Single Pile (Move Size) Dynamic Blocking Nim

Abstract: Several authors have written papers dealing with a class of combinatorial games consisting of one-pile counter pickup games for which the maximum number of counters that can be removed on each move is a function of the number of counters that was removed on the last move, see [1], [2], $\cdots$. These authors have been able to consolidate a massive number of games into one general theorem. The purpose of this paper is to add a new feature to this class of games in which on each turn before the moving player moves the opposing player can block some of his moves. We will analyze in detail one example of such a blocking game. We will then state without proof some lemmas concerning another more complicated example.

This second game, which appears to be only slightly more complex than the first game, nevertheless has a much stranger and wilder set of safe positions.

All of this leads us to believe that there are endless examples of these dynamic blocking games that can be brought under control. Also, the methods used to accomplish this are analogous to the methods used in this paper. However, we believe that these blocking games must mostly be analyzed ad-hoc on a case by case basis and that very little of the massive consolidation that was accomplished with the non-blocking version can be achieved with the blocking version.

Rules of First Game: Two players alternate subtracting positive whole numbers from a given initial positive integer and then replacing the integer on each turn by the new smaller integer that results, observing the following rules. An ordered pair (N, X), where N is an integer and X is a positive integer, is called a position. The number N represents the integer that we are dealing with and X represents the greatest whole number that can be subtracted from N on the next move. We allow $X < N, X = N, X > N$. Of course, initially N is positive.

The function $f(n) = 2n$ is also given which determines the maximum size of the next move in terms of the current move. Thus a move in the game is an ordered pair of positions $(N, X) \rightarrow (N-K, f(k)) = (N-k, 2k)$, where $1 \leq k \leq X$, subject to the second rule. Note that we are allowing the moving player to overshoot $N = 0$. The second rule states that on each turn before the moving player moves, the opposing player can block up to two of his moves. He can do this in any way that he chooses including the option of blocking less than two moves. The game ends as soon as one of the following things happen to (N, X) :

1. N is non-positive

2. $X \leq 2$.

The winner is the one who makes the last move in the game. If the game starts with (N, X) where $1 \leq X \leq 2$, we agree that the second moving player is the winner since the first blocking player can prevent the first moving player from moving. The reader might note that if the game does not start with $X = 1$ then it is impossible for $f(n) = 2n$ to ever equal 1.

Base: Let us define the sequence $(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3), \cdots$ as follows. We call this the base of the game.

1. $\forall_i, b_i = a_i + 1, ci = b_i + 1$.

2. $(a_1, b_1, c_1) = (1, 2, 3), (a_2, b_2, c_2) = (4, 5, 6)$,

   $(a_3, b_3, c_3) = (7, 8, 9), (a_4, b_4, c_4) = (12, 13, 14)$

3. $\forall_i \geq 5, a_i = c_{i-1} + a_{i-2}$.

Thus a few more terms are

$(a_5, b_5, c_5) = (21, 22, 23)$

$(a_6, b_6, c_6) = (35, 36, 37)$

$((a_7, b_7, c_7) = (58, 59, 60), \cdots$.

Obviously, $a_1 < b_1 < c_1 < a_2 < b_2 < c_2 < a_3 < b_3 < c_3 < a_4 < b_4 < c_4 < \cdots$.

Lemma 1: Suppose $c_i < N < a_{i+1})$ where N is an integer, then $N - c_i < a_{i-1}$.

Proof: If $i \leq 3$, the only two possibilities are $i = 3, c_i = c_3 = 9, a_{i+1} = a_4 = 12, N = 10$ or 11.

Obviously, $N - c_i \in \{1, 2\}$, and $N - c_i \leq 2 < a_{i-1} = a_2 = 4$. So assume $i \geq 4$.

Now since $i \geq 4$, $a_{i+1} = c_i + a_{i-1}$.

Algorithm: $\forall$ positive integer N, we will express N in the above base by the following recursive algorithm. Of course, the algorithm does this uniquely. We express $N = 1$ in the base by $N = a_1 = 1$. Suppose we have expressed $1, 2, 3, \cdots$, $N - 1$ in the base by the algorithm where $N - 1 \geq 1$, we now wish to express N in the base.

If $N = a_i$ or $N = b_i$ or $N = c_i$ for some i, we agree that N is already expressed in the base. Otherwise suppose $c_i < N < a_{i+1}$ for some i.

Let us now write $N - c_i + (N - c_i)$. By lemma 1, we know that $i \leq N - c_i < a_{i-1}$. Now by induction we know that the recursive algorithm has expressed $N - c_i = \phi_{i_2} + \phi_{i_3} + \cdots + \phi_{i_4}$ where $i - 1 > i_2 > i_3 > \cdots > i_4$ and where each $\phi_{i_j} \in \left\{a_{i_j}, b_{i_j}, c_{i_j}\right\}$. Calling $c_i = \phi_{i_1}$, we have $N = \phi_{i_1} + \phi_{i_2} + \cdots \phi_{i_t}$, where $i_1 > i_2 > i_3 > \cdots > i_t$ and where each $\phi_{i_j} \in \left\{a_{i_j}, b_{i1_j}, c_{i_j}\right\}$. The reader can note that $\phi_{it} \in \{a_{it}, b_{it}, c_{it}\}$. However all of the <u>other</u> $\phi'_{ij}S$ must equal c_{i_j}. We now prove a second lemma for the base of the game (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3),$\cdots$.

