ ■

Theorem 3. Two normal rational quadratics Q,Q' are linearly similar if and only if Q and Q' have the same signature (θ,ϕ).

Proof. Of course, normality is equivalent to $\theta \neq 0$. Also, if $Q \sim Q'$ then Q and Q' must have the same signature (θ, ϕ) since (θ, ϕ) is an invariant under linear fractional transformation.

Also, from Corollary 1, if $\phi = 0$ then $Q \sim Q'$ if and only if Q and Q' have the same signature $(\theta, \phi) = (\theta, 0)$.

Therefore, we now assume that Q and Q' have the same signature (θ, ϕ) where $\theta \neq 0, \phi \neq 0$ and we show that $Q \sim Q'$. From Theorem 2 we know that Q and Q' are each linearly similar to normal RQ's of the form $Q \sim \overline{A}x + \overline{B} + \frac{1}{x}, Q' \sim A^*x + B^* + \frac{1}{x}$ where $\overline{A} \neq 0, A^* \neq 0$.

Now if $(\theta, \phi) = (-27, 16)$ then from Lemma 7 and Note 3 we know that $\overline{A}x + \overline{B} + \frac{1}{x} = A^*x + B^* + \frac{1}{x} = -\frac{1}{2}x + \frac{1}{x}$.

Therefore, $Q \sim Q'$ when $(\theta, \phi) = (-27, 16)$.

Let us now assume that $(\theta, \phi) \neq (-27, 16)$.

From Lemma 7 we know that the equation $(**)$ of Section 6 must have exactly two or exactly three distinct roots which we call A_1, A_2 or A_1, A_2, A_3. Therefore, there are exactly two or exactly three RQ's of the form $A_i x \pm B_i + \frac{1}{x}$, where $i \in \{1, 2\}$ or $i \in \{1, 2, 3\}$, that have this signature (θ, ϕ). Of course, by Lemma 8, $A_i x + B_i + \frac{1}{x} \sim A_i x - B_i + \frac{1}{x}$.

We now show that any $Ax + B + \frac{1}{x}, A \neq 0$, having a signature $(\theta, \phi), \theta \neq 0, \phi \neq 0, (\theta, \phi) \neq (-27, 16)$ is linearly similar to at least one other $\overline{A}x + \overline{B} + \frac{1}{x}, \overline{A} \neq 0$, where $A \neq \overline{A}$. This will imply that each of $A_i x \pm B_i + \frac{1}{x}, i \in \{1, 2\}$ or $i \in \{1, 2, 3\}$, is linearly similar to all of the others and this will imply that $Q \sim Q'$ which finishes the proof.

If we now use the case (a) Section 3 transformation with $\frac{Ax^2+Bx+1}{x}$ where $A = A, B = B, C = D = 1, H = E = 0$, we have $\frac{Ax^2+Bx+1}{x} \sim \frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{H}x^2+\overline{D}x+\overline{E}}$ where $\overline{A} = a\left[Ad^2 - Bd + 1\right] - bd, \overline{H} = [A-1]\, d^2 - Bd + 1$ and $\overline{D} = a\left[d^2 + Bd - 2\right] + b\left[(1-2A)\, d + B\right]$.

Since $\theta \neq 0$, we know from section 6 that $A \neq 1$. Letting $a = 1, b = 0$ we have $\overline{A} = Ad^2 - Bd + 1, \overline{H} = (A-1)\, d^2 - Bd + 1$ and $\overline{D} = d^2 + Bd - 2$.

Now since $\theta \neq 0$ we know from the definition of the numerator of θ that $\overline{H}\,(d)$ must have two distinct roots $d_1 \neq d_2$. Also, $d_1 \neq 0, d_2 \neq 0$. Therefore, $\begin{vmatrix} a & b \\ 1 & d \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & d \end{vmatrix} \neq 0$.

Now $\overline{H}\,(d) = 0$ implies $\overline{A} = d^2$ and $\overline{D} = Ad^2 - 1$.

Therefore, $(***)$ $\frac{Ax^2+Bx+1}{x} \sim \frac{d_i^2 x^2 + \overline{B}_i x + \overline{C}_i}{\left(Ad_i^2 - 1\right)x + \overline{E}_i}$ where $i \in \{1, 2\}$ with $d_1 \neq d_2, d_1 \neq 0, d_2 \neq 0$.

Since the signature (θ, ϕ) of $\frac{Ax^2+Bx+1}{x}$ satisfies $\phi \neq 0$ we know that $\frac{Ax^2+Bx+1}{x}$ is not linearly similar to a rational quadratic polynomial. Therefore, $Ad^2 - 1 \neq 0$ when $d \in \{d_1, d_2\}$. Now from Lemma 5 $\frac{d_i^2 x^2 + \overline{B}_i x + \overline{C}_i}{\left(Ad_i^2 - 1\right)x + \overline{E}_i} \sim \frac{d_i^2 x}{Ad_i^2 - 1} + B_i^* + \frac{1}{x}$ where $d_i \in \{d_1, d_2\}$.

Now if $d_1^2 \neq d_2^2$ then $\frac{d_1^2}{Ad_1^2 - 1} \neq \frac{d_2^2}{Ad_2^2 - 1}$ and this finishes the proof.

