Win-loss Sequences for Multiple Round Tournaments

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Abstract Each of n teams numbered $1, 2, \cdots, n$ play each of the other $n-1$ teams exactly t times. Thus, each team plays $t(n-1)$ games, and the total number of games is $tc_2^n = \frac{tn(n-1)}{2}$. Each game $\{a, b\}$ produces a win for one team and a loss for the other team. Define $a_i, i = 1, 2, \cdots, n$, to be the win records for the n teams. That is, for each $i = 1, 2, \cdots, n$, team i wins a total of a_i games where $0 \le a_i \le t(n-1)$. Of course, $\sum_{i=1}^{n} a_i = \frac{tn(n-1)}{2}$.

Suppose $a_i, i = 1, 2, \cdots, n$, are arbitrarily specified win records for the teams $1, 2, \cdots, n$ subject only to the two conditions (1) $0 \le a_i \le t(n-1)$ and (2) $\sum_{i=1}^{n} a_i = \frac{tn(n-1)}{2}$.

In this paper, we prove necessary and sufficient conditions that $a_i, i = 1, 2, \cdots, n$, must satisfy so that $a_i, i = 1, 2, \cdots, n$, is realizable. In [1] we solved this problem for the special case $t = 1$, and in this paper we solve the general case by reducing it to this special case $t = 1$. We have not come even remotely close to solving the general case by modifying the proof given in [1].

1. Finding necessary conditions on $a_i, i = 1, 2, \cdots, n$.

   Suppose for $1 \le k \le n$ we choose any combination $\{n_1, n_2, \cdots, n_k\}$ of k teams from the collection of n teams. Now these k teams play $t \cdot C_2^k = \frac{tk(k-1)}{2}$ games among themselves. Therefore, the total number of wins among themselves for these k teams equals $\frac{tk(k-1)}{2}$.

   Also, each of the k teams plays each of the $n-k$ remaining teams t times for a total of $tk(n-k)$ games. Therefore, $(3')$ is a necessary condition. $(3')$. For each $1 \le k \le n$, any combination of k teams $\{n_1, n_2, \cdots, n_k\}$ must satisfy $\sum_{i=1}^{k} a_{n_i} \le \frac{tk(k-1)}{2} + tk(n-k) = \frac{tk}{2}(2n-k-1)$.

   If we agree to write $t(n-1) \ge a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$, then the above necessary condition $(3')$ is equivalent to the following (3).

   (3). $\forall k \in \{1, 2, \cdots, n\}, \sum_{i=1}^{k} a_i \le \frac{tk}{2}(2n-k-1)$.

Note 1 The following condition (3^*) is also obviously necessary. However, condition (3^*) is not used in this paper, and we leave it as an easy exercise for the reader to prove that conditions $\{(1),(2),(3^*)\}$ are equivalent to conditions $\{(1),(2),(3')\}$.

(3^*) $\forall k\in\{1,2,\cdots,n\},\forall\{n_1,n_2,\cdots,n_k\}\subseteq\{1,2,\cdots,n\}\; tc_2^k=\frac{tk(k-1)}{2}\leq\sum_{i=1}^{k}a_{n_i}$

2. Stating the necessary and sufficient conditions on $a_i, i=1,2,\cdots,n$. In this paper, we prove that the conditions developed in section 1 are both necessary and sufficient for $a_i, i=1,2,\cdots,n$, to be realizable.

   Writing $t(n-1)\geq a_1\geq a_2,\cdots,\geq a_n\geq 0$, this means that we prove the following conditions (1), (2), (3) are both necessary and sufficient for $a_i, i=1,2,\cdots,n$, to be realizable.

   1. $0\leq a_i\leq t(n-1), i=1,2,\cdots,n.$
   2. $\sum_{i=1}^{n}a_i=\frac{tn(n-1)}{2}.$
   3. $\forall k\in\{1,2,\cdots,n\}\sum_{i=1}^{k}a_i\leq\frac{tk}{2}(2n-k-1)$

   As stated in section 1, (3) is equivalent to $(3')$ which we write as $(3)\leftrightarrow(3')$.

   $(3')$ $\forall k\in\{1,2,\cdots,n\},\forall\{n_1,n_2,\cdots,n_k\}\subseteq\{1,2,\cdots,n\},\sum_{i=1}^{k}a_{n_i}\leq\frac{tk}{2}(2n-k-1)$

3. Plan for proving the if and only if conditions on $a_i, i=1,2,\cdots,n$.

   In the paper [2], we proved that (1), (2), (3) are necessary and sufficient for $a_i, i=1,2,\cdots,n$, to be realizable when $t=1$. At this time, it seems to us to be a hopeless task to modify the proof given in [2] to take care of the general case where t is arbitrary. So we will solve the general case by breaking up the general case in such a way that we can use the solution for $t=1$ that is given in [2] to solve the general case. This means that we need to solve the following problem.

Problem 1 *t and n are arbitrary but fixed positive integers and $a_1\geq a_2,\cdots,\geq a_n\geq 0$ are non-negative integers that satisfy the following conditions.*

1. $\forall i\in\{1,2,\cdots,n\},0\leq a_i\leq t(n-1).$
2. $\sum_{i=1}^{n}a_i=t\cdot C_2^n=\frac{tn(n-1)}{2}.$
3. $\forall k\in\{1,2,\cdots,n\},\sum_{i=1}^{k}a_i\leq tF(k)$ *where* $F(k)=\frac{k}{2}(2n-k-1).$

Using this hypothesis, we wish to find t. Sequences $a_{\theta 1}, a_{\theta 2}, \cdots, a_{\theta n}, \theta = 1, 2, \cdots, t$, of non-negative integers that satisfy condition (4) as well as conditions (1), (2), (3′).

