Linearly Similar Polynomials

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A standard technique for solving the recursion $x_{n+1} = g(x_n)$, where $g : \mathbb{C} \to \mathbb{C}$ is a complex function, is to first find a fairly simple function $\overline{g} : \mathbb{C} \to \mathbb{C}$ and a bijection (i.e., a 1-1 onto function) $f : \mathbb{C} \to \mathbb{C}$ such that $g = f^{-1} \circ \overline{g} \circ f$ where $\circ$ is the composition of functions and f^{-1} is the inverse function of f. Then $x_n = g^n(x_0) = (f^{-1} \circ \overline{g}^n \circ f)(x_0)$ where g^n and $\overline{g}^n$ are the n-fold compositions and $\overline{g}^n$ is fairly easy to compute. With this motivation we are in general interested in studying all pairs of rational functions $g, \overline{g}$ such that for some $a, b, c, d \in \mathbb{C}$

$$g = \left(\frac{ax+b}{cx+d}\right)^{-1} \circ \overline{g} \circ \left(\frac{ax+b}{cx+d}\right)$$

where a, b, c, d satisfy $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

In this paper we study an intermediate problem by finding all pairs of quadratic polynomials $g, \overline{g}$ and also all pairs of cubic polynomials $g, \overline{g}$ such that for some $a, b \in \mathbb{C}, a \neq 0, g = (ax+b)^{-1} \circ \overline{g} \circ (ax+b)$. We denote such pairs by $g \approx \overline{g}$ and we show that $g \approx \overline{g}$ if and only if g and $\overline{g}$ have the same *signature* which we define as those invariants that quadratic and cubic polynomials have under the linear transformation $g = (ax+b)^{-1} \circ \overline{g} \circ (ax+b)$.

1 Introductory Concepts

Let $\mathbb{C}$ be the set of complex numbers. If $g : \mathbb{C} \to \mathbb{C}$ and $\overline{g} : \mathbb{C} \to \mathbb{C}$ are arbitrary functions from $\mathbb{C}$ into $\mathbb{C}$, we say that g and $\overline{g}$ are similar (denoted by $g \sim \overline{g}$) if there exists a bijection (i.e., a 1-1 onto function) $f : \mathbb{C} \to \mathbb{C}$ such that $g = f^{-1} \circ \overline{g} \circ f$ where $\circ$ is the composition of functions. From elementary set theory, we know that the word similar means exactly what it says. Thus, for example, suppose that g and $\overline{g}$ are similar bijections on $\mathbb{C}$. If we break each of g and $\overline{g}$ down into its cycles$\ldots, g^{-2}(x), g^{-1}(x), g^0(x) = x, g(x), g^2(x), \ldots$, then the types of these cycles will be the same in both g and $\overline{g}$.

For example, if all of the cycles in g are the 3-cycles $x, g(x), g^2(x), g^3(x) = x$, then all of the cycles in $\overline{g}$ will also be the 3-cycles $x, \overline{g}(x), \overline{g}^2(x), \overline{g}^3(x) = x$. If g and $\overline{g}$ are rational functions (i.e., the quotient of polynomials) and

$$g = \left(\frac{ax+b}{cx+d}\right)^{-1} \circ \overline{g} \circ \left(\frac{ax+b}{cx+d}\right),$$

then we encounter a stronger type of similarity which we will call algebraic similarity. Intuitively algebraic similarity means that g and $\overline{g}$ have similar algebraic properties. However, we have not been able to define exactly what this means. None-the-less we have still been able to use this intuitive concept to heuristically derive a technique for computing all of the invariants that g and $\overline{g}$ must have when g and $\overline{g}$ are either polynomials or rational functions of any arbitrary degree and $g = \left(\frac{ax+b}{cx+d}\right)^{-1} \circ \overline{g} \circ \left(\frac{ax+b}{cx+d}\right)$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

We will show the reader how to do this when $g = (ax+b)^{-1} \circ \overline{g} \circ (ax+b)$ and $g, \overline{g}$ are polynomials of degree 2 or degree 3. These same ideas also generalize for higher degree polynomials. However, the invariants quickly become so complex that they must all be proved with a computer. Indeed, one of the three invariants used in this paper was (for convenience only) computer proved.

