UNC Charlotte 2002 Comprehensive with solutions

March 4, 2002

1. It takes 852 digits to number the pages of a book consecutively. How many pages are there in the book?

   (A) 184 **(B)** 235 **(C)** 320 **(D)** 368 **(E)** 425

   (C) Pages 1 through 9 use 9 digits and 10 through 99 use $90 \times 2 = 180$ digits, for a total of 189 digits for pages 1 through 99. That leaves 663 digits remaining to make the required total of 852 digits. These are obtained by going 221 pages beyond page 99, through page 320.

2. Solve the equation $8^{\frac{1}{6}} + x^{\frac{1}{3}} = \frac{7}{3-\sqrt{2}}$.

   (A) 24 **(B)** 27 **(C)** 32 **(D)** 64 **(E)** none of **A**, **B**, **C** or **D**

   (B) Note that $8^{\frac{1}{6}} = (2^3)^{\frac{1}{6}} = 2^{\frac{1}{2}}$. Rationalizing the denominator of the right side gives $\frac{7}{3-\sqrt{2}} = \frac{7(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})} = \frac{21+7\sqrt{2}}{7} = 3+\sqrt{2}$. Thus the equation reduces to $x^{\frac{1}{3}} = 3$ or $x = 27$. Alternatively, you can use a calculator to solve this problem.

3. The fraction $\frac{5x-11}{2x^2+x-6}$ was obtained by adding the two fractions $\frac{A}{x+2}$ and $\frac{B}{2x-3}$. Find the value of $A+B$.

   (A) -4 **(B)** -2 **(C)** 1 **(D)** 2 **(E)** 4

   (D) Note that $\frac{5x-11}{2x^2+x-6} = \frac{A}{x+2} + \frac{B}{2x-3} = \frac{A(2x-3)}{(x+2)(2x-3)} + \frac{B(x+2)}{(x+2)(2x-3)}$. Also, note that $5x-11 = (2A+B)x - (3A-2B)$, $\quad 2A+B = 5$ and $3A-2B = 11$. Solving this system of equations we obtain $A = 3$ and $B = -1$, so $A+B = 2$

4. The slope of the line through the points that satisfy $y = 8 - x^2$ and $y = x^2$ is

   (A) 2 **(B)** 4 **(C)** 0 **(D)** -2 **(E)** -4

   (C) The slope is 0 because the two parabolas are intersecting even functions. In fact they are reflections (through the line $y = 4$) of each other. Alternatively, $8 - x^2 = x^2 \Rightarrow x = \pm 2, y = 4$.

5. The product of the zeros of $f(x) = (2x - 24)(6x - 18) - (x - 12)$ is

   (A) -72 **(B)** 5 **(C)** 6 **(D)** 37 **(E)** 432

   (D) The roots are 12 and 37/12 because $f(x) = 12 \cdot (x-12) \cdot (x-3) - (x-12) = (x-12)(12x-37)$ so the product is $12 \cdot (37/12) = 37$.

6. Factor $x^4 + 4y^4$ over the real numbers. Hint: Add and subtract $4x^2y^2$.

   (A) $(x^2 - 2xy + 2y^2)(x^2 + 2xy + 2y^2)$

   (B) $(x^2 + 2xy + 2y^2)^2$

   (C) $(x^2 + 2xy - 2y^2)(x^2 + 2xy + 2y^2)$

   (D) $(x^2 - 2xy - 2y^2)(x^2 + 2xy + 2y^2)$

   (E) none of **A**, **B**, **C**, or **D**

   (A) Add and subtract $4x^2y^2$ to get $x^4 + 4y^4 = x^4 + 4x^2y^2 + 4y^4 - 4x^2y^2 = (x^2 + 2y^2)^2 - (2xy)^2 = (x^2 - 2xy + 2y^2)(x^2 + 2xy + 2y^2)$.