It turns out that it is not convenient to require $a_{\theta 1} \geq a_{\theta 2} \geq \cdots \geq a_{\theta n} \geq \theta$, and this is the reason that we will be using (3′) instead of (3) in the list below.

4. $\forall i \in \{1, 2, \cdots, n\}, \sum_{\theta=1}^{t} a_{\theta i} = a_i$.

(1) $\forall \theta \in \{1, 2, \cdots, t\}, \forall i \in \{1, 2, \cdots, n\}, 0 \leq a_{\theta i} \leq n - 1$.

(2) $\forall \theta \in \{1, 2, \cdots, t\}, \sum_{i=1}^{n} a_{\theta i} = C_2^n = \frac{n(n-1)}{2}$.

(3′) $\forall \theta \in \{1, 2, \cdots, t\}, \forall k \in \{1, 2, \cdots, n\}, \forall \{n_1, n_2, \cdots, n_k\} \subseteq \{1, 2, \cdots, n\}, \sum_{i=1}^{k} a_{\theta, n_i} \leq F(k) = \frac{k}{2}(2n - k - 1)$.

Of course, each of these t sequences $a_{\theta i}, i = 1, 2, \cdots, n$, satisfies the hypothesis (1), (2), (3)↔(3′) of this paper when $t = 1$, and the paper [1] shows that (1), (2), (3)↔(3′) are necessary and sufficient conditions for each win sequence $a_{\theta i}, i = 1, 2, \cdots, n$, to be realizable when $t = 1$. After, we use [1] to deal with each sequence $a_{\theta 1}, a_{\theta 2}, \cdots, a_{\theta n}, \theta \in \{1, 2, \cdots, t\}$, we use condition (4) to put all of these t sequences together which solves the main problem of this paper.

4 Lemmas needed to solve Problem 1.

We first prove Lemma 1. Lemma 1, however, is not very convenient for solving Problem 1 since it requires the rearranging of terms.

We then prove the trivial Lemma 2. Lemma 2 then allows us to state Lemma 1 as Lemma 3. Lemma 3 does not require the rearranging of terms, and this is very convenient when we solve Problem 1. For completeness, we also state the trivial companion Lemma 3. Lemmas 3, 3′ will be the main machinery that we use to solve Problem 1.

Lemma 2 *Suppose $2 \leq \bar{t} \leq n$ are fixed positive integers, and k is a variable positive integer that satisfies $1 \leq k \leq \bar{t} - 1$. As always, define $F(k) = \frac{k}{2}(2n - k - 1)$.*

Suppose $a_1, a_2, \cdots, a_{\bar{t}} \geq 0$ are non-negative integers.

Also, $\forall k \in \{1, 2, \cdots, \bar{t} - 1\}, \sum_{i=1}^{k} a_i \leq F(k)$ and $\sum_{i=1}^{\bar{t}} a_i < F(\bar{t})$.

Define $a_{\bar{t}}^* = a_{\bar{t}} + 1$. Also define $\{\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_{\bar{t}}\} = \{a_1, a_2, \cdots, a_{\bar{t}-1}, a_{\bar{t}}^*\}$ with $\bar{a}_1 \geq \bar{a}_2 \geq \cdots \geq \bar{a}_{\bar{t}}$. Then (a) $\forall k \in \{1, 2, \cdots, \bar{t}\}, \sum_{i=1}^{k} \bar{a}_i \leq F(k)$.

Proof. Of course, if $a_1 \geq a_2 \cdots \geq a_{\bar{t}-1} \geq a_{\bar{t}}^* = a_{\bar{t}} + 1$, then there is nothing to prove since $\bar{a}_i = a_i, i = 1, 2, \cdots, \bar{t} - 1, \bar{a}_{\bar{t}} = a_{\bar{t}}^* = a_{\bar{t}} + 1$ and $\sum_{i=1}^{\bar{t}} a_i < F(\bar{t})$.

Therefore, suppose that $a_1 \geq a_2 \geq, \cdots, \geq a_{r-1} > a_r = a_{r+1} = a_{r+2} = \cdots = a_{\bar{t}-1} = a_{\bar{t}}$ where $1 \leq r \leq \bar{t} - 1$.

Let us call $a_r = a_{r+1} = a_{r+2} = \cdots = a_{\bar{t}-1} = a_{\bar{t}} = w$.

Then $\overline{a}_1 = a_1, \overline{a}_2 = a_2, \cdots, \overline{a}_{r-1} = a_{r-1}, \overline{a}_r = w + 1$ and $\overline{a}_{r+1} = \overline{a}_{r+2} = \cdots = \overline{a}_{\bar{t}} = w$.

We must show that $\forall \theta \in \left\{r, r+1, \cdots, \bar{t}\right\}, (*) \left(\sum_{i=1}^{\theta} a_i\right) + 1 = \sum_{i=1}^{\theta} \overline{a}_i \leq F(\theta)$.

Now by hypothesis we know that $(*)$ is true when $\theta = \bar{t}$ since $\sum_{i=1}^{\bar{t}} a_i < F\left(\bar{t}\right)$ and, of course, $\sum_{i=1}^{\bar{t}} \overline{a}_i = \left(\sum_{i=1}^{\bar{t}} a_i\right) + 1 \leq F\left(\bar{t}\right)$.

Therefore, we must show that $(*)$ is true for $r \leq \theta \leq \bar{t} - 1$.

We do this by showing that $\forall r \leq \theta \leq \bar{t} - 1$, it is impossible for all of (1), (2), (3) to be true. (1), (3) are true by hypothesis,

1. $\sum_{i=1}^{\theta-1} a_i \leq F(\theta - 1)$.

2. $\sum_{i=1}^{\theta} a_i = F(\theta)$.

3. $\sum_{i=1}^{\bar{t}} a_i < F\left(\bar{t}\right)$.