Therefore, let us assume that $d_1^2 = d_2^2$. Of course, this is true if and only if $B = 0$ and when $B = 0$ the roots of $\overline{H}\,(d)$ satisfy $d^2 = \frac{1}{1-A}$.

Using $(***)$ the proof is complete if we can show that $A \neq \frac{d^2}{Ad^2-1}$ when $d^2 = \frac{1}{1-A}$. Now $A = \frac{d^2}{Ad^2-1}$ is true if and only if $A = \frac{1}{2A-1}$ which is equivalent to $(2A+1)(A-1) = 0$. From Section 6 we know that $A = 1$ implies $\theta = 0$, a contradiction. Therefore, suppose $A = -\frac{1}{2}$. Since $B = 0$ we see from Section 6 that $\theta = \frac{-4\left(-\frac{1}{2}-1\right)^3}{\frac{-1}{2}} = -27$ and $\phi = \frac{-2\left[2\left(\frac{-1}{2}\right)-1\right]^2}{-\frac{1}{2}} = 16$ and this is also a contradiction. ■

10 Some Applications

Let us define $f(x) = \frac{1}{\sqrt{m}} \tan x, g(x) = 2x + \tan^{-1}(\sqrt{m}c), f^{-1}(x) = \tan^{-1}(\sqrt{m}x)$.

Also, $\overline{g}(x) = f(x) \circ g(x) \circ f^{-1}(x)$.

By straightforward calculations we see that $\overline{g}(x) = \frac{-cmx^2+2x+c}{-mx^2-2cmx+1}$ and therefore we are able to compute $(\overline{g}(x))^n$ by $(\overline{g}(x))^n = f(x) \circ [2x + \tan^{-1}(\sqrt{m}c)]^n \circ f^{-1}(x)$.

We might think for a moment that we have accomplished something of significance. However, a calculation shows that the signature of $\overline{g}(x)$ is the following since $A = -cm, B = 2, C = c, H = -m, D = -2cm, E = 1$.

$$\begin{aligned}
\theta &= \frac{-27m^2c^2 + 18(-m)(cm)(-1)c - 4(cm)^3 c + 4m(-1)^3 + (cm)^2(-1)^2}{(-cm+cm)^2 + (2+2c^2m)(-2m-2c^2m^2)} \\
&= \frac{-9m^2c^2 - 4m^3c^4 - 4m + m^2c^2}{-4(1+mc^2)(m+m^2c^2)} \\
&= \frac{-8m^2c^2 - 4m^3c^4 - 4m}{-4m(1+mc^2)^2} \\
&= \frac{m^2c^4 + 2mc^2 + 1}{(1+mc^2)^2} = 1.
\end{aligned}$$

Also, by similar calculations we see that $\phi = 0$. Therefore, since the signature of x^2 is also $(\theta, \phi) = (1, 0)$, we see that $\overline{g}(x) = \frac{-cmx^2+2x+c}{-mx^2-2cmx+1} \sim x^2$. Of course, $\left[\frac{-cmx^2+2x+c}{-mx^2-2cmx+1}\right]^n \sim (x^2)^n = x^2 \circ x^2 \circ \ldots \circ x^2$ and the recursion $x_{n+1} = \overline{g}(x_n)$ where $x_n = \overline{g}^n(x_0)$ can be easily solved. The reader might like to compute the signatures of some standard RQ's such as $\cos(2\cos^{-1} x) = 2x^2 - 1, \tanh\left(2\tanh^{-1} x\right) = \frac{2x}{1+x^2}, \coth\left(2\coth^{-1} x\right) = \frac{x^2+1}{2x}, \cot(2\cot^{-1} x) = \frac{x^2-1}{2x}, \tan(2\tan^{-1} x) = \frac{2x}{1-x^2}$.

We note that if $Ax^2 + Bx + C, A \neq 0$, is any rational quadratic polynomial then there exists a unique complex number C^* such that $Ax^2 + Bx + C \sim x^2 + C^*$. Also, the reader might like to show that $\frac{(a+c)x^2+2mx+cm}{x^2+2cx+(m-ac)} \sim \frac{x^2+s}{2x-t} \sim x^2$ and $\frac{Ax^2-2Bx+C}{x^2-2Ax+B} \sim \frac{x^2-2rx+t}{-2x+r} \sim \frac{1}{x^2}$.

11 Discussion

Using the same basic heuristic techniques that we used in this paper it appears fairly easy to believe that we could in general compute at least some of the invariants for the following general linear fractional transformations.

$\frac{ax+b}{x+d} \circ \frac{\sum_{i=0}^n a_i x^i}{\sum_{i=0}^n b_i x^i} \circ \frac{b-dx}{x-a} = \frac{\sum_{i=0}^n \overline{a}_i x^i}{\sum_{i=0}^n \overline{b}_i x^i}$. However, it would quickly become impractical to try to prove that the proposed invariants will actually work using a computer. Also, since there are only three degrees of freedom in $\frac{ax+b}{x+d}$, it hardly seems worth the effort to go too high even if the invariants could be verified.

References

[1] Weisner, Louis, Introduction to the Theory of Equations, The MacMilian Company, New York, 1949.