Is this Modified Nim Game a Game of Chance?

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Abstract. Basically there are two types of games, namely games that do and games that do not involve chance. Classical n-pile Bouton's nim is an example of a game that does not involve chance. The coin matching game is a game that does involve chance. In the coin matching game, two players Art and Beth have a single coin each. They independently decide whether to lay down heads or tails. After deciding they simultaneously lay down their coins. If the two coins are placed the same way (i.e., head-head, tail-tail) then Art wins, and if the coins are different then Beth wins. In this paper we will give a complete mathematical strategy for a modified n-pile nim game that appears to be a game of pure chance. Indeed, intuitively it appears that this modified nim game is not any different from the coin matching game. Also, at the end of the paper we will show that the parameters of this modified nim game can be changed to turn it into a game that does involve chance.

Notation. If $a \le b$ are integers then $[a, b] = \{x : x \text{ is an integer and } a \le x \le b\}$. Also, $(a, b] = [a, b] \setminus \{a\}$. Also, $N = \{0, 1, 2, 3, \cdots\}$. Arthur, I think we don't use this notation anywhere in the paper.

Game 1. Four integers m, M, t, and T are given integers that satisfy $m \le M, t \le T$, and $1 \le m + t$. Also, k piles of counters are placed on the table. Art and Beth alternate moving. Suppose it is Art's move. Art's move consists of the following four steps.

1. First, Art chooses any non-empty pile, and he tells Beth his choice.
2. Next, unknown to each other, Art writes down on a card any $x \in [m, M]$, while Beth writes down any $\overline{x} \in [t, T]$. Note that each player cannot see what the other player is writing down on his\her card. This step is the same as in the coin matching game.
3. Art and Beth simultaneously lay down their chosen $x, \overline{x}$.
4. Let n be the number of counters in the chosen pile. If $n > x + \overline{x}$ then Art removes exactly $x+\overline{x}$ counters from the chosen pile, leaving $n-x-\overline{x}$ counters. If $1 \le n \le x+\overline{x}$, Art removes all of the counters in the chosen pile which makes that pile empty.

   Since $1 \le m + t$ it follows that $1 \le x + \overline{x}$. Suppose it is Beth's move. Beth's move consists of the same four steps with Beth and Art switched. For example, on step 2, Beth chooses any $x \in [m, M]$ and Art chooses any $\overline{x} \in [t, T]$. In other words, the game is symmetric or impartial. The winner of the game is the player who makes the last move. That is, the winner is the player who removes the last counter.

The complete analysis of this game involves a concept called the generalized Sprague-Grundy values of a game. However, many readers will not be familiar with this concept. Therefore, in this paper we will assume that $1 \leq t = m \leq T = M$ since the analysis of this particular game does not involve a deep knowledge of Sprague-Grundy values.

Analysis. First consider a single pile of n counters where $0 \leq n$. Define a function $g : N \to \{0, 1\}$, called the generalized Sprague-Grundy function for the single pile as follows:

1. $g(0) = 0$,
2. $\forall n \geq 1$, suppose $n \in (p(m+M), (p+1)(m+M)]$ where $p \in N$ and $(a, b]$ is specified in notation 1.

   Then $g(n) = 1$ if p is even and $g(n) = 0$ if p is odd.

Suppose there are k piles and suppose these k piles have respectively $n_1, n_2, \cdots, n_k$ counters where each n_i satisfies $0 \leq n_i$.

Then we can denote this position as a multiset $\{n_1, n_2, \cdots, n_k\}$. For example, the position $\{0, 1, 2, 2, 3, 3, 3, 7\}$ means there are 8 piles and the pile sizes are $0, 1, 2, 2, 3, 3, 3$ and 7. Of course, $\{0, 1, 2, 2, 3, 3, 3, 7\}$ is the same position as $\{1, 2, 2, 3, 3, 3, 7\}$. Let us say that a position $\{n_1, n_2, \cdots, n_k\}$ is *balanced* if $\sum_{i=1}^{k} g(n_i)$ is even and *unbalanced* if $\sum_{i=1}^{k} g(n_i)$ is odd.

Suppose the game starts with the position $\{n_1, n_2, \cdots, n_k\}$. We will show that if $\{n_1, n_2, \cdots, n_k\}$ is unbalanced, then the player who moves first (Art) can always win with perfect play. Also, if $\{n_1, n_2, \cdots, n_k\}$ is balanced, then the player who moves second (Beth) can always win with perfect play.

First, observe that the game has only one terminal position which is $\{0, 0, 0, \cdots, 0\}$ and this terminal position is balanced.

Case 1. First, suppose $\{n_1, n_2, \cdots, n_k\}$ is unbalanced. We prove that the first moving player (Art) can move to a balanced position. Now $\sum_{i=1}^{k} g(n_i)$ is odd. This means that for some pile i with n_i counters, $g(n_i) = 1$. This means $n_i \in (p(m+M), (p+1)(m+M)]$ where p is even.

Art chooses such a pile. Now if $n_i \in (0, m+M]$, Art writes down $x = M$. No matter what $\overline{x} \in [m, M]$ the opposing player (Beth) writes down, we know that $x + \overline{x} \geq n_i$ which means that Art will empty the pile n_i. Since $g(0) = 0$ and $g(n_i) = 1$, we know that the game is balanced after Art moves.