2 Very Linearly Similar Quadratic Polynomials

Definition 1 *Suppose g and $\overline{g}$ are nth degree complex polynomials. We say that g and $\overline{g}$ are very linearly similar (denoted by $g \approx \overline{g}$) if there exists $a, b \in \mathbb{C}$, $a \neq 0$, such that $g = (ax+b)^{-1} \circ \overline{g} \circ (ax+b)$.*

Theorem 1 *The relation $\approx$ is an equivalence relation on the collection of all complex polynomials.*

Proof. We let the reader verify the following three conditions which define an equivalence relation.

a. $g \approx g$.(reflexive condition).

b. $g \approx \overline{g}$ implies $\overline{g} \approx g$. (symmetric condition).

c. $g \approx \overline{g}$ and $\overline{g} \approx g^*$ implies $g \approx g^*$. (transitive condition).

■

Problem 1 *Suppose $g(x) = Ax^2 + Bx + C, A \neq 0$, is a quadratic polynomial.*

Define $\left(\frac{x}{a} - \frac{b}{a}\right) \circ (Ax^2 + Bx + C) \circ (ax+b) = \overline{g}(x) = \overline{A}x^2 + \overline{B}x + \overline{C}$ where $a \neq 0$, b are arbitrary complex numbers and $(ax+b)^{-1} = \frac{x}{a} - \frac{b}{a}$.

We wish to compute an invariant for $g(x)$ and $\overline{g}(x)$. This means that we wish to find an expression involving A, B, C that remains unchanged when we substitute $\overline{A}, \overline{B}, \overline{C}$ for A, B, C.

Solution 1 *Suppose that $Ax + Bx + C$ and $\overline{A}x^2 + \overline{B}x + \overline{C}, A \neq 0, \overline{A} \neq 0$ are any arbitrary complex quadratics.*

We now show that we can linearly transform $Ax^2 + Bx + C$ into $\overline{A}x^2 + \overline{B}x + \overline{C}$ by

$$(*) \qquad \overline{A}x^2 + \overline{B}x + \overline{C} = \left(\frac{x}{a} - \frac{b}{a}\right) \circ \left(Ax^2 + Bx + C\right) \circ (ax+b)$$

(where $a \neq 0$) if and only if a certain condition is met, and this condition will define the invariance relation that (A, B, C) and $\left(\overline{A}, \overline{B}, \overline{C}\right)$ have under the above linear transformation $(*)$.

Now

$$\begin{aligned} & \left(\frac{x}{a} - \frac{b}{a}\right) \circ \left(Ax^2 + Bx + C\right) \circ (ax + b) \\ = & \left(\frac{x}{a} - \frac{b}{a}\right) \circ \left[A (ax + b)^2 + B (ax + b) + C\right] \\ = & \left(\frac{x}{a} - \frac{b}{a}\right) \circ \left(Aa^2x^2 + (2Aab + Ba)\, x + Ab^2 + Bb + C\right) \\ = & Aax^2 + (2Ab + B)\, x + \frac{Ab^2 + (B-1)\, b + C}{a} \\ = & \overline{A}x^2 + \overline{B}x + \overline{C}. \end{aligned}$$

Now $Aa = \overline{A},\ 2Ab + B = \overline{B}$ *implies* $a = \frac{\overline{A}}{A}, b = \frac{\overline{B}-B}{2A}$.

Therefore,

$$\begin{aligned} \overline{C} &= \frac{1}{a}\left[Ab^2 + (B-1)\, b + C\right] \\ &= \frac{A}{\overline{A}}\left[A\left(\frac{\overline{B}-B}{2A}\right)^2 + (B-1)\left(\frac{\overline{B}-B}{2A}\right) + C\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} 4\overline{AC} &= \left(\overline{B} - B\right)^2 + 2 (B-1) \left(\overline{B} - B\right) + 4AC \\ &= \overline{B}^2 - 2\overline{B}B + B^2 + 2B\overline{B} - 2\overline{B} - 2B^2 + 2B + 4AC. \end{aligned}$$

This implies $(B^2 - 2B) - 4AC = \left(\overline{B}^2 - 2\overline{B}\right) - 4\overline{AC}$. *Therefore,* $(B-1)^2 - 4AC = \left(\overline{B} - 1\right)^2 - 4\overline{AC}$, *and this expression must be the invariant relation when* $Ax^2 + Bx + C \approx \overline{A}x^2 + \overline{B}x + \overline{C}$. *This invariant can also easily be checked (by hand) by substituting* $\overline{A} = Aa,\ \overline{B} = 2Ab + B,\ \overline{C} = \frac{Ab^2+(B-1)b+C}{a}$ *for* A, B, C *in* $(B-1)^2 - 4AC$ *and showing that* $(B-1)^2 - 4AC$ *remains unchanged.*

Definition 2 *We call* $\theta = (B-1)^2 - 4AC$ *the signature of* $Ax^2 + Bx + C$.