7. What is the remainder when $x^2 + 3x - 5$ is divided by $x - 1$?

   (A) -5 **(B)** -2 **(C)** -1 **(D)** 0 **(E)** 1

   (C) By long division or the Remainder Theorem, the remainder is -1.

8. Jeremy starts jogging at a constant rate of five miles per hour. Half an hour later, David starts running along the same route at seven miles per hour. For how many minutes must David run to catch Jeremy?

   (A) 75 minutes **(B)** 80 minutes **(C)** 90 minutes

   (D) 95 minutes **(E)** 105 minutes

   (A) David runs at 7 mph for t hours while Jeremy runs for $t + 1/2$ hours at 5 mph. They run the same distance, so $5 \cdot (t + 1/2) = 7t$, which yields $t = 5/4$ hours, or 75 minutes.

9. For the final exam in Professor Ahlin's class, the average (= arithmetic mean) score of the group of failing students was 62 and the average score among the passing students was 92. The overall average for the 20 students in the class was 80. How many students passed the final?

   (A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

   (D) Translating the information into equations, where x represents the number of students who passed the final,

   $$62(20 - x) + 92(x) = 20 \cdot 80 = 1600.$$

   Solve this for x to get $x = 12$.

10. Fifteen numbers are picked from the set $\{1, 2, 3, \ldots 20, 21\}$. Find the probability that at least three of those numbers are consecutive.

    (A) 0.1 **(B)** 0.2 **(C)** 0.4 **(D)** 0.5 **(E)** 1.0

    (E) Imagine putting the 15 numbers into seven boxes labelled 123, 456, 789, etc. Each number is put into the box that it helps to label. After all 15 numbers have been distributed among the boxes, some box must have three balls, by the Pigeon-Hole Principle. Thus the probability that some box has three consecutive numbers is 1.

11. Cara has 162 coins in her collection of nickels, dimes, and quarters, which has a total value of \$22.00. If Cara has twelve fewer nickels than quarters, how many dimes does she have?

    (A) 50 **(B)** 60 **(C)** 70 **(D)** 74 **(E)** 78

    (C) Solve simultaneously the three equations $q - 12 = n$, $n + d + q = 162$ and $5n + 10d + 25q = 2200$ to get $d = 70$.

12. Let (h, k) denote the center and let r denote the radius of the circle given by $x^2 + 2x + y^2 - 4y = 4$. What is the sum $h + k + r$?

    (A) 2 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 10

    (B) Complete the square to write the equation in the form $(x+1)^2+(y-2)^2 = 3^2$, at which point it is easy to see that $h = -1, k = 2$, and $r = 3$, so the sum is $-1 + 2 + 3 = 4$.

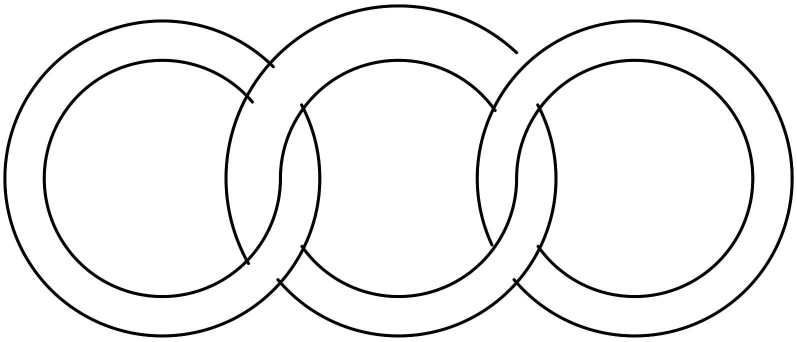
13. Let N denote the smallest four-digit number with all different digits that is divisible by each of its digits. What is the sum of the digits of N?

(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

(D) Trying the smallest digits first, we are led to numbers of the form $12xy$, which in turn leads to 1236.

14. A chain with two links is 13 cm long. A chain made from three links of the same type is 18 cm long. How long is a chain made from 25 such links?

