# UNCC 2001 Algebra II

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1. Compute the sum of the roots of $x^2 - 5x + 6 = 0$.

   **(A)** 3 **(B)** 7/2 **(C)** 4 **(D)** 9/2 **(E)** 5

   **(E)** The sum of the roots of the quadratic $ax^2 + bx + c = 0$ is $-b/a$ which, for this example, is 5. Alternatively, factor the quadratic into $(x - 3)(x - 2) = 0$ and find the roots.

2. Compute the sum of all the roots of $(2x + 3)(x - 4) + (2x + 3)(x - 6) = 0$.

   **(A)** 7/2 **(B)** 4 **(C)** 7 **(D)** 13 **(E)** none of **A**, **B**, **C** or **D**

   **(A)** Factor to get $(2x + 3)(2x - 10) = 0$, so the two roots are $-3/2$ and 5.

3. The radius of the circle given by

$$x^2 - 6x + y^2 + 4y = 36$$

   is

   **(A)** 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 36

   **(C)** Complete the squares by adding 9 and 4 to both sides to get

$$x^2 - 6x + 9 + y^2 + 4y + 4 = 36 + 9 + 4 = 49 = 7^2.$$

   So the radius is 7.

4. Suppose that $\sqrt{x + 1} = 1 - x$. Which of the following statements is correct?

   **(A)** There are no solutions.

   **(B)** There are two solutions. The larger solution is greater than 2.

   **(C)** There are two solutions. The larger solution is less than or equal to 2.

   **(D)** There is only one solution. This solution is greater than 2.

   **(E)** There is only one solution. This solution is less than 2.

   **(E)** Square both sides of the equation and reduce to get $x^2 - 3x = 0$. This has two solutions, $x = 3$ and $x = 0$. But $x = 3$ is extraneous for the given equation. Thus there is only one solution, and it is less than 2.

5. Solve the following equation for $x$:

$$\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} = 3.$$

**(A)** 0 **(B)** 5/3 **(C)** 1 **(D)** 3/5 **(E)** 3

**(B)** Clearing fractions gives

$$\sqrt{x+1}+\sqrt{x-1} = 3\sqrt{x+1} - 3\sqrt{x-1},$$

from which it follows that $4\sqrt{x-1} = 2\sqrt{x+1}$. Divide by two, square both sides, and simplify to get $4x-4 = x+1$, from which it follows that $3x = 5$, and $x = 5/3$.

6. If $2^{10x-1} = 1$, what is $\log x$?

**(A)** $-1$ **(B)** 0 **(C)** 1 **(D)** 2 **(E)** 3

**(A)** Note that $2^{10x-1} = 1$ implies $10x = 1$, $x = 0.1$ So $\log x = -1$.

7. The equation $\sqrt{(x+7)}+x=13$ has

**(A)** no roots **(B)** one root **(C)** two roots **(D)** three roots **(E)** none of **A**, **B**, **C**, or **D**

**(B)** Subtract $x$ from both sides and then square both sides to get

$$x+7=169-26x+x^2$$

which is equivalent to $x^2-27x+162=0$. This can be factored to give the roots $x=18$ and $x=9$, but the former is extraneous, so there is just one root.

8. How many real solutions of the system $x-y=3$, $x^2-y=-1$ are there?

**(A)** none **(B)** 1 **(C)** 2 **(D)** 3 **(E)** infinitely many

**(A)** Solve the first equation for $y$, and replace that value in the second to get

$$x^2-(x-3)=-1.$$

Then solve this by the quadratic formula to get

$$x=\frac{1\pm\sqrt{1-16}}{2}.$$

Since the discriminant is negative, it follows that there are no real solutions.

9. Find the base $a$ for which $\log_a 2=\sqrt[3]{8}$

**(A)** $a=\sqrt{2}$ **(B)** $a=2$ **(C)** $a=4$ **(D)** $a=\sqrt[3]{2}$ **(E)** $a=\dfrac{1}{3}$

**(A)** We are simply solving $\log_a 2=2$ for a positive number $a$. Equivalently, $a^2=2$ which is true if and only if $a=\sqrt{2}$.

10. If the sides of a square are each increased by 12 inches, the area is increased by 200 square inches. The length of a side of the original square is

**(A)** 2 inches **(B)** $2\frac{1}{3}$ inches **(C)** $10\frac{1}{2}$ inches **(D)** $3\frac{2}{3}$ inches **(E)** $2\frac{1}{4}$ inches

**(B)** From what is given , we can write

$$(x+12)^2=x^2+200$$

where $x$ is the original side length. This is equivalent to $x^2+24x+144=x^2+200$ from which it follows that $x=\frac{56}{24}=2\frac{1}{3}$.