In solving Problem 1, we have also proved Theorem 2.

Theorem 2 *Suppose* $Ax^2 + Bx + C,\ \overline{A}x^2 + \overline{B}x + \overline{C},\ A \neq 0,\ \overline{A} \neq 0$, *are arbitrary complex quadratics. Then* $Ax^2+Bx+C \approx \overline{A}x^2+\overline{B}x+\overline{C}$ *if and only if* Ax^2+Bx+C *and* $\overline{A}x^2+\overline{B}x+\overline{C}$ *have the same signature* θ.

2.1 A note on the Discriminant of a Polynomial

If $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_0$ is any single variable polynomial, then from algebra we know that there is a standard expression (denoted by $\rho(P(x), P'(x))$ which is called the discriminant of $P(x)$. See p. 99, [1].

When the discriminant of $P(x)$ is zero, we know from algebra that $P(x)$ has a repeated root, and when the discriminant of $P(x)$ is non-zero, we know that the n roots of $P(x)$ are all distinct.

Of course, the discriminant of the quadratic $P(x) = a_0x^n + a_1x + a_2$ is $a_1^2 - 4a_0a_2$.

We observe that the invariant of the quadratic $Ax^2 + Bx + C,\ A \neq 0$, that was derived in the last section is also the discriminant of $Ab^2 + (B-1)b + C$, when we substitute $b = x$ for b and where $\overline{A}x^2 + \overline{B}x + \overline{C} = Aax^2 + (2Ab+B)x + \frac{Ab^2+(B-1)b+C}{a}$.

This simple observation (with one slight modification) appears to generalize for arbitrary degree polynomials, and it will soon be used to compute the second invariant of cubic polynomials.

2.2 Computing the Linear Transformation of Cubic Polynomials

By straightforward calculations we see that

$$\begin{aligned}
& \left(\frac{x}{a} - \frac{b}{a}\right) \circ \left(Ax^3 + Bx^2 + Cx + D\right) \circ (ax+b) \\
= & \left(\frac{x}{a} - \frac{b}{a}\right) \circ \left(A(ax+b)^3 + B(ax+b)^2 + C(ax+b) + D\right) \\
= & \left(\frac{x}{a} - \frac{b}{a}\right) \circ \left[\begin{array}{c} Aa^3x^3 + \left(3Aa^2b + Ba^2\right)x^2 \\ + \left(3Aab^2 + 2Bab + Ca\right)x + Ab^3 + Bb^2 + Cb + D \end{array}\right] \\
= & Aa^2x^3 + (3Aab + Ba)x^2 + \left(3Ab^2 + 2Bb + C\right)x + \left(Ab^3 + Bb^2 + (C-1)b + D\right)/a \\
= & \overline{A}x^3 + \overline{B}x^2 + \overline{C}x + \overline{D}.
\end{aligned}$$

2.3 Two Invariants of Cubic Polynomials under Linear Transformation

Problem 2 *Suppose $g(x) = Ax^3 + Bx^2 + Cx + D, A \neq 0$, is a cubic polynomial.*

Define $\left(\frac{x}{a} - \frac{b}{a}\right) \circ \left(Ax^3 + Bx^2 + Cx + D\right) \circ (ax+b) = \overline{g}(x) = \overline{A}x^3 + \overline{B}x^2 + \overline{C}x + \overline{D}$ where $a \neq 0,\ b$ are arbitrary complex numbers. We wish to compute two invariants for $g(x)$ and $\overline{g}(x)$. This means that we wish to find two expressions involving A, B, C, D that remain unchanged when we substitute $\overline{A}, \overline{B}, \overline{C}, \overline{D}$ for A, B, C, D.