(A) 120 **(B)** 128 **(C)** 136 **(D)** 144 **(E)** 150



(B) Look at the diagram, and note the length of a chain with n links is $2R + 2(n-1)r$ where R is the outer radius and r is the inner radius. Thus, $2R + 2r = 13$ and $2R + 4r = 18$, which yields $2r = 5/2$ and $2R = 8$. Thus the length of a 25 link chain is $2R + 24r = 8 + 120 = 128$. Alternatively, note that each link adds 5 cm. to the length of the chain, for a total of $8 + 5 \cdot 24 = 128$.

15. What is the sum of the three positive integers a, b, and c that satisfy

$$a + \frac{1}{b + \frac{1}{c}} = 7.5?$$

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 11

(D) First note that $a = 7$ and that $\frac{1}{b+\frac{1}{c}} = 0.5$. So $b + \frac{1}{c} = 2$ and it follows that $b = c = 1$. Thus $a + b + c = 9$.

16. A circle C contains the points $(0,6)$, $(0,10)$, and $(8,0)$. What is the x-coordinate of the center?

    (A) 6.75 **(B)** 7.25 **(C)** 7.50 **(D)** 7.75 **(E)** 8.25

    (D) The center (h,k) of C must lie on the line $y = 8$, because the center is the same distance from $(0,6)$ as it is from $(0,10)$. Thus the circle satisfies $(x-h)^2 + (y-8)^2 = r^2$, for some number h. It must also lie on the line that perpendicularly bisects the segment from $(0,6)$ to $(8,0)$, an equation for which is $y - 3 = \frac{4}{3}(x-4)$. Since $y = 8$, it follows that $h = 7.75$. Alternatively, the distance from $(h,8)$ to $(0,6)$ must be the same as the distance from $(h,8)$ to $(8,0)$, which we can solve for h.

17. The number 839 can be written as $19q + r$ where q and r are positive integers. What is the largest possible value of $q - r$?

    (A) 37 **(B)** 39 **(C)** 41 **(D)** 45 **(E)** 47

    (C) Divide 839 by 19 to get $839 = 44 \cdot 19 + 3$. One pair of positive value is $q = 44$ and $r = 3$. If we try to make q any bigger, we are forced to make r negative.

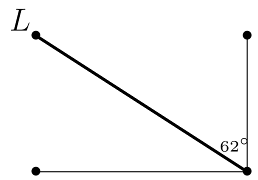
18. The four angles of a quadrilateral form an arithmetic sequence. The largest is 15 degrees less than twice the smallest. What is the degree measure of the largest angle?

    (A) 95° **(B)** 100° **(C)** 105° **(D)** 115° **(E)** 125°

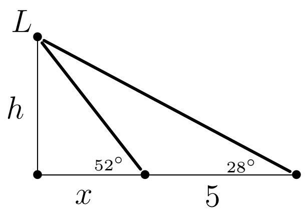
    (D) Let the angles be $a, a+d, a+2d$, and $a+3d$. Then $4a + 6d = 360$ and $a + 3d + 15 = 2a$. Thus $a = 3d + 15$, so we can replace a with $3d + 15$ in the first equation. Solve this to get $3d = 50$ and $a = 50 + 15 = 65$, so the largest angle is $65 + 50 = 115°$.

19. A family is traveling due west on a straight road that passes a famous landmark, L in the figure below. At a given time, the bearing on the landmark is $62°$ west of north. After the family has travelled 5 miles farther, the landmark is $38°$ west of north. What is the closest the family can come to the landmark if they remain on the road?

(A) 4.25 **(B)** 4.34 **(C)** 4.55 **(D)** 4.76 **(E)** 4.85



(C) Let h denote the distance from the landmark to the point on the road nearest the landmark. Then $\tan 52 = h/x$ and $\tan 28 = h/(x+5)$. Solving each of these for x reveals that $h/\tan 52 = h/\tan 28 - 5$ which we can solve for h to get $h = 4.55$.