Of course, we know by hypothesis that $\sum_{i=1}^{\theta} a_i \leq F(\theta)$. So if $\sum_{i=1}^{\theta} a_i < F(\theta)$ is true, then we know that $(*)$ is true. This is why we are assuming that (2) is true.

Now (2) is equivalent to $\left(\sum_{i=1}^{\theta-1} a_i\right) + w = F(\theta)$.

Therefore, using (1), we see that (a) $w = F(\theta) - \left(\sum_{i=1}^{\theta-1} a_i\right) \geq F(\theta) - F(\theta - 1)$.

Also, (3) is equivalent to $\left(\sum_{i=1}^{\theta} a_i\right) + \left(\sum_{i=\theta+1}^{\bar{t}} a_i\right) < F\left(\bar{t}\right)$ which is equivalent to $F(\theta) + \left(\bar{t} - \theta\right) w < F\left(\bar{t}\right)$.

This is equivalent to (b) $\left(\bar{t} - \theta\right) w < F\left(\bar{t}\right) - F(\theta)$.

Using (a) and copying (b) we see that (c) $\left(\bar{t} - \theta\right)(F(\theta) - F(\theta - 1)) \leq \left(\bar{t} - \theta\right) w < F\left(\bar{t}\right) - F(\theta)$.

We show that (c) is impossible by showing that (d) $F\left(\bar{t}\right) - F(\theta) \leq \left(\bar{t} - \theta\right)(F(\theta) - F(\theta - 1))$ when $r \leq \theta \leq \bar{t} - 1 \leq n - 1$.

Using the definition of F, we see that (d) is true if and only if $\frac{\bar{t}}{2}\left(2n - \bar{t} - 1\right) - \frac{\theta}{2}(2n - \theta - 1) \leq \left(\bar{t} - \theta\right)\left[\frac{\theta}{2}(2n - \theta - 1) - \left(\frac{\theta-1}{2}\right)(2n - \theta)\right]$.

This is true if and only if $\bar{t}\left(2n - \bar{t} - 1\right) - \theta(2n - \theta - 1) \leq \left(\bar{t} - \theta\right)(2n - 2\theta)$. This is true if and only if $2n\bar{t} - \bar{t}^2 - \bar{t} - 2n\theta + \theta^2 + \theta \leq 2n\bar{t} - 2\theta\bar{t} - 2n\theta + 2\theta^2$ which is true if and only if $0 \leq \bar{t}^2 + \bar{t} + \theta^2 - \theta - 2\theta\bar{t}$.

Now $\theta \leq \bar{t}-1$ means that $\bar{t}=\theta+1+\phi$, where $\phi \geq 0$. Therefore, we show that $0 \leq (\theta+1+\phi)^2 + (\theta+1+\phi) + \theta^2 - \theta - 2\theta(\theta+1+\phi)$. This is true if and only if $0 \leq \theta^2 + 1 + \phi^2 + 2\theta + 2\phi + 2\theta\phi + \theta + 1 + \phi + \theta^2 - \theta - 2\theta^2 - 2\theta - 2\theta\phi$.

This is true if and only if $0 \leq \phi^2 + 3\phi + 2$ which is true. ■

Lemma 3 *Suppose $\bar{a}_1 \geq \bar{a}_2 \geq \cdots \geq \bar{a}_{\bar{t}} \geq 0$ are non-negative integers.*

Then the following two conditions are equivalent.

(a) $\forall k \in \left\{1,2,\cdots,\bar{t}\right\}, \sum\limits_{i=1}^{k} \bar{a}_i \leq F(k)$.

(b) $\forall k \in \left\{1,2,\cdots,\bar{t}\right\}, \forall \{n_1,n_2,\cdots,n_k\} \subseteq \left\{1,2,\cdots,\bar{t}\right\}, \sum\limits_{i=1}^{k} \bar{a}_{n_i} \leq F(k)$.

Proof. (b) obviously implies (a). Also (a) implies (b) since $\bar{a}_1 \geq \bar{a}_2 \geq \cdots \geq \bar{a}_{\bar{t}}$ implies $\sum\limits_{i=1}^{k} \bar{a}_{n_i} \leq \sum\limits_{i=1}^{k} \bar{a}_i$. ■

Observation 1 Since Lemma 2 states that (a) and (b) are equivalent, in Lemma 1 we do not need to rearrange the terms $\left[a_1,a_2,\cdots,a_{\bar{t}-1},a^*_{\bar{t}}\right]$ to define, $\{\bar{a}_1,\bar{a}_2,\bar{a}_3,\cdots,\bar{a}_{\bar{t}}\} = \left\{a_1,a_2,,a_{\bar{t}-1},a^*_{\bar{t}}\right\}$ with $\bar{a}_1 \geq \bar{a}_2 \geq \bar{a}_3 \geq \cdots \geq \bar{a}_{\bar{t}}$. All we have to do is use (b) instead of (a) and then we do not have to rearrange anything. This becomes much more convenient when we solve problem 1.

Lemma 1 can bow be stated as Lemma 3. For completeness we also state the trivial companion Lemma 3′.

Lemma 4 *Suppose $2 \leq \bar{t} \leq n$ are fixed positive integers and k is a variable positive integer that satisfies $1 \leq k \leq \bar{t}-1$. As always, define $F(k) = \frac{k}{2}(2n-k-1)$.*

Suppose $a_1,a_2,\cdots,a_{\bar{t}-1},a_{\bar{t}}$ are non-negative integers and $\min\{a_1,a_2,\cdots,a_{\bar{t}-1}\} \geq a_{\bar{t}}$ *where* $\min\{\}$ *is the smallest member of the set.*

Also, suppose (b) $\forall k \in \left\{1,2,\cdots,\bar{t}-1\right\}, \forall \{n_1,n_2,\cdots,n_k\} \subseteq \left\{1,2,\cdots,\bar{t}-1\right\}, \sum\limits_{i=1}^{k} a_{n_i} \leq F(k)$.