<u>Lemma 2</u>: $\forall i \geq 3$, it is true that

1. $2 \cdot a_i \geq c_{i+1}$ and

2. $2 \cdot c_{i-1} < c_{i+1}$.

<u>Proof</u>: Obviously, this is true for $i = 3$. Therefore, suppose $i \geq 4$. Since $i \geq 4, a_{i+1} = c_i + a_{i=1}$. Therefore, $a_{i+1} = a_i + a_{i-1} + 2$ and $c_{i+1} = a_i + a_{i-1} + 4$.

Now $2a_i \geq c_{i+1}$ is true if $2a_i \geq a_i + a_{i-1} + 4$ if $a_i \geq a_{i-1} + 4$, which is true since $i \geq 4$.

Now $2c_{i-1} < c_{i+1}$ is true if $2 \cdot (a_{i-1} + 2) < a_i + a_{i-1} + 4$ if $a_{i-1} < a_i$, which is always true.

<u>Safe and unsafe positions:</u>

For all positive integers N, X, the position (N, X) is determined to be safe or unsafe by the following rules.

First, express $N = Q_{i_1} + Q_{i_2} + \cdots + Q_{i_4}$, where $i_1 > i_2 > i_3 > \cdots i_t$, by the above algorithm,

1. If $Q_{i_t} = a_1 = 1$, then (N, X) is unsafe if $x \geq 3$ and safe if $1 \leq X \leq 2$.

2. If $Q_{i_t} = b_1 = 2$, then (N, X) is unsafe if $X \geq 3$ and safe if $1 \leq X \leq 2$.

3. If $Q_{i_t} = c_1 = 3$ then (N, X) is unsafe if $X \geq 4$ and safe if $1 \leq X \leq 3$.

4. If $Q_{i_t} = a_2 = 4$, then (N, X) is unsafe if $X \geq 5$ and safe if $1 \leq X \leq 4$.

5. if $Q_{i_t} = b_2 = 5$, then (N, X) is unsafe if $X \geq 6$ and safe if $1 \leq X \leq 5$.

6. If $Q_{i_t} \in \{c_2, a_3, b_3, c_3, a_4, b_4, c_4, a_5, b_5, c_5, \cdots\}$, then (N, X) is unsafe if $X \geq Q_{i_t}$ and safe if $1 \leq X \leq Q_{i_t} - 1$.

Remark: Note that we have only stated the safe and unsafe positions of the game. We have not stated the strategy of the game. The strategy of the game will be studied after the proof of the safe and unsafe positions.

Proof of safe and unsafe positions: Let us first plot the ordered pairs (N, X), where $N \in \{-3 - 1, 0, 1, 2, 3, \cdots\}$ and $X \in \{1, 2, 3, \cdots\}$, as Cartesian co-ordinates in the usual way with N plotted as the abscissa and X plotted as the ordinate. Note that X is being plotted vertically.

For each such pair (N, X), we will assign (N, X) the value of $F(N, X) \in \{0, 1\}$, where $F(N, X) = 0$ if (N, X) is a safe position and $F(N, X) = 1$ if (N, X) is an unsafe position.

All $(-2,X)'s, (-1,X)'s, (0,X)'s, (N,1)'s$ and $(N,2)'s$ will be assigned the value of 0 since by the rules of the game they will be safe terminal positions. These will be the only terminal positions that we will need to include in order to completely analyze the game.

Now when $N \geq 1, X \geq N+2$ we know that $F(N,X) = 1$. This is because the first moving player can subtract at least one of $N+2, N = 1, N$ from N and win since the first blocking player can only block two of these three moves.

Remember, the games is over as soon as the integer N becomes non-positive and the winner is the one who makes the last move in the game. So we can imagine that all ordered pairs $(N,X), N, X \in \{1,2,3,\cdots\}, X \geq N+2$, have already been assigned the value of 1 in addition to the $(-2,X)'s, (-1,X)'s, (0,X)'s, (N,1)'s$ and $(N,2)'s$ which have already been assigned the value of 0. We must now compute $F(N,X)$ for the other $(N,X)'s$. We will compute $F(N,X)$, where $N \geq 2, 3 \leq X \leq N+1$, recursively as follows. $(XXX)F(N,X) = 1$ when the list $F(N-1,2), F(N-2,4), F(N-3,6), \cdots F(N-X,2X)$ contains at least three $0's$. It must contain at least three $0's$ because otherwise the blocking player can block all moves that go to two or fewer $0's$. $F(N,X) = 0$ when this list contains at most two $0's$. We compute the $F(2,X)'s, X = 1,2,3,4,\cdots$, first. Then we compute the $F(3,X)'s$, then the $F(4,X)'s$, etc.