20. Vic can beat Harold by one tenth of a mile in a two mile race. Harold can beat Charlie by one fifth of a mile in a two mile race. If Vic races Charlie, how far ahead will Vic finish?

(A) 0.15 miles **(B)** 0.22 miles **(C)** 0.25 miles

(D) 0.29 miles **(E)** 0.33 miles

(D) Vic is 20/19 times as fast as Harold. Harold is 20/18 times as fast as Charlie, so Vic is $(20/19)(20/18) \approx 1.1696$ times as fast as Charlie. When Vic runs 2 miles, Charlie will have run $2/1.1696 \approx 1.71$ miles. Vic will finish 0.29 miles ahead of Charlie.

21. For what positive value of x is there a right triangle with sides $x+1, 4x$, and $4x+1$?

    (A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

    (B) The number x must satisfy $(x+1)^2 + (4x)^2 = (4x+1)^2$ since $4x+1$ is certainly the largest of the three numbers. Thus $x^2 + 2x + 1 + 16x^2 - 16x^2 - 8x - 1 = x^2 - 6x = 0$ from which it follows that $x = 6$.

22. What is the probability of obtaining an ace on both the first and second draws from an ordinary deck of 52 playing cards when the first card is not replaced before the second is drawn? There are four aces in the deck.

    (A) 1/221 **(B)** 4/221 **(C)** 1/13 **(D)** 1/17 **(E)** 30/221

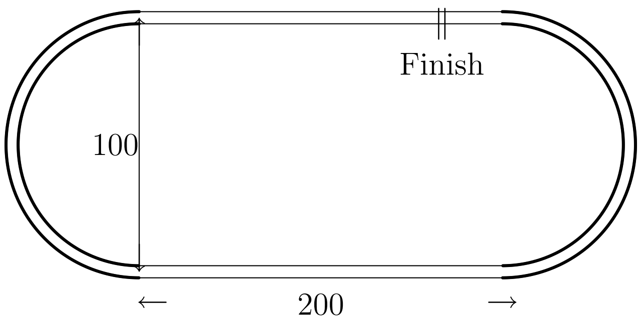
    (A) The probability of obtaining an ace on the first draw is $4/52 = 1/13$. If the first card drawn is an ace there are 3 aces remaining in the deck, which now consists of 51 cards. Thus, the probability of getting an ace on the second draw is $3/51 = 1/17$. The required probability is the product of the two, which is $1/221$.

23. Let (m, n) be an ordered pair of integers such that to $5m^2 + 2n^2 = 2002$. Which of the following digits could be the units digit of n?

    (A) 2 **(B)** 3 **(C)** 4 **(D)** 7 **(E)** 9

    (E) Note that $5m^2 = 2(1001 - n^2)$. This implies that m is even, so m^2 is a multiple of 4. Thus $1001 - n^2$ is a multiple of 10, and it follows that the units digit of n^2 is 1. The number $(10k + d)^2$ has a units digit of 1 only when the digit d is 1 or 9. Since 1 is not an option, the answer is 9. In fact one solution is $(16, 19)$.

24. A running track has the shape shown below. The ends are semicircular with diameter 100 yards. Suppose that the lanes are each 1 yard wide and numbered from the inside to the outside. The competitor in the inside lane runs 700 yards counter clockwise. The other runners start ahead of the inside lane runner, and also run 700 yards, with all runners finishing at the same place. Approximately how much of a head start should a runner in the fifth lane receive over a runner in the first lane?



(A) 15 yards **(B)** 20 yards **(C)** 25 yards **(D)** 30 yards **(E)** 35 yards

(C) In order to run 700 yards, the runners must traverse both semicircular ends. The runner in the first lane has an inner radius of 50 yards, while the runner in the fifth lane has an inner radius of 54 yards. The difference in distance is $2\pi(54 - 50) = 8\pi \approx 25$ yards.