11. Determine $m$ such that $x^3 - 5x^2 + 7x + (m-5)$ is divisible by $(x-4)$

**(A)** $-7$ **(B)** 0 **(C)** 5 **(D)** 7 **(E)** 17

**(A)** The polynomial $f(x) = x^3 - 5x^2 + 7x + (m-5)$ is divisible by $(x-4)$ precisely when $f(4) = 0$ by the Factor Theorem. Thus

$$f(4) = 4^3 - 5 \cdot 4^2 + 7 \cdot 4 + m - 5 = 64 - 80 + 28 - 5 + m = 7 + m = 0,$$

which happens precisely when $m = -7$.

12. If $x^2+2x+n > 10$ for all real numbers $x$, then which of the following conditions must be true?

**(A)** $n > 11$ **(B)** $n < 11$ **(C)** $n = 10$ **(D)** $n = \infty$ **(E)** $n > -11$

**(A)** Complete the square to find that $f(x) = x^2+2x+n = (x+1)^2-1+n > 10$ if and only if $n > 11$. Alternatively, the minimum value of $x^2 + 2x + n$ occurs at the vertex of the parabola, whose $x$-coordinate is given by $-b/2a = -2/2 = -1$. Thus $f(-1) = (-1)^2 + 2(-1) + n > 10$ if and only if $n > 11$.

13. The product of four different integers, exactly three of which are odd, is 7!. The sum of the four integers is 63. What is the largest of the four integers?

**(A)** 35 **(B)** 48 **(C)** 64 **(D)** 72 **(E)** 105

**(B)** Since only one of the numbers is even, the even one must have all the even factors of 7!, so the even number is either 16 or 48. Matching 16 with the other factors does not lead to collections of factors with sum 63. But the set $\{48, 3, 5, 7\}$ works. The sum is 63 and the product is 7!.

14. What is the largest integer $k$ such that

$$\frac{3}{2} \cdot \frac{2}{1} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \cdots \frac{k}{k+1} \geq \frac{1}{8}$$

**(A)** 20 **(B)** 21 **(C)** 23 **(D)** 24 **(E)** 26

**(C)** The inequality is equivalent to $\frac{3}{k+1} \geq \frac{1}{8}$ which is true if and only if $k + 1 \leq 24$. Thus $k = 23$ is the largest integer satisfying the inequality.

15. What is the sum of all integers $x$ that satisfy

$$-5 \le x/\pi \le 10?$$

**(A)** 312 **(B)** 324 **(C)** 346 **(D)** 376 **(E)** 412

**(D)** The sum is $-15-14-13-\cdots 0+1+2\cdots+31$, which is just the sum of 16 consecutive integers $16+17+\cdots+31 = 16(\frac{16+31}{2}) = 376$.

16. It takes 6 hours for vote counter $A$ to count a bucket of votes. If vote counter $B$ is assigned to help $A$ with the count, it takes 4 hours. How long does it take vote counter $B$ to count a bucket of votes alone?

**(A)** 2 hours **(B)** 10 hours **(C)** 12 hours **(D)** 24 hours **(E)** 8 hours

**(C)** Together the vote counters can count 1/4 buckets per hour. Since vote counter $A$ counts votes at 1/6 buckets per hour, counter $B$ counts votes at the rate of $1/4 - 1/6 = 1/12$ buckets per hour. Hence it will take vote counter $B$ 12 hours to count a bucket of votes.

17. A circle $C$ contains the points $(0,6), (0,10)$, and $(8,0)$. What is the second $x$-intercept?

**(A)** 7.00 **(B)** 7.25 **(C)** 7.50 **(D)** 7.75 **(E)** 9.00

**(D)** The center of $C$ must lie on the line $y = 8$, because the center is the same distance from $(0,6)$ as it is from $(0,10)$. It must also lie on the line that perpendicularly bisects the segment from $(0,6)$ to $(8,0)$, an equation for which is $y-3 = \frac{4}{3}(x-4)$. Solving these two equations simultaneously gives $x = 7.75$.

18. What is the $x$-intercept of the line $L$ satisfying

- $L$ is perpendicular to the line defined by $3x - 2y = 6$, and
- the $y$-intercept of $L$ is 2.

**(A)** 1 **(B)** 2 **(C)** 2.4 **(D)** 3 **(E)** 3.2

**(D)** The slope of $L$ is $-1/(3/2) = -2/3$ so an equation for $L$ is $y = -2x/3+2$. Set $y = 0$ to find the $x-$intercept: $-2x/3 = -2$, and $x = 3$.

19. The number of zeroes at the end of 45! is

**(A)** 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

**(C)** Among the integers from 1 to 45 there are 9 that are divisible by 5 and one that is divisible by 25 so the prime decomposition of 45! contains $5^{10}$. Since there are plenty of 2's in the prime factorization, there are 10 zeros at the end of the number.