Note that for a fixed $N \in \{1, 2, 3, \cdots\}$ and a variable $X \in \{1, 2, 3, \cdots\}$, the infinite sequence $F(N, 1) = 0, F(N, 2) \leq 0, F(N, 3), F(N, 4), \cdots$ always consists of a finite string of consecutive $0's$ followed by an infinite string of consecutive $1's$. This is true by (XXX) because once this sequence first switches from 0 to 1 it must always retain the value of 1 thereafter. For each fixed $N \in \{1, 2, 3, \cdots\}$, let us define $g(N)$ to be the smallest $X \in \{1, 2, 3, \cdots\}$ such that $F(N, X) = 1$. Since $F(N, 2) = 0$ and $F(N, N+2) = 1$, we know that $3 \leq g(N) \leq N+2$. We note that once we know the value of $g(N)$ then by the definition of $g(N)$ we know that $F(N, X) = 0$ when $1 \leq X \leq g(N) - 1$ and $F(N, X) = 1$ when $g(N) \leq X$.

For any $N \in \{1, 2, 3, \cdots\}$, it is easy to see that $g(N)$ is the smallest positive integer having the property that exactly three members of the following list have a value of 0: $F(N-1, 2), F(N-2, 4), F(N-3, 6), \cdots, F(N-g(N), 2g(N))$. We will now prove lemma 3 for the base $(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3), \cdots$. This will complete the proof of the safe and unsafe positions.

Lemma 3: 1. $g(a_1) = g(1) = 3, g(b_1) = g(2) = 3,$

$g(c_1) = g(3) = 4, g(a_2) = g(4) = 5,$

$g(b_2) = g(5) = 6,$

$g(c_2) = g(6) = 6.$

2. $\forall i \geq 3, g(a_i) = a_i, g(b_i), g(c_i) - c_i$

3. $\forall N \in \{1, 2, 3, 4, \cdots\} \setminus \{a_1 b_1 b_1 a_2 b_2 c_2 a_3 b_3 c_3, \cdots\}$, (i.e. $\forall$ positive

integer N that is not a member of the base), $3 \leq g(N) < N$.

4. $\forall$ positive integer N, if $c_i < N < a_{i+1}$, then $N - c_i + (N - c_i)$, where $1 \leq N - c_i < a_{i-1}$, and $g(N) = g(N - c_i)$.

Remark: Once we have proved Lemma 3, the proof of the safe and unsafe positions follows readily. Remember, $\forall$ positive integer N, we write $N = \phi_{i_1} + \phi_{i_2} + \phi_{i_3} + \cdots + \phi_{i_t}$, where the $\phi'_{i_j}s$ are determined by the algorithm and where $i_1 > i_2 > i_3 > \cdots i_t$. Remember, all $\phi'_{i_j}S$ equal c_{i_j}except possibly ϕ_{i_t} which can equal either $a_{i_t}, b_{i_t}, c_{i_t}$. Now from condition 4 of lemma 3 and from the way the algorithm specified the $\phi'_{i_j}s$, we see that $g(N) = g(\phi_{i_1} + (\phi_{i_2} + \cdots + \phi_{i_t})) = g(\phi_{i_2} + (\phi_{i_3} + \cdots + \phi_{i_t})) = g(\phi_{i_3}, (\phi_{i_4} + \cdots + \phi_{i_t})) = \cdots = g(\phi_{i_t})$. using conditions $1, 2$ of lemma 3 with $g(\phi_{i_t})$ along with the definition of g completes the proof of the safe and unsafe positions.

Proof of Lemma 3: Note at the start that Lemma 1 takes care of the statement in condition 4 that $1 \leq N - c_i < a_{i-1.}$In Fig.2, we have computed the value of

$F(N, X)$ for a few values of N.

Note that conditions 1,2,3,4 of Lemma 3 are obviously true for the $N's$ in Fig. 2. So we can now use induction on N.

Let us now suppose that conditions 1,2 are true for all $a_i \in \{a_1, a_2, a_3, \cdots, a_k\}, b_i \in \{b_1, b_2, b_3, \cdots, b_k\}, c_i \in \{c_1, c_2, c_3, \cdots, c_k\}$ and conditions 3,4 are true for all.

$N \in \{1, 2, 3, 4, \cdots, a_k, b_k, c_k\} \setminus \{a_1, b_1, c_1, \cdots, a_k, b_k, c_k\}$, where $k \geq 4$. We now show that condition 2 is true for $a_{\ i} = a_{k+1}, b_{\ i} = b_{k+1}, c_i = c_{k+1}$ and conditions 3,4 are true for all $N \in \{c_k + 1, c_k + 2, c_k + 3, \cdots, a_{k+1} - 1 = c_k + a_{k-1} - 1\}$.

Note that $a_{k+1} = c_k + a_{k-1}$ since $k \geq 4$. Let us illustrate this in Fig.3. In Fig.3 note first that condition 2 of Lemma 3 is true for $a_{k-1}, b_{k-1}, c_{k-1}, a_k, b_k, c_k$ since $k \geq 4$. Also carefully observe how integer lines $A, 1, 2, 3, 4, A', 1', 2', 3', 4'$ are defined in the drawing. For example, line A is the set of integer points on the upward vertical line through $(0, 0)$. Line 1 is the set of integer points $\{(t - 1, 2), (t - 2, 4), (t - 36), \cdots\}$. Line 2 is the set of integer points $\{(a_{k-1} - 1, 2), (a_{k-1} - 2, 2), (a_{k-1} - 3, 6), \cdots\}$.