Also, suppose $\sum\limits_{i=1}^{\bar{t}} a_i < F(\bar{t})$ *which, of course, is equivalent to* $\left(\sum\limits_{i=1}^{\bar{t}-1} a_i\right) + (a_{\bar{t}}+1) \leq F(\bar{t})$.

Define $\bar{a}_1 = a_1, \bar{a}_2 = a_2, \cdots, \bar{a}_{\bar{t}-1} = a_{\bar{t}-1}, \bar{a}_{\bar{t}} = a_{\bar{t}}+1$. *Then (b′)*$\forall k \in \left\{1,2,\cdots,\bar{t}\right\}, \forall \{n_1,n_2,\cdots,n_k\} \subseteq \left\{1,2,3,\cdots\bar{t}\right\}, \sum\limits_{i=1}^{\bar{t}} \bar{a}_{n_i} \leq F(k)$.

Lemma 3′ Suppose $2 \leq t \leq n$ are fixed positive integers and k is a variable positive integer that satisfies $1 \leq k \leq \bar{t}-1$. As always, define $F(k) = \frac{k}{2}(2n-k-1)$.

Suppose $a_1,a_2,\cdots,a_{\bar{t}-1},a_{\bar{t}}$ are non-negative integers and $\min\{a_1,a_2,\cdots,a_{\bar{t}-1}\} \geq a_{\bar{t}}$.

Also, suppose (b) $\forall k \in \left\{1,2,\cdots,\bar{t}-1\right\}, \forall \{n_1,n_2,\cdots,n_k\} \subseteq \left\{1,2,\cdots\bar{t}-1\right\}, \sum\limits_{i=1}^{k} a_{n_i} \leq F(k)$.

Also, suppose $\sum_{i=1}^{\bar{t}} a_i \leq F(k)$.

Then (b′) $\forall k \in \{1, 2, \cdots, \bar{t}\}, \forall \{n_1, n_2, \cdots, n_k\} \subseteq \{1, 2, 3, \cdots \bar{t}\}, \sum_{i=1}^{k} a_{n_i} \leq F(k)$.

Lemmas 3, 3′ along with an inequality $(***)$ that we soon develop will be our main machinery for solving Problem 1.

5. Solving Problem 1.

   Using the hypothesis of Problem 1, let us first observe that the t identical sequences $\frac{a_1}{t} \geq \frac{a_2}{t} \geq \frac{a_3}{t} \geq \cdots \geq \frac{a_n}{t}$ satisfy the following (1), (2), (3), (4).

   (1) $0 \leq \frac{a_i}{t} \leq n - 1, i - 1, 2, \cdots, n$.

   (2) $\sum_{i=1}^{n} \frac{a_i}{t} = C_2^n$.

   (3) $\forall k \in \{1, 2, \cdots, n\}, \sum_{i=1}^{k} \frac{a_i}{t} \leq F(k) = \frac{k}{2}(2n - k - 1)$.

   (4) $\forall i \in \{1, 2, \cdots, n\}, a_i = \frac{a_i}{t} + \frac{a_i}{t} + \cdots + \frac{a_i}{t}\,(t - time)$.

   Of course, if $\frac{a_i}{t}$ is an integer for all $i = 1, 2, \cdots, n$ then the solution to Problem 1 is obvious. However, in general not all of the $\frac{a_i}{t}$'s will be integers. Our plan is to modify this sequence to form t sequences which together satisfy the 4 conditions required in Problem 1.

Notation 5 *$\forall i \in \{1, 2, \cdots, n\}$, let $\frac{a_i}{t} = \left\lfloor \frac{a_i}{t} \right\rfloor + \frac{R_i}{t}$ where $\lfloor\rfloor$ is the floor function and $0 \leq R_i < t$ is an integer. Of course, $\left\lfloor \frac{a_i}{t} \right\rfloor$ is the quotient and R_i is the remainder when a_i is divided by t. Also, of course, $\left\lfloor \frac{a_1}{t} \right\rfloor \geq \left\lfloor \frac{a_2}{t} \right\rfloor \geq \cdots \geq \left\lfloor \frac{a_n}{t} \right\rfloor$.*

Now $\forall k \in \{1, 2, \cdots, n\}, \sum_{i=1}^{k} \frac{a_i}{t} = \left(\sum_{i=1}^{k} \left\lfloor \frac{a_i}{t} \right\rfloor\right) + \left(\sum_{i=1}^{k} \left\lfloor \frac{R_i}{t} \right\rfloor\right) \leq F(k)$.

This implies $\forall k \in \{1, 2, \cdots, n\}, \left(\sum_{i=1}^{k} \left\lfloor \frac{a_i}{t} \right\rfloor\right) \leq F(k) - \sum_{i=1}^{k} \frac{R_i}{t}$.

$\forall k \in \{1, 2, \cdots, n\}$, define b_k to be the non-negative integer satisfying $b_k - 1 < \sum_{i=1}^{k} \frac{R_i}{t} \leq b_k$.

Then since $F(k)$ and $\sum_{i=1}^{k} \left\lfloor \frac{a_i}{t} \right\rfloor$ are both integers, we see that $(*) \sum_{i=1}^{k} \left\lfloor \frac{a_i}{t} \right\rfloor \leq F(k) - b_k$.

We also easily see that $\sum_{i=1}^{n} \frac{R_i}{t}$ is an <u>integer</u> since $C_2^n = \sum_{i=1}^{n} \frac{a_i}{t} = \left(\sum_{i=1}^{n} \frac{a_i}{t}\right) + \left(\sum_{i=1}^{n} \frac{R_i}{t}\right)$ and C_2^n and $\sum_{i=1}^{k} \left\lfloor \frac{a_i}{t} \right\rfloor$ are both integers.