Solution 2 *The first invariant can be computed exactly as we did in solving Problem 1. From the previous section; we must have $Aa^2 = \overline{A}, 3Aab + Ba = \overline{B}, 3Ab^2 + 2Bb + C = \overline{C}$ and $\frac{Ab^3+Bb^2+(C-1)b+D}{a} = \overline{D}$.*

Now $a^2 = \overline{A}/A,\ b = \frac{\overline{B}-Ba}{3Aa}$, from which it follows that $\overline{C} = 3A\left[\frac{\overline{B}-Ba}{3Aa}\right]^2 + 2B\left[\frac{\overline{B}-Ba}{3Aa}\right] + C = \frac{\left(\overline{B}-Ba\right)^2}{3Aa^2} + \frac{2B\left(\overline{B}-Ba\right)}{3Aa} + C$. Hence, $\overline{C} = \frac{\overline{B}^2 - 2B\overline{B}a + B^2a^2 + 2B\overline{B}a - 2B^2a^2 + 3Aa^2C}{3Aa^2}$. Thus $3Aa^2\overline{C} =$

$\overline{B}^2 - B^2a^2 + 3Aa^2C$. Therefore, $3A\left(\frac{\overline{A}}{A}\right)\overline{C} = \overline{B}^2 - B^2\left(\frac{\overline{A}}{A}\right) + 3A\left(\frac{\overline{A}}{A}\right)C$. Thus $3\overline{AC} = \overline{B}^2 - \frac{B^2\overline{A}}{A} + 3\overline{A}C$. Finally it follows that $3C - \frac{B^2}{A} = 3\overline{C} - \frac{\overline{B}^2}{\overline{A}}$.

We will call $\theta = C - \frac{B^2}{3A}$ the first invariant of $g(x) = Ax^3 + Bx^2 + Cx + D$ under the linear transformation $\overline{g}(x) = (ax+b)^{-1} \circ g(x) \circ (ax+b)$.

We observe that θ is also equivalent to $\rho(3Ax^2 + 2Bx + C)/A$ where we substitute $b = x$ in $\overline{C} = 3Ab^2 + 2Bb + C$. This invariant θ can also easily be verified by showing that θ remains unchanged when we substitute $\overline{A} = Aa^2, \overline{B} = 3Aab + Ba$ and $\overline{C} = 3Ab^2 + 2Bb + C$ in θ for A, B, C respectively. This can easily be done by hand. When we try to compute the second invariant for $Ax^3 + Bx^2 + Cx + D \approx \overline{A}x^3 + \overline{B}x^2 + \overline{C}x + \overline{D}$ in this exact same way, we run into insurmountable difficulty. Therefore, we will simply conjecture that the second invariant called $\phi = \rho(Ax^3 + Bx^2 + (C-1)x + D)/A^k$ where k is decided by using a specific example. This division by A^k is the modification that we referred to earlier.

Now the standard discriminant for the cubic polynomial $P(x) = a_0x^3 + a_1x^2 + a_2x + a_3$ is $\rho(P(x), P'(x)) = -27a_0^2a_3^2 + 18a_0a_1a_2a_3 - 4a_1^3a_3 - 4a_0a_2^3 + a_1^2a_2^2$. Therefore, $\phi = (-27A^2D^2 + 18AB(C-1)D - 4B^3D - 4A(C-1)^3 + B^2(C-1)^2)/A$ since a specific example shows that $A^k = A$ must be true.

Professor Ben Klein of Davidson College has verified that ϕ is an invariant by using the Mathematica software.

Definition 3 *We define the signature of the cubic polynomial $Ax^3 + Bx^2 + Cx + D, A \neq 0$, to be the ordered pair (θ, ϕ) where θ and ϕ are the invariants specified above.*

The rest of this paper is devoted mainly to proving that two cubic polynomials $g(x)$ and $\overline{g}(x)$ are very linearly similar if and only if $g(x)$ and $\overline{g}(x)$ have the same signature (θ, ϕ).

3 Proving our Main Results for Cubic Polynomials

Theorem 3 *The signature of the cubic polynomial $x^3 + Cx + D$ is $(\theta, \phi) = \left(C, -27D^2 - 4(C-1)^3\right)$.*

Proof. Using $A = 1, B = 0, C = C, D = D$ in the formulas for θ, ϕ gives the signature (θ, ϕ). ■

Corollary 1 *Suppose $x^3 + Cx + D$ has a signature (θ, ϕ).*

Then $C = \theta$ and $D \in \left\{\sqrt{\frac{-\phi - 4(\theta-1)^3}{27}}, -\sqrt{\frac{-\phi - 4(\theta-1)^3}{27}}\right\}$.