Let us deal first with $N \in \{c_k + 1, c_k + 2, \cdots, a_{k+1} - 1 = c_k + a_{k-1} - 1\}$. For these $N's$, we need to prove conditions 3,4 of lemma 3. Of course, once we prove condition 4, condition 3 will immediately follow from this. This is because if $N = c_k + (N - c_k)$, where $1 \leq N - c_k < a_{k-1}$and $g(N) = g(N - c_{k)}$, then the following is true. We combine conditions 1,2,3 of lemma 3 with our just mentioned inductive assumption about k and the fact that $1 \leq N - c_k < a_{k-1} < c_k$ to see that $g(N) = g(N - c_k) \leq (N - c_k) + 2 < N$. So let us now prove condition 4. Reread the definition of $g(N)$ before continuing. Since $k \geq 4$,we know that $g(a_k) = a_k \geq a_4 = 12, g(b_k) = b_k \geq b_4 = 13$ and $g(c_k) = c_k \geq c_4 = 14$.

From this, it is easy to see from Fig.3 and the definition of g that $g(c_k + 1) = g(1) = 3, g(c_k + 2) = g(2) = 3, g(c_k + 3) = g(3) = 4, g(c_k + 4) = g(4) = 5$ and

$g(c_k+5)=g(5)=6$. Therefore, suppose we have proved condition 4 for all $N \in \{c_k+1, c_k+2, \cdots, c_k+t-1\}$, where $5 \leq + -1 \leq a_{k-1}-2$. We now prove that condition 4 is met for $N = c_k + t$, where $6 \leq + \leq a_{k-1}-1$. This means we know that $g(1)=g(c_k+1), g(2)=g(c_k+2), \cdots, g(t-1)=g(c_k+t-1)$, and we wish to prove $g(t)=g(c_k+t)$. Now $g(t)$ is the smallest positive integer X such that the list $F(t-1,2), F(F2,4), F(F3,6), \cdots, F(t-X,2X)$ contains exactly three $0's$.

Also, $g(c_k+t)$ is the smallest positive integer X such that the list $F(c_k+t-1,2), F(c_k+t-2,4), F(c_k+t-3,6), \cdots F(c_k+t-X,2X)$ contains exactly three $0's$.

Now since we are assuming that $g(1)=g(c_k+1), g(2)=g(c_k+2), g(3)=g(c_k+3), \cdots, g(t-1)=g(c_k+t=1)$, we know that the above two lists must be identical as long as $1 \leq X \leq t-1$. This follows from the definition of g since $g(N)$ tells us that $F(N,X)=0$ when $1 \leq X \leq g(N)-1$ and $F(N,X)=1$ when $g(N) \leq X$. This is illustrated by line1 and line $1'$ in Fig.3.

Of course, $t \notin \{a_1, b_1, c_1, a_2, b_2\}$ since $t \geq 6$. Now if $t \notin \{c_2, a_3, b_3, c_3, a_4, b_4, c_4, \cdots a_{k-2}, b_{k-2}, c_{k-2}\}$, by the inductive assumption on k we know that $g(t) < t$, this is condition 3 of lemma 3. This readily tells us that $g(t)=g(c_k+t)$. This is because the list $F(t-1,2), F(t-2,4), \cdots, F(t-X,2X)$ will contain three $0's$ <u>before</u> X reaches $X=t$, and this lists is identical to the second list $F(c_k+t-1,2), F(c_k+t-2,4), \cdots, F(c_k+t-X,2X)$ as long as $1 \leq X \leq t-1$. Now if $t \in \{c_2, a_3, b_3, c_3, a_4, b_4, c_4, \cdots, a_{k-2}, b_{k=2}, c_{k=2}\}$, by the inductive assumption on k, we know that $g(t) = t$ from condition 2 of lemma 3 and the fact that $g(c_2) = c_2$. Let us now study lines 1 and 1′ in Fig.3. We know by lemma 2 that $\forall_i \geq 3, 2 \cdot c_{i=1} < c_{i+1}$. Therefore, since $k \geq 4$, we know that $2 \cdot c_{k-2} < c_k$.