Let us now point out that in set theory a set can be specified in two ways. We can specify the set explicitly using the language of set theory or we can specify the set by showing the pattern. Thus, $\{x : x \text{ is a positive integer}\} = \{1, 2, 3, 4, 5, 6, 7, \cdots\}$.

In this paper, we define our basic algorithm by using the second method since the first method is very confusing while the second method is not confusing at all.

Let us now start with the following t identical sequences.

1. $\left\lfloor \frac{\alpha_1}{t} \right\rfloor \geq \left\lfloor \frac{\alpha_2}{t} \right\rfloor \geq \left\lfloor \frac{\alpha_3}{t} \right\rfloor \geq \left\lfloor \frac{\alpha_4}{t} \right\rfloor \geq \cdots \geq \left\lfloor \frac{\alpha_n}{t} \right\rfloor.$

2. $\left\lfloor \frac{\alpha_1}{t} \right\rfloor \geq \left\lfloor \frac{\alpha_2}{t} \right\rfloor \geq \left\lfloor \frac{\alpha_3}{t} \right\rfloor \geq \left\lfloor \frac{\alpha_4}{t} \right\rfloor \geq \cdots \geq \left\lfloor \frac{\alpha_n}{t} \right\rfloor \cdots$

t. $\left\lfloor \frac{\alpha_1}{t} \right\rfloor \geq \left\lfloor \frac{\alpha_2}{t} \right\rfloor \geq \left\lfloor \frac{\alpha_3}{t} \right\rfloor \geq \left\lfloor \frac{\alpha_4}{t} \right\rfloor \geq \cdots \geq \left\lfloor \frac{\alpha_n}{t} \right\rfloor.$

We will now increase by 1 some of the members of these sequences $1, 2, \cdots, t$ to befine t new sequences. We will then use Lemma 3, 3′ and the inequality $(***)$ that we soon develop to prove that these t new sequences satisfy conditions (1), (2), (3′), (4) of Problem 1. We will add 1's according to the following pattern, and we illustrate the complete pattern by using $t = 7, n = 6, R_1 = 3, R_2 = 2, R_3 = 3, R_4 = 4, R_5 = 5, R_6 = 4$. We observe that $\sum_{i=1}^{n=6} \frac{R_i}{t} = \frac{3+2+3+4+5+4}{7} = 3$ which is an integer as we know it must be. This illustration obviously defines the general pattern for arbitrary θ, t, n.

1. $\left\lfloor \frac{\alpha_1}{t} \right\rfloor + 1 \left\lfloor \frac{\alpha_2}{t} \right\rfloor \left\lfloor \frac{\alpha_3}{t} \right\rfloor + 1 \left\lfloor \frac{\alpha_4}{t} \right\rfloor \left\lfloor \frac{\alpha_n}{t} \right\rfloor + 1 \left\lfloor \frac{\alpha_n}{t} \right\rfloor$

2. $\left\lfloor \frac{\alpha_1}{t} \right\rfloor + 1 \left\lfloor \frac{\alpha_2}{t} \right\rfloor \left\lfloor \frac{\alpha_3}{t} \right\rfloor \left\lfloor \frac{\alpha_4}{t} \right\rfloor + 1 \left\lfloor \frac{\alpha_n}{t} \right\rfloor + 1 \left\lfloor \frac{\alpha_n}{t} \right\rfloor$

3. $\left\lfloor \frac{\alpha_1}{t} \right\rfloor + 1 \left\lfloor \frac{\alpha_2}{t} \right\rfloor \left\lfloor \frac{\alpha_3}{t} \right\rfloor \left\lfloor \frac{\alpha_4}{t} \right\rfloor + 1 \left\lfloor \frac{\alpha_n}{t} \right\rfloor + 1 \left\lfloor \frac{\alpha_n}{t} \right\rfloor$

4. $\left\lfloor \frac{\alpha_1}{t} \right\rfloor \left\lfloor \frac{\alpha_2}{t} \right\rfloor + 1 \left\lfloor \frac{\alpha_3}{t} \right\rfloor \left\lfloor \frac{\alpha_4}{t} \right\rfloor + 1 \left\lfloor \frac{\alpha_n}{t} \right\rfloor \left\lfloor \frac{\alpha_n}{t} \right\rfloor + 1$

5. $\left\lfloor \frac{\alpha_1}{t} \right\rfloor \left\lfloor \frac{\alpha_2}{t} \right\rfloor + 1 \left\lfloor \frac{\alpha_3}{t} \right\rfloor \left\lfloor \frac{\alpha_4}{t} \right\rfloor + 1 \left\lfloor \frac{\alpha_n}{t} \right\rfloor \left\lfloor \frac{\alpha_n}{t} \right\rfloor + 1$

6. $\left\lfloor \frac{\alpha_1}{t} \right\rfloor \left\lfloor \frac{\alpha_2}{t} \right\rfloor \left\lfloor \frac{\alpha_3}{t} \right\rfloor + 1 \left\lfloor \frac{\alpha_4}{t} \right\rfloor \left\lfloor \frac{\alpha_n}{t} \right\rfloor + 1 \left\lfloor \frac{\alpha_n}{t} \right\rfloor + 1$

7. $\left\lfloor \frac{\alpha_1}{t} \right\rfloor \left\lfloor \frac{\alpha_2}{t} \right\rfloor \left\lfloor \frac{\alpha_3}{t} \right\rfloor + 1 \left\lfloor \frac{\alpha_4}{t} \right\rfloor \left\lfloor \frac{\alpha_n}{t} \right\rfloor + 1 \left\lfloor \frac{\alpha_n}{t} \right\rfloor + 1$

We point out that if $R_i = 0$, then in column i we don't add 1 to any of the members of that column.