Let us now consider $p(m+M) < n_i \leq (p+1)(m+M)$ with $p \geq 2$ and p even. We will consider two subcases, and we ignore subcase 2 when $m = M$.

Subcase 1. $n_i = p(m+M) + \theta, 1 \leq \theta \leq 2m,$

Subcase 2. $n_i = p(m+M) + \theta, 2m < \theta \le m + M$.

Subcase 1 The moving player (Art) writes down $x = m$ on his card. The opposing player (Beth) can write down any $\overline{x} \in [m, M]$ on her card. We now show that $n_i - x - \overline{x} \in ((p-1)(m+M), p(m+M)]$. Since $p-1$ is odd this implies $g(n_i - x - \overline{x}) = 0$, and this balances the game.

We must show that $(p-1)(m+M) < n_i - x - \overline{x} \le p(m+M)$. We first show that $n_i - x - \overline{x} \le p(m+M)$. Since $n_i = p(m+M) + \theta, 1 \le \theta \le 2m$, this inequality is true if and only if $\theta \le x + \overline{x}$, and this is true since $\theta \le 2m, x = m$ and $m \le \overline{x}$. Next, we show that $(p-1)(m+M) < n_i - x - \overline{x}$. This is true if and only if $x + \overline{x} < m + M + \theta$. This is true since $x = m, \overline{x} \le M$ and $1 \le \theta$.

Subcase 2 $n_i = p(m+M)+\theta, 2m < \theta \le m+M$. Let us write $n_i = p(m+M)+m+\phi, m < \phi \le M$. The moving player (Art) writes down ϕ on his card, and Beth can write down any $\overline{x} \in [m, M]$ on her card. We show that $n_i - x - \overline{x} \in ((p-1)(m+M), p(m+m)]$ which again implies that $g(n - x - \overline{x}) = 0$. This balances the game.

We must show that $(p-1)(m+M) < n_i - x - \overline{x} \le p(m+M)$. We first show that $n_i - x - \overline{x} \le p(m+M)$. Since $n_i = p(m+M)+m+\phi$, this is true if and only if $m+\phi \le x+\overline{x}$. This is true since $x = \phi, m \le \overline{x}$.

Next, we show $(p-1)(m+M) < n_i - x - \overline{x}$. This is true if and only if $x + \overline{x} < m + \phi + m + M$, which is true since $x = \phi, \overline{x} \le M, 1 \le m$.

Case 2. Last, suppose $\{n_1, n_2, \cdots, n_k\}$ is balanced. If $n_1 = n_2 = \cdots = n_k = 0$, the game is already over. So suppose at least one $n_i \ne 0$. We show that the opposing player (Beth) with perfect play can always force the moving player (Art) to move to an unbalanced position. First, Art chooses a nonempty pile i with $n_i \ne 0$.

We consider two subcases.

Subcase 1. $n_i \in (0, m+M]$. This means $g(n_i) = 1$.

Subcase 2. $n_i \in (p(m+M), (p+1)(m+M)], p \ge 1$.

In Subcase 1, Beth writes down $\overline{x} = M$ on her card. Then no matter what $x \in [m, M]$ Art writes down, Art must wipe out pile i. This changes $g(n_i) = 1$ to $g(0) = 0$ and forces the game to become unbalanced.

In Subcase 2, we do not care whether p is odd or even. We observe that Art and Beth are choosing numbers from the same internal $[m, M]$. This symmetry means that we can use the same analysis as in Case 1 to show that Beth can choose $\overline{x} \in [m, M]$ so that no matter what $x \in [m, M]$ Art chooses, it will be true that $n_i - x - \overline{x} \in ((p-1)(m+M), p(m+M)]$.

(Of course, in Case 1, p was always even, but this does not effect the reasoning.)

Since $p-1$ has the opposite parity of p, this implies that $g(n_i)$ and $g(n_i - x - \overline{x})$ will have opposite parities. Therefore, after the move, the game becomes unbalanced. Since the player who is destined to win can always force the $\sum_{i=1}^{k} g(n_i)$ values of the game to switch parity after each move and since $\sum_{i=1}^{k} g(0) = 0$ is even, it is obvious that the first moving player can win when the initial position is unbalanced and the second moving player can win when the initial position is balanced. ■

Modifying the parameters The rules of this game are the same as before with one change. This game is also symmetric or impartial. By symmetry suppose it is Art's turn to move. Then Art chooses any $x \in \{1, 4\}$ and Beth chooses any $\overline{x} \in \{1, 2\}$. Of course, if it is Beth's move, Beth chooses any $x \in \{1, 4\}$ and Art chooses any $\overline{x} \in \{1, 2\}$. The rest of the game is the same as before.

To prove that there is no complete strategy for this game, suppose that a single pile remains. Let 1's denote the winning positions for the first moving player and 0's denote the winning positions for the second moving player. We note that $n = 11$ is an uncertain position since the outcome depends on chance. Note that the moving player always wants to land on a 0, and the opponent of the moving player always wants to force him to land on a 1.

0 1 1 1 1 1 0 0 0 1 1 ?

Fig. 1

Misère Game The misère game of the games specified in this paper are played by the same rules with only one change. The player who removes the last counter is the loser instead of the winner. We challenge the reader to solve the misère game when $1 \leq m = t \leq T = M$.

References

[1] Berlekamp, Conway, and Guy, Winning Ways, Academic Press, New York, 1982.

[2] Richard K. Guy, Fair Game, 2nd ed., COMAP, New York, 1989.