If $t \in \{c_2, a_3, b_3, c_3, a_4, b_4, c_4, \cdots, a_{k-2}, b_{k=2}, c_{k=2}\}$, then since $c_2 < a_3 < b_3 < c_3 < a_4 < b_4 < c_4, < \cdots$ we know that $2t \leq 2c_{k-2}$ and $2c_{k-2} < c_k$ implies $2t < c_k$. This means that line 1′ in Fig.3 will intersect line A' at a $0F$-value the same way that line 1 intersects line A at a $0F$-value. Combining this with the assumption that we are making on t, we know that the F-values along line 1 and 1′ are identical up to and including the $0F$-values at the intersections of line 1 and A and lines 1′ and A'. Since $g(t) = t$, we know that the $0F$-value at the intersection of lines 1 and A is the third $0F$-value on line 1′. Therefore, the $0F$-value at the intersection of lines 1′ and A' is the third $0F$-value on line 1′. $g(c_k + t) = g(t) = t$ must be true by the definition of g. Of course, $g(c_k + t) = g(t)$ is what we wished to prove. Let us now deal with $a_{k+1}, b_{k+1}, c_{k+1}$, we wish to prove that condition 2 of lemma 3 is true for $a_{k+1}, b_{k+1}, c_{k+1}$. Let us first show that $g(a_{k+1}) = a_{k+1}$, we know by the inductive assumption on k that $g(a_{k-1}) = a_{k-1}$ since $k \geq 4$. In Fig.3, the F-values of the points along line 2 are used to compute $g(a_{k-1})$. Since $g(a_{k-1}) = a_{k-1}$, we know from the definition of g that the $0F$-value at the point of g that the $0F$-value at the point of intersection of line 2 and line A must be the third 0 on line 2.

Now $a_{k+1} = c_k + a_{k-1}$ means that $a_{k-1} = a_{k+1} - c_k$. We know from lemma 2 that $2a_{k-1} \geq c_k$ since $k \geq 4$. We use the F-values along line $2'$ to compute $g(a_{k+1})$. Since we have proved $g(t) = g(c_k + t)$ for all $t = 1, 2, 3, \cdots, a_{k-1} - 1$, we know by the definition of g that the F-values along line $2'$ must be identical to the F-values of the corresponding points along line 2 right up until line 2 intersects line A and line $2'$ intersects line A'. Now since $2a_{k-1} \geq c_k, a_{k-1} = a_{k+1} - c_k$ and $g(c_k) = c_k$, we know that line $2'$ intersects line A' at a $1F$-value. This means that line $2'$ has exactly two $0's$ on it up to and including the point where it intersects line A' since line 2 has exactly three $0's$ on it up to and including the point where it intersects line A. Now as line $2'$ continues to rise upward to the left, it will encounter more $1F$-values right up until it intersects line A. At the intersection of line $2'$ and line A, we find a third $0F$-value on line $2'$. Therefore, from the definition of g, we know that $g(a_{k+1}) = a_{k+1}$. Next, we will show that $g(b_{k+1}) = b_{k+1}$. In Fig.3, the F-values of the points along line 3 are used to compute $g(b_{k-1})$. We know $g(b_{k-1}) = b_{k-1}$ since $k \geq 4$. Since$g(b_{k-1}) = b_{k-1}$, we know from the definition of g that the $0F$-value at the point of intersection of line 3 and line A must be the third $0F$-value along line 3.

Now $b_{k-1} = b_{k+1} - c_k$ since $a_{k-1} = a_{k+1} - c_k$. Of course, since $2a_{k-1} \geq c_k$, we know that $2b_{k-1} \geq c_k$. We use the F-values along line $3'$ to compute $g(b_{k+1})$.

We have proved that $g(1) = g(c_k + 1), g(2) = g(c_k + 2), \cdots g(a_{k-1} - 1) = g(c_k +$

$a_{k-1}-1) = g(a_{k+1}-1)$. Also, we know $g(a_{k+1}) = a_{k+1} \geq 21$, since $k \geq 4$. From this we easily see that the F-values of the points along line 3 must be identical to the F-values of the corresponding points along line 3 right up until line 3 intersects line A' and line 3 intersects line A. Now since $2b_{k-1} > c_k, b_{k-1} = b_{k+1} - c_k$and $g(c_k) = c_k$, we know that line $3'$ intersects line A' at a $1F$-value . Of course, line 3 intersects line A at a $0F$-value which is the third $0F$-value on line 3. This means that line $3'$ must have exactly two $0F$-values on it up to and including the point where line $3'$ continues to rise upward to the left, it will encounter more $1F$-values until it intersects line A. At the intersection of line 3 and line A we find the third $0F$-value on line $3'$. Therefore, from the definition of g, we know that $g(b_{k+1}) = b_{k+1}$.

Last, let us show that $g(c_{k+1}) = c_{k+1}$. In Fig.3, the F-values of the points along line 4 are used to compute $g(c_{k-1})$. Since $g(c_{k-1}) = c_{k-1}$, we know from the definition of g that the $0F$-value at the point of intersection of line 4 and line A must be the third $0F$-value along line 4.