The pattern that we have used should now be self-explanatory, and this example should make the general pattern clear for an arbitrary number of rows, t, an arbitrary number of columns, n, and for arbitrary integers $R_1, R_2, \cdots, R_n$ subject only to $0 \leq R_i < t, i = 1, 2, \cdots, n$, and $t | \sum_{i=1}^{n} R_i$. In this example, let us note in column 3 that we have added 1 to the bottom two $\left\lfloor \frac{\alpha_3}{t} \right\rfloor$'s and then we add 1 to the top $\left\lfloor \frac{\alpha_3}{t} \right\rfloor$. We have now defined the following 7 sequences and in general we use this same pattern to define the t sequences $\{a_{\theta_i} : i = 1, 2, \cdots, n\}$ as θ varies over $\theta = 1, 2, \cdots, t$.

1. $a_{11} \quad a_{12} \quad a_{13} \quad a_{14} \quad a_{15} \quad a_{16}$

2. a_{21} a_{22} a_{23} a_{24} a_{25} a_{26}

3. a_{31} a_{32} a_{33} a_{34} a_{35} a_{36}

4. a_{41} a_{42} a_{43} a_{44} a_{45} a_{46}

5. a_{51} a_{52} a_{53} a_{54} a_{55} a_{56}

6. a_{61} a_{62} a_{63} a_{64} a_{65} a_{66}

7. a_{71} a_{72} a_{73} a_{74} a_{75} a_{76}

In general it is obvious from the way that the algorithm for adding 1*'s has been defined that the following is true for each row* $\theta \in \{1, 2, \cdots, t\}$*. In row* θ*, suppose exactly* r *of the members* $\left\lfloor \frac{\alpha_1}{t} \right\rfloor, \left\lfloor \frac{\alpha_2}{t} \right\rfloor, \left\lfloor \frac{\alpha_3}{t} \right\rfloor, \cdots, \left\lfloor \frac{\alpha_k}{t} \right\rfloor$ *have been increased by* 1 *where* $0 \le r \le k \le n$*.*

Then the only way that this can happen is for $(**)\, r - 1 < \sum_{i=1}^{n} \frac{R_i}{t} \le r + 1$*. To see this, first note that if* $\sum_{i=1}^{k} \frac{R_i}{t} < r - 1$*, then in row* θ *at most* $r - 1$ *of the members* $\left\lfloor \frac{\alpha_1}{t} \right\rfloor, \left\lfloor \frac{\alpha_2}{t} \right\rfloor, \left\lfloor \frac{\alpha_3}{t} \right\rfloor, \cdots, \left\lfloor \frac{\alpha_k}{t} \right\rfloor$ *could have been increased by* 1*. Also, if* $r + 1 \le \sum_{i=1}^{k} \frac{R_i}{t}$*, then in row* θ *at least* $r + 1$ *of the members* $\left\lfloor \frac{\alpha_1}{t} \right\rfloor, \left\lfloor \frac{\alpha_2}{t} \right\rfloor, \left\lfloor \frac{\alpha_3}{t} \right\rfloor, \cdots, \left\lfloor \frac{\alpha_k}{t} \right\rfloor$ *must have been increased by* 1*. Using inequality* $(**)$ *with inequality* $(*)$*, we see that* $(**)$ *implies* $\sum_{i=1}^{k} \left\lfloor \frac{a_i}{t} \right\rfloor \le F(k) - r$*. to see this, note that the inequality* $(**)$ *implies that in the hypothesis for inequality* $(*)$ *we must have either* $b_k = r$ *or* $b_k = r + 1$*.*

Applying this to the t *sequences* $\{a_{\theta i} : i = 1, 2, \cdots, n\}$*,* $\theta \in \{1, 2, \cdots, t\}$*, we see that* $(***)\, \forall \theta \in \{1, 2, \cdots, t\}, \forall k \in \{1, 2, \cdots, n\}, \sum_{i=1}^{k} a_{\theta i} \le F(k)$*. Inequality* $(***)$ *and Lemmas 3, 3′ will now be our main machinery. We now show that the 7 sequences* $\{a_{\theta i} : i = 1, 2, \cdots, 6\}, \theta \in \{1, 2, 3. \cdots, 7\}$ *and in general that* t *sequences* $\{a_{\theta i} : i = 1, 2, \cdots, n\}, \theta \in \{1, 2, \cdots, t\}$*, satisfy the four conditions (1), (2), (3′), (4) required in Problem 1. We first prove condition (4).*

Condition 4. We show (4). $\forall i \in \{1, 2, \cdots, n\}, \sum_{\theta=1}^{t} a_{\theta i} = a_i$.

Condition 6 *From the general pattern, we note that* $\forall i \in \{1, 2, \cdots, n\}$ *exactly* R_i *members of column* i *have been changed from* $\left\lfloor \frac{\alpha_i}{t} \right\rfloor$ *to* $\left\lfloor \frac{\alpha_i}{t} \right\rfloor + 1$*. Thus, the sum of the* t *members of column* i *goes from* $t \cdot \left\lfloor \frac{\alpha_i}{t} \right\rfloor$ *to* $t \cdot \left\lfloor \frac{\alpha_i}{t} \right\rfloor + R_i = a_i$ *since* $\frac{a_i}{t} = \left\lfloor \frac{\alpha_i}{t} \right\rfloor + \frac{R_i}{t}$*. This means that (4)* $\forall i \in \{1, 2, \cdots, n\}, \sum_{\theta=1}^{t} a_{\theta i} = a_i$*.*

Condition 1. We show (1). $\forall \theta \in \{1, 2, \cdots, t\}, \forall i \in \{1, 2, \cdots, n\}, 0 \le a_{\theta i} \le n - 1$. We consider two cases.

Case 1. Suppose $\frac{\alpha_i}{t}$ is an integer for some $i \in \{1, 2, \cdots, n\}$. Then $R_i = 0$ which means that $\forall \theta \in \{1, 2, \cdots, t\}, a_{\theta i} = \frac{a_i}{t}$.