Proof. Follow from Theorem 3. ■

Theorem 4 *Suppose $Ax^3 + Bx^2 + Cx + D, A \neq 0$, is any arbitrary cubic polynomial. Then there exists a cubic polynomial $x^3 + \overline{C}x + \overline{D}$ such that $Ax^3 + Bx^2 + Cx + D \approx x^3 + \overline{C}x + \overline{D}$.*

Proof. Now

$$\begin{aligned}
& \left(\frac{x}{a}-\frac{b}{a}\right) \circ \left(Ax^3+Bx^2+Cx+D\right) \circ (ax+b) \\
= \; & Aa^2x^3 + (3Aab+Ba)\,x^2 + \left(3Ab^2+2Bb+C\right)x \\
& + \frac{Ab^3+Bb^2+(C-1)\,b+D}{a} \\
= \; & x^3 + 0\cdot x^2 + \overline{C}x + \overline{D}.
\end{aligned}$$

Let a, b be defined so that $Aa^2 = 1, 3Aab + Ba = 0$. Therefore, $a = \pm\sqrt{\frac{1}{A}}, b = \frac{-B}{3A}$ which completes the proof. ■

Theorem 5 *Suppose $x^3 + Cx + D$ and $x^3 + \overline{C}x + \overline{D}$ have the same signature (θ, ϕ). Then $x^3 + Cx + D \approx x^3 + \overline{C}x + \overline{D}$.*

Proof. From Corollary 1 we know that $C = \overline{C} = \theta$. Also, from Corollary 1, $D, \overline{D} \in \left\{\sqrt{\frac{-\phi-4(\theta-1)^3}{27}} - \sqrt{\frac{-\phi-4(\theta-1)^3}{27}}\right\}$.

Therefore, $D = \overline{D}$ or else $D = -\overline{D}$.

Therefore, we can complete the proof by assuming that $D = -\overline{D}$. Using $A = 1, B = 0, C = C, D = D, \overline{A} = 1, \overline{B} = 0, \overline{C} = \overline{C}, \overline{D} = \overline{D}$ in the proof of Theorem 4, we have $a = \pm 1, b = 0$. Let us use $a = -1$. Therefore, $ax+b = -x$ and we have $(-x) \circ (x^3 + Cx + D) \circ (-x) = (-x) \circ (-x^3 - Cx + D) = x^3 + Cx - D$.

Therefore, $x^3 + Cx + D \approx x^3 + Cx - D = x^3 + \overline{C}x + \overline{D}$. ■

Main Theorem 6. Two cubic polynomials $g(x)$ and $\overline{g}(x)$ are very linearly similar if and only if $g(x)$ and $\overline{g}(x)$ have the same signature (θ, ϕ).

Proof. Of course, if $g(x)$ and $\overline{g}(x)$ are very linearly similar, then they must have the same signature (θ, ϕ) since θ and ϕ are invariants under $\approx$.

Conversely, if $g(x)$ and $\overline{g}(x)$ have the same signature (θ, ϕ), then we know from Theorems 4, 5 and from the equivalence relation properties of $\approx$ that were proved in Theorem 1 that $g(x) \approx \overline{g}(x)$ must be true. ■

4 Some Concluding Remarks

We note that the signature of $\overline{g}(x) = x^3 = 1 \cdot x^3 + 0 \cdot x^2 + 0 \cdot x + 0$ is $(\theta, \phi) = (0, 4)$. Also, we note that the recursion $x_{n+1} = x_n^3$ can easily be solved in a closed form. Suppose $g(x) = Ax^3 + Bx^2 + Cx + D, A \neq 0$, is any cubic polynomial that has a signature $(\theta, \phi) = (0, 4)$. From this we know that $g(x) \approx x^3$, and we can now easily solve the recursion $x_{n+1} = g(x_n)$ in a closed form.

Also this paper can be generalized as follows. Suppose $g(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ and $\overline{g}(x) = \overline{a}_0x^n + \overline{a}_1x^{n-1} + \cdots + \overline{a}_n$ are very linearly similar n^{th} degree complex polynomials. That is, $g(x) = (ax+b)^{-1} \circ \overline{g}(x) \circ (ax+b)$ for some $a, b \in \mathbb{C}, a \neq 0$.

Then we can almost certainly derive the $n-1$ invariants that $g(x)$ and $\overline{g}(x)$ must have. We do this exactly as we did in this paper, and we then prove that the invariants are correct by using a computer program such as Mathematica.

References

[1] Barbeau, E. J. Polynomials, (Problem Books in Mathematics) (Paperback), Springer Verlag, New York, 1989.

[2] Weisner, Louis, Introduction to the Theory of Equation, The MacMillan company, New York, 1949.