Now $c_{k-1} = c_{k+1} - c_k$ since $a_{k-1} = a_{k+1} - c_k$. Of course, since $2a_{k-1} \geq c_k$, we know that $2c_{k-1} > c_k$. We use the F-values along line $4'$ to compute $g(c_{k+1})$. We have proved $g(1) = g(c_k+1), g(2) = g(c_k+2), \cdots g(a_{k-1}-1) = g(c_k+a_{k-1}-1) = g(a_{k-1}-1)$ in addition to $g(a_{k+1}) = a_{k+1} \geq 21$, since $k \geq 4$, and $g(b_{k+1}) = b_{k+1} \geq 22$, since $k \geq 4$. Also $g(a_{k-1}) = a_{k-1} \geq 7$, since $k \geq 4$, and $g(b_{k-1}) = b_{k-1} \geq 8$, since $k \geq 4$. From this we see that the F-values of the points along line 4 must be identical to the F-values of the corresponding points along line 4 right up until line $4'$ intersects line A and line 4 intersects line A. Now since $2c_{k-1} > c_k, c_{k-1} = c_{k+1} - c_k$and $g(c_k) = c_k$, we know that line $4'$ intersects line A' at a $1F$-value. Of course, line 4 intersects line A at a $0F$-value, which is the third $0F$-value on line 4, this means that line 4 must have exactly two $0F$-values on it up to and including the point where line $4'$ intersects line A'. Now as line $4'$ continues to rise upward to the left, it will encounter more $1F$-values until it intersects line A. At the intersection of line $4'$ and line A we find the third $0F$-value on line $4'$. Therefore, from the definition of g we know that $g(c_{k+1}) = c_{k+1}$.

The next part of this paper deals with the actual strategy. Before, we only dealt with the safe and unsafe positions but not the strategy.

Definition: Let N be a positive integer. Let us call the following finite sequence the strategy sequence for N : $F(N-1,2), F(N-2,4), F(N-3,6), \cdots, F(N-g(N), 2g(N))$. Of course, from the definition of $g(N)$, we know that $F(N-g(N), 2g(N)$ is the third 0 in this sequence.

The next part of this paper shows how to compute the strategy sequence for each positive integer N. Once we have the strategy sequences for all N, the strategy (both moving and blocking) is almost self-evident. Note, as an example that the strategy sequence for $N = 12$ is $0,1,0,1,1,1,1,1,1,1,1,0$. Suppose $(N, X) = (12, 6)$. Now this position is a safe position, and before the moving player moves, the blocking player will block the subtraction of 1 and 3. He does this because the first two $0's$ in the strategy sequence occupy the first and third positions. This forces the moving player to move to a 1(or unsafe position). On the other hand, if $(N, X) = (12, 13)$, we see that (N, X) is an unsafe position and no matter how the blocking player moves, the moving player can always move to a 0 position in the sequence since the third 0 in the sequence occupies the $12^{\underline{th}}$ position and $12 < 13$.

Lemma 4: Suppose $c_i < N < a_{i+1}$, where N is an integer. Of course, from lemmas 1, 3 we know that $1 \le N - c_i < a_{i-1}$ and $g(N) = g(N - c_i)$. We now prove that the strategy sequence for N is identical to the strategy sequence for $N - c_i$. That is, the following two sequences are identical:

(a) $F(N-1, 2), F(N-2, 4), F(N-3, 6), \cdots, F(N - g(N), 2g(N))$

(b) $F(N - c_i - 1, 2), F(N - c_i - 2, 4), F(N - c_i - 3, 6), \cdots, F(N - c_i - g(N - c_i), 2g(N - c_i))$

Proof: Basically all we have to do is go through the entire proof of lemma 3 paying special attention to the proof of condition 4.

First, observe in Fig.2 that $N = 10, 11$ are the only $N's$ in Fig.2 that apply to lemma 4. Of course, $c_3 = 9 < 10 < 11 < a_4 = 12$.

Now $g(10) = g(1) = 3$ and $< F(10-1, 2), F(10-2, 4), F(10-3, 6) > = < 0, 0, 0 >$

Also, $< F(1-1, 2), F < 1-2, 4), F(1-3, 6) > = < 0, 0, 0 >$. So the strategy sequences for 10 and 1 are identical.

Now $g(11) = g(2) = 3'$.

Also, $< F(11-1,2), F(11-2,4), F(11-3,6) >=< 0,0,0 >$.

Also, $< F(2-1,2), F(2-2,4), F(2-3,6) =< 0,0,0 >$. So the strategy sequences for 11 and 2 are identical.

As in lemma 3, this starts the induction. As we continue through the rest of the proof of lemma 3 with an emphasis on condition 4, we also easily see that the two strategy sequences (a) and (b) are always identical, especially note lines $1, 1'$ in Fig.3.

Lemma 5: Suppose N is any arbitrary positive integer. Using the previous algorithm to express N in the base of the game, we have $N = \phi_{i_1} + \phi_{i_2} + \phi_{i_3} + \cdots + \phi_{i_t}$, when $i_1 > i_2 > i_3 > \cdots i_t$.

Recall that $\phi_{i_t} \in \{a_{i_t}, b_{i_t}, c_{i_t}\}$, but $\phi_{i_j} = c_{i_t}$ for the other $i_j's$.

From the way that the algorithm specified the $\phi i_j's$ and from the fact the $\phi_{i_1} = c_{i_1}, \phi_{i_2} = c_{i_2}, \cdots, \phi_{i_{t-1}} = c_{i_{t-1}}, \phi_{i_t} \in \{a_{i_t}, b_{i_t}, c_{i_t}\}$, we see that lemma 4 can be used over and over to prove that the strategy sequence for N is identical to the strategy sequence for ϕ_{i_t}.

Lemma 5 combined with lemma 6 will now complete the computation of the strategy sequences for the positive integers.

Lemma 6: The strategy sequence for $a_1 = 1$ is $0,0,0$. The strategy sequence for $b_1 = 2$ is $0,0,0$.