Now by hypothesis (1) of Problem 1, $0 \leq a_i \leq t \cdot (n-1)$ which implies $0 \leq \frac{a_i}{t} \leq n-1$.

Case 2. Suppose $\frac{a_i}{t}$ is not an integer for some $i \in \{1, 2, \cdots, n\}$.

Then $1 \leq R_i < t$.

Also, $\forall \theta \in \{1, 2, \cdots, t\}, a_{\theta i} = \left\lfloor \frac{a_i}{t} \right\rfloor$ or $a_{\theta_i} = \left\lfloor \frac{a_i}{t} \right\rfloor + 1$. Now obviously $0 \leq a_{\theta i}$. We show that $a_{\theta i} \leq n-1$ by showing that $\left\lfloor \frac{a_i}{t} \right\rfloor + 1 \leq n-1$.

Now $a_i \leq t(n-1)$ and $\frac{a_i}{t}$ is not an integers $\left\lfloor \frac{a_i}{t} \right\rfloor < \frac{a_i}{t} \leq n-1$. This implies $\frac{a_i}{t} \leq n-2$ which implies $\left\lfloor \frac{a_i}{t} \right\rfloor + 1 \leq n-1$.

Condition 2. We show (2). $\forall \theta \in \{1, 2, \cdots, t\}, \sum\limits_{i=1}^{n} a_{\theta i} = C_2^n = \frac{n(n-1)}{2}$.

Now since $t | \sum\limits_{i=1}^{n} R_i$, we can see from the algorithm for defining the $a_{\theta i}$'s that the following is true.

In each row $\theta, \theta \in \{1, 2, \cdots, t\}$, exactly $\frac{1}{t} \cdot \sum\limits_{i=1}^{n} R_i$ of the members $\left\lfloor \frac{a_i}{t} \right\rfloor, i = 1, 2, \cdots, n$, of row θ have been increased by 1. Therefore, $\forall \theta \in \{1, 2, \cdots, t\}, \sum\limits_{i=1}^{n} a_{\theta i} = \left(\sum\limits_{i=1}^{n} \left\lfloor \frac{a_i}{t} \right\rfloor \right) + \left(\sum\limits_{i=1}^{n} \frac{R_i}{t} \right)$.

From hypothesis (2) of Problem 1, we see that $\sum\limits_{i=1}^{n} \frac{a_i}{t} = \frac{1}{t} \cdot \sum\limits_{i=1}^{n} a_i = \frac{1}{t} \cdot t C_2^n = C_2^n$.

Now since $\forall i \in \{1, 2, \cdots, n\}, \frac{a_i}{t} = \left\lfloor \frac{a_i}{t} \right\rfloor + \frac{R_i}{t}$, we see that $\sum\limits_{i=1}^{n} \frac{a_i}{t} = \left(\sum\limits_{i=1}^{n} \left\lfloor \frac{a_i}{t} \right\rfloor \right) + \left(\sum\limits_{i=1}^{n} \frac{R_i}{t} \right) = C_2^n$.

Therefore, $\sum\limits_{i=1}^{n} a_{\theta i} = C_2^n$.

Condition 3′. We show (3′). $\forall \theta \in \{1, 2, \cdots, t\}, \forall k \in \{1, 2, \cdots, n\}, \forall \{n_1, n_2, \cdots, n_k\} \subseteq \{1, 2, \cdots, n\}, \sum\limits_{i=1}^{k} a_{\theta, n_i} \leq F(k) = \frac{k}{2}(2n - k - 1)$.

Considering $\theta \in \{1, 2, \cdots, t\}$ to be arbitrary but fixed, we prove (3′) by proving the following sequentially for each $\overline{n} = 1, 2, 3, \cdots, n$. For each $\overline{n}$, we show that $\forall k \in \{1, 2, \cdots, \overline{n}\}$ and $\forall \{n_1 n_2, \cdots, n_k\} \subseteq \{1, 2, \cdots, \overline{n}\}, \sum\limits_{i=1}^{k} a_{\theta, n_i} \leq F(k) = \frac{k}{2}(2n - k - 1)$.

We use Lemma 3 or Lemma 3′ at each step, and we also use the fact that $\forall \theta \in \{1, 2, \cdots, t\}, \forall k \in \{1, 2, \cdots, n\}, (***) \sum\limits_{i=1}^{k} a_{\theta i} \leq F(k)$.

In the example that we have been using with $t = 7, n = 6, R_1 = 3, R_2 = 2, R_3 = 3, R_4 = R_5 = 5, R_6 = 4$, we will now go through the proof for row $\theta = 3$.

We will also explain the general theory at each step which gives the proof for arbitrary t, n. We recall that $\left\lfloor \frac{\alpha_1}{t} \right\rfloor \geq \left\lfloor \frac{\alpha_2}{t} \right\rfloor \geq \left\lfloor \frac{\alpha_3}{t} \right\rfloor \geq \cdots \geq \left\lfloor \frac{\alpha_n}{t} \right\rfloor$. Since each $a_{\theta i} = \left\lfloor \frac{\alpha_i}{t} \right\rfloor$ or $a_{\theta i} = \left\lfloor \frac{\alpha_i}{t} \right\rfloor = 1$, we see that $\forall i, j \in \{1, 2, \cdots, n\}$, if $j < i$ then $a_{\theta i} \geq \left\lfloor \frac{\alpha_i}{t} \right\rfloor$. Therefore, $\min \{a_{\theta 1}, a_{\theta 2}, \cdots, a_{i-1}\} \geq \left\lfloor \frac{\alpha_i}{t} \right\rfloor$.