20. How many positive divisors does 6! have?

**(A)** 4 **(B)** 6 **(C)** 10 **(D)** 20 **(E)** 30

**(E)** $6! = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 2^4 \cdot 3^2 \cdot 5$, so it has $(4+1)(2+1)(1+1) = 30$ positive integer divisors.

21. If $x$ is 150% of $y$, what percent of $3x$ is $4y$? Round your answer to the nearest whole number.

**(A)** 75 **(B)** 79 **(C)** 89 **(D)** 92 **(E)** 112

**(C)** The numbers $x$ and $y$ satisfy $3/2 = x/y = 1.5$; so $4x/3y = 4/3 \cdot y/x = 4/3 \cdot 2/3 = 8/9 \approx 0.8888 \approx 0.89 = 89\%$.

22. How many positive integers can be represented as a product of two distinct members of the set $\{1, 2, 3, 4, 5, 6\}$?

**(A)** 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

**(E)** Count them one at a time. They are $2, 3, 4, 5, 6, 8, 10, 12, 15, 18, 20, 24$, and 30.

23. What is the largest possible product of three distinct members of the set $\{2/3, -2/3, 4/5, 1, -1, 5/4\}$?

**(A)** $-1$ **(B)** $-2/3$ **(C)** $4/5$ **(D)** 1 **(E)** $5/4$

**(D)** The product is negative if one or three of the numbers are negative, so we must choose either three positive numbers or one positive and two negative numbers. The largest product of the first type is $4/5 \cdot 1 \cdot 5/4 = 1$ and the largest of the second type is $-2/3 \cdot -1 \cdot 5/4 < 1$, so 1 is the largest possible product.

24. Another way to write $\left(a^{-1} + b^{-1}\right)^{-1}$ is

**(A)** $\dfrac{(a+b)}{ab}$ **(B)** $\dfrac{1}{a} + \dfrac{1}{b}$ **(C)** $\dfrac{ab}{(a+b)}$ **(D)** $a + b$ **(E)** $ab$

**(C)** Note that $\left(a^{-1} + b^{-1}\right)^{-1} = \frac{1}{a^{-1}+b^{-1}} = \frac{1}{\frac{b+a}{ab}} = \frac{ab}{a+b}$.

25. Determine $k$ so that the roots of $x^2 + 2kx - 1 = 2k$ will be equal.

**(A)** $-1$ **(B)** 1 **(C)** $i$ **(D)** $-i$ **(E)** $\pm\sqrt{2}$

**(A)** The roots are equal if and only if the discriminant $D$ is 0. Thus $D = (2k)^2 + 4(2k+1) = 0$ from which it follows that $k^2 + 2k + 1 = 0$, which is equivalent to $(k+1)^2 = 0$. Thus $k = -1$.

26. The coefficient of the term involving $x^8$ in the expansion of $\left(x^2 + 3y\right)^{10}$ is:

**(A)** $\left(3^7\right)(70)$ **(B)** $(3)(70)$ **(C)** $\left(3^7\right)(5)$ **(D)** $\left(3^6\right)(5)$ **(E)** none of **A**, **B**, **C** or **D**

**(A)** Use the binomial theorem to get $\left(x^2 + 3y\right)^{10} = (x^2)^{10} + 10(x^2)^9(3y) + \frac{9\cdot 10}{2}(x^2)^8(3y)^2 + \ldots + \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1}(x^2)^4(3y)^6 + \ldots$, so the coefficient of the $x^8$ term is $(3^7)(70)$

27. Three packages of coffee cost a total of \$10.20. The first costs \$0.30 more than the second, and the second costs \$0.66 less than the third. How much did the second package cost?

**(A)** \$3.52 **(B)** \$2.86 **(C)** \$3.82 **(D)** \$3.08 **(E)** \$3.38

**(D)** We have three equations with three unknowns, $C_1, C_2$, and $C_3$:

$$\begin{aligned} C_1 + C_2 + C_3 &= 10.20 \\ C_1 &= C_2 + 0.30 \\ C_2 &= C_3 - 0.66 \end{aligned}$$

Replacing $C_1$ and $C_3$ with their equals yields

$$(C_2 + .30) + C_2 + (C_2 + .66) = 10.20$$

from which it follows that $C_2 = 3.08$.

28. How many pounds of $H_2O$ must be evaporated from 50 pounds of a 3% solution so that the remaining solution will be 5% salt?

**(A)** 1.6 **(B)** $5\frac{1}{3}$ **(C)** 9.6 **(D)** $13\frac{1}{3}$ **(E)** 20

**(E)** Salt in solution $= 0.03 \cdot 50 = 1.5$ lbs. Let $x$ denote the number of pounds of $H_2O$ which must be evaporated. Then $50 - x$ is the number of pounds of solution remaining, and the amount of salt will be unchanged. Since 5% of the solution will be salt, $.05(50 - x) = 1.5$ implies $x = 20$.