The strategy sequence for $c_1 = 3$ is $0, 1, 0, 0$. The strategy sequence for $a_2 = 4$ is $0, 1, 1, 0, 0$. The strategy sequence for $b_2 = 5$ is $0, 1, 1, 1, 0, 0$. The strategy sequence for $c_2 = 6$ is $0, 0, 1, 1, 1, 0$. $\forall_i \in \{4, 6, 8, 10, 12, 14, \cdots\}$, the strategy sequence for a_i is $010111111 \cdots 10$. The last 0 in the sequence is the $a_i\underline{th}$ member of the sequence, $\forall_i \in \{4, 6, 8, 10, 12, 14, \cdots\}$, the strategy sequence for b_i is $0110111111 \cdots 10$. The last 0 is the $b_i\underline{th}$ member of the sequence. $\forall_i \in \{4, 6, 8, 10, 12, 14, \cdots\}$, the strategy sequence for c_i is $00111111 \cdots 10$. The last 0 is the $c_i\underline{th}$ member of the sequence. $\forall_i \in \{3, 5, 7, 9, 11, 13, \cdots\}$, the strategy sequence for a_i is $001111111 \cdots 10$. The last 0 in the sequence is the $a_i\underline{th}$ member of the sequence.$\forall_i \in \{3, 5, 7, 9, 11, 13, \cdots\}$, the strategy sequence for b_i is $001111111 \cdots 10$. The last 0 in the sequence is the $b_i\underline{th}$ member of the sequence. $\forall_i \in \{3, 5, 7, 9, 11, 13, \cdots\}$, the strategy sequence for c_i is $001111111 \cdots 10$. The last 0 is the $c_i\underline{th}$ member of the sequence.

Proof: The proof for $a_1, b_1, c_1, a_2, b_2, c_2$, is trivial and uses Fig.2.

The proof for $i \in \{4, 6, 8, 10, 12, \cdots\}$ and $i \in \{3, 5, 7, 9, 11, \cdots\}$ consists of first showing that the above strategy sequences are correct for a_4, b_4, c_4 and a_3, b_3, c_3. The reader can check this by using Fig.2. The rest of the proof consists of observing Fig.3 that was used in the proof of lemma 3. We observe in Fig.3 that lines $2, 2'$, lines $3, 3'$ and line $4, 4'$ are analogous. This means that the F-values along line 2 and $2'$ are identical in the beginning. The same is true for $3, 3'$ and $4, 4'$. Further observe that lines $2, 3, 4$ are associated with $a_{k-1}, b_{k-1}, c_{k-1}$ while lines $2', 3', 4'$ are associated with $a_{k+1}, b_{k+1}, c_{k+1}$. Since $(k+1)-(k-1)=2$, this explains why the strategy sequences for $i \in \{4,6,8,10,12,14,\cdots\}$ are similar and the strategy sequences for $i \in \{3,5,7,9,11,\cdots\}$ are similar. This concludes our analysis of the first game.

Rules of Second Game: The rules of the second game are identical to the rules of the first game with the following changes:

(a) Instead of $f(n)=2n$, in this game we use $f(n)=4n$. Thus a move in this game is $(N,X) \to (N-k, 4k), 1 \le k \le X$, subject to the blocking rule.

(b) On each turn, before the moving player moves, the opposing player can block up to one of the moving player's moves including the option of not blocking any moves at all. In the first game, the blocking player could block up to two moves.

The game ends as soon as N becomes non-positive, and the winner is the one who makes the last move in the game. If the game starts at $(N,X)=(N,1)$, we agree that the second player (i.e. the 2nd moving player) is the winner since by blocking the subtraction of $X=1$ the first blocking player can prevent the first moving player from moving. We now state without proof lemmas for the second game that are analogous to the corresponding lemmas for the first game.

Base: Let us define the sequence $(a_1,b_1),(a_2,c_2,b_2),(a_3,b_3),(a_4,a_4),(a_5,c_5,b_5),\cdots$ as follows, we call this sequence the base of the game.

1. If $i \in \{2,5,6,7,8,11,12,13,14,17,18,19,20,\cdots\}$, we use (a_i,c_i,b_i) for this i.

2. If $i \in \{1,3,4,9,10,15,16,21,22,27,28,33,34,\cdots\}$, we use (a_i,b_i) for this i.

Note that aside from (a_i, b_i),(a_2, c_2, b_2), we always have two successive $(a_i, b_i)'s$ followed by four successive $(a_i, c_i, b_i)'s$ followed by two successive $(a_i, b_i)'s$ followed by four successive $(a_i, c_i, b_i)'s$ followed by two successive $(a_i, b_i)'s$, etc.

3. In all $(a_i, b_i)'s$, we define $b_i = a_i + 1$.

4. In all $(a_i, c_i, b_i)'s$, we define $c_i = a_i + 1, b_i = c_i + 1$.

5. $(a_i, b_i) = (1, 2), (a_2, c_2, b_2) = (3, 4, 5)$,

   $(a_3, b_3) = (6, 7), (a_4, b_4) = (8, 9)$,

   $(a_5, c_5, b_5) =)10, 11, 12), (a_6, c_6, b_6) = (13, 14, 15)$,

   $(a_{-1}, c_{-1}, b_{-1}) = (17, 18, 19), (a_8, c_8, b_8) = (22, 23, 24)$,

   $(a_9, b_9) = (29, 30)$.