<u>Row $\theta = 3$</u>. The members of row $\theta = 3$ are

$a_{31} = \left\lfloor \frac{\alpha_1}{t} \right\rfloor + 1, a_{32} = \left\lfloor \frac{\alpha_2}{t} \right\rfloor,$
$a_{33} = \left\lfloor \frac{\alpha_3}{t} \right\rfloor, a_{33} = \left\lfloor \frac{\alpha_4}{t} \right\rfloor + 1,$
$a_{35} = \left\lfloor \frac{\alpha_5}{t} \right\rfloor + 1, a_{36} = \left\lfloor \frac{\alpha_6}{t} \right\rfloor.$

As always, from $(***)$, $\sum\limits_{i=1}^{k=1} a_{3i} = a_{31} \leq F(1)$. Therefore, condition $3'$ trivially holds for all $k \in \{1\}$ and all $\{n_1, n_2, \cdots, n_k\} = \{n_1\} \in \{1\}$ since $\sum\limits_{i=1}^{k=1} a_{3i} = a_{31} \leq F(1)$.

Now $a_{32} = \left\lfloor \frac{\alpha_2}{t} \right\rfloor$. As always, from $(***)$, $a_{31}+a_{32} = a_{31}+\left\lfloor \frac{\alpha_2}{t} \right\rfloor \leq F(2)$. Now $\min\{a_{31}\} = a_{31} \geq \left\lfloor \frac{\alpha_2}{t} \right\rfloor = a_{32}$. Therefore, Lemma $3'$ shows that condition $3'$ holds for all $k \in \{1, 2\}$ and all $\{n_1, \cdots, n_k\} \subseteq \{1, 2\}$.

Now $a_{33} = \left\lfloor \frac{\alpha_3}{t} \right\rfloor$. As always, from $\int ***, a_{31} + a_{32} + a_{33} = a_{31} + a_{32} + \left\lfloor \frac{\alpha_3}{t} \right\rfloor \leq F(3)$.

Also, $\min\{a_{31}, a_{32}\} \geq \left\lfloor \frac{\alpha_3}{t} \right\rfloor = a_{33}$. Therefore, Lemma $3'$ shows that condition $3'$ holds for all $k \in \{1, 2, 3\}$ and all $\{n_1, \cdots, n_k\} \subseteq \{1, 2, 3\}$.

Now $a_{34} = \left\lfloor \frac{\alpha_4}{t} \right\rfloor + 1$. As always, from $(***)$, $a_{31} + a_{32} + a_{33} + a_{34} = a_{31} + a_{32} + a_{33} + \left(\left\lfloor \frac{\alpha_4}{t} \right\rfloor + 1\right) \leq F(4)$.

Therefore. $a_{31} + a_{32} + a_{33} + \left(\left\lfloor \frac{\alpha_4}{t} \right\rfloor\right) \leq F(4)$.

Also, $\min\{a_{31} + a_{32} + a_{33}\} \geq \left\lfloor \frac{\alpha_4}{t} \right\rfloor$

Therefore, Lemma 3 shows that condition $3'$ holds for all $k \in \{1, 2, 3, 4\}$ and all $\{n_1, n_2, \cdots, n_k\} \subseteq \{1, 2, 3, 4\}$.

Now $a_{35} = \left\lfloor \frac{\alpha_5}{t} \right\rfloor + 1$. As always, fro $(***)$, $a_{31} + a_{32} + a_{33} + a_{34} + a_{35} = a_{31} + a_{32} + a_{33} + a_{34} + \left(\left\lfloor \frac{\alpha_5}{t} \right\rfloor + 1\right) \leq F(5)$.

Therefore, $a_{31} + a_{32} + a_{33} + a_{34} + \left\lfloor \frac{\alpha_5}{t} \right\rfloor < F(5)$.

Also, $\min\{a_{31} + a_{32} + a_{33} + a_{34}\} \geq \left\lfloor \frac{\alpha_5}{t} \right\rfloor$.

Therefore, Lemma 3 shows that condition $3'$ holds for all $k \in \{1, 2, 3, 4, 5\}$ and all $\{n_1, n_2, n_3, n_4\} \subseteq \{1, 2, 3, 4, 5\}$.

Now $A_{36} = \left\lfloor \frac{\alpha_6}{t} \right\rfloor$. As always, from $(***)$, $a_{31} + a_{32} + a_{33} + a_{34} + a_{35} + a_{36} = a_{31} + a_{32} + a_{33} + a_{34} + a_{35} + \left\lfloor \frac{\alpha_6}{t} \right\rfloor \leq F(6)$.

Also, $\min\{a_{31} + a_{32} + a_{33} + a_{34} + a_{35}\} \geq \left\lfloor \frac{\alpha_6}{t} \right\rfloor = a_{36}$.

Therefore, Lemma $3'$ shows that condition $3'$ holds for all $k \in \{1, 2, 3, 4, 5, 6\}$ and all $\{n_1, n_2, \cdots, n_k\} \subseteq \{1, 2, \cdots, 6\}$ which is what we wished to prove for the sequence in row $\theta = 3$.

In the above proof, we observe the following general pattern when $a_{31} = \left\lfloor \frac{\alpha_i}{t} \right\rfloor$ we used Lemma $3'$ at that step in the proof. Also, when $a_{3i} = \left\lfloor \frac{\alpha_i}{t} \right\rfloor + 1$, we used Lemma 3 at that step in the proof. From the inequality $(***) \sum\limits_{i=1}^{\overline{n}} a_{\theta i} \leq F(\overline{n})$, from the fact that $\min\{a_{\theta 1}, a_{\theta 2}, \cdots, a_{\theta, \overline{n}-1}\} \geq \left\lfloor \frac{\alpha_{\overline{n}}}{t} \right\rfloor$, and from the fact that the induction on $\overline{n}$ has proved that condition $3'$ is true for $\{1, 2, \cdots, \overline{n}-1\}$, we know in general that Lemma 3 or Lemma $3'$ will prove that condition $3'$ is true for $\{1, 2, 3, \cdots, \overline{n}\}$. This gives the proof for arbitrary θ, t, n, and this completes the solution to Problem 1.

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