6. $\forall_i \geq 10, a_i = b_{i-1} + a_{i-6}$.

   Thus a few more terms are

   $(a_{10}, b_{10}) = (38, 39), (a_{11}, c_{11}, b_{11}) = (49, 50, 51)$,

   $(a_{12}, c_{12}, b_{12}) = (64.65.66), (a_{13}, c_{13}, b_{13}) = (83, 84, 85) \cdots$

   Obviously $a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < \cdots$.

Lemma 1′: Suppose $b_i < N < a_{i+1}$, where N is an integer, then $N - b_i < a_{i-4}$. Also if $i \geq 8$ then $N - b_i < a_{i-5}$.

The proof of this is easy to see.

Lemma 2′:

1. $\forall_i \geq 3$, it is true that $49_i \geq b_{i+5}$.

2. $\forall_i \geq 2$, it is true that $49_{i-1} \geq b_{i+5}$.

The definition of g in the second game is the same as in the first game.

Definition: Suppose for some i we have $(a_i, c_i, b_i), (a_{i+1}, c_{i+1}, b_{i+1}), (a_{i+2}, c_{i+2}, b_{i+2}), (a_{i+3}, c_{i+3}, b_{i+3})$. Remember, the triples occur in foursomes. We say that (a_i, c_i, b_i) is the $1^{\underline{st}}$ member of its foursome, $(a_{i+1}, c_{i+1}, b_{i+1})$ is the $2^{\underline{nd}}$ member of its foursome, $(a_{i+2}, c_{i+2}, b_{i+2})$ is the $3^{\underline{rd}}$ member of its foursome and $(a_{i+3}, c_{i+3}, b_{i+3})$ is the $4^{\underline{th}}$ member of its foursome.

The following lemma $3'$ brings the game under control.

Lemma $3'$:

1. $g(a_1) = g(1) = 2, g(b_1) = g(2) = 3,$

   $g(a_2) = g(3) = 4, g(c_2) = g(4) = 5,$

   $g(b_2) = g(5) = 5.$

2. $\forall_i \geq 3, g(a_i) = a_i, g(b_i) = b_i$.

3. $\forall_i \geq 3$, if for this i we have (a_i, c_i, b_i), then $g(c_i)$ is computed as follows. $G(c_i) = 2$ if (a_i, c_i, b_i) is the $1^{\underline{st}}$ or $2^{\underline{nd}}$ member of its foursome. $G(c_i) = 3$ if (a_i, c_i, b_i) is the $3^{\underline{rd}}$ member of its foursome, and $g(c_i) = 4$ if (a_i, c_i, b_i) is the $4^{\underline{th}}$ member of its foursome.

4. $\forall$ positive integer N if N is not a member of the base, then $2 \leq g(N) < N$.

5. $\forall$ positive integer N if $b_i < N < a_{i+1}$, then $N = b_i + (N - b_i)$, where $1 \leq N - b_i < a_{i-4}$ and $1 \leq N - b_i < a_{i+5}$ when $i \geq 8$

a) If for this i, we have (a_i, b_i), then $g(N) = g(N - b_i)$.

b) If for this i, we have (a_i, c_i, b_i), then $g(N) = g(N - b_i)$, when $6 \leq N - b_i$

and $g(N) = 3$ if $N - b_i = 1$,

$g(N) = 4$ if $N - b_i = 2$,

$g(N) = 5$ if $N - b_i = 3$,

$g(N) = 4$ if $N - b_i = 4$,

$g(N) =$ if $N - b_i = 5$,

Conclusion: As stated in the abstract, we believe there are endless examples of these move size dynamic single pile blocking games that can be brought under control. However, after observing the two games specified in this paper, it seems very unlikely that anyone can achieve the massive consolidation that was accomplished with the non-blocking version, this means that methods very similar to the methods used in this paper must be used ad-hoc to analyze these games on a case by case basis. As another example, the reader might like to analyze the game where the opposing player can block one move and $\in \{1, 2, 3, 4, \cdots\} \rightarrow \{1, 2, 3, 4, \cdots\}$ is specified as follows: $\in (n) = \frac{3}{2}n$, when n is even, and $\in (n) = \frac{3n-1}{2}$ when n is odd.

Finally, the Misere version can also be analyzed by methods that are similar to the methods used in this paper. Also, the rules used in this paper can be modified, e.g. not allowing the moving player to overshoot $N = 0$.

References

[1] Berlekamp E., J. Conway, and R. Guy, Winning Ways, Academic Press, New York, 1982.

[2] Richard K. Guy, Fair Game, 2nd. ed., COMAP, New York, 1989.

[3] Holshouser, Reiter, and Rudzinski, *(Last Move) Dynamic One-Pile Nim*, submitted.

[4] Holshouser, Reiter, and Rudzinski, *Dynamic One-Pile Nim*, submitted.

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