25. Dick and Nick share their food with Albert. Dick has 5 loaves of bread, and Nick has 3 loaves. They share the bread equally. Albert gives Dick and Nick 8 dollars which they agree to share fairly. How should they divide the \$8 between them?

    (A) Dick should get \$3 of Albert's money. **(B)** Dick should get \$4 of Albert's money.

    (C) Dick should get \$5 of Albert's money. **(D)** Dick should get \$6 of Albert's money.

    (E) Dick should get \$7 of Albert's money.

    (E) Albert pays \$8 for his 8/3 loaves, so loaves must be worth \$3 each. Nick eats all but 1/3 of a loaf of his bread while Dick gives up 7/3 loaves. Thus Dick should get \$7 of Albert's money.

26. A 12×12 square is divided into n^2 congruent squares by equally spaced lines parallel to its sides. Circles are inscribed in each of the squares. Find the sum of the areas of the circles.

    (A) 6π **(B)** 12π **(C)** 24π **(D)** 36π **(E)** the answer depends on n

    (D) The diameter of each of the n^2 circles is $12/n$, and each radius is $6/n$. Therefore, the area of each circle is $\pi(6/n)^2 = 36\pi/n^2$ and the total area of the n^2 circles is $(36\pi/n^2) \times n^2 = 36\pi$.

27. Consider a square with side length s. Let a denote the area of the inscribed circle (which touches all four edges) and let A denote the area of the circumscribed circle (which goes through all four corners). Which of the following holds?

    (A) $a = \frac{1}{2}A$ **(B)** $a = \frac{1}{3}A$ **(C)** $a = \frac{2}{3}A$ **(D)** $a = \frac{2}{\pi}A$ **(E)** $a = \frac{3}{4}A$

    (A) Note that $a = \frac{1}{2}A$, since the radius of the inscribed circle is $r = \frac{1}{2}s$, while by the Pythagorean theorem, the radius of the circumscribed circle is $R = \sqrt{(\frac{1}{2}s)^2 + (\frac{1}{2}s)^2} = s/\sqrt{2}$, so $a = \pi r^2 = \pi\frac{1}{4}s^2$, $A = \pi R^2 = \pi\frac{1}{4} \cdot 2s^2 = 2a$.

28. What is the positive zero of the function

$$f(x) = \log\sqrt{5x+5} + \frac{1}{2}\log(2x+1) - \log 15?$$

(A) 2 **(B)** $\log 15$ **(C)** 3 **(D)** 4 **(E)** f has no positive zero

(D) To find the zero, first solve $\log\sqrt{5x+5} + \frac{1}{2}\log(2x+1) = \log 15$. Then $\frac{1}{2}\{\log(5x+5)(2x+1)\} = \log 15$. Hence $\log(10x^2+15x+5) = \log(15^2)$. It follows that $10x^2+15x+5 = 15^2 = 225$ which is equivalent to $2x^2+3x-44 = 0$, the left side of which can be factored into $(2x+11)(x-4)$. Thus, $x = 4$ is the positive zero of f.

29. Suppose that the equation $x^2 - px + q = 0$ has roots $x = a$ and $x = b$. Which of the following equations has roots $x = a + \frac{1}{b}$ and $x = b + \frac{1}{a}$?

(A) $x^2 - \left(p + \dfrac{p}{q}\right)x + \left(q + \dfrac{1}{q} + 2\right) = 0.$ **(B)** $x^2 - \left(q + \dfrac{p}{q}\right)x + \left(p + \dfrac{1}{q} + 2\right) = 0.$

(C) $x^2 - \left(p + \dfrac{q}{p}\right)x + \left(q + \dfrac{1}{p} + 2\right) = 0.$ **(D)** $x^2 - \left(p + \dfrac{q}{p}\right)x + \left(p + \dfrac{1}{p} + 2\right) = 0.$

(E) $x^2 - \left(q + \dfrac{q}{p}\right)x + \left(q + \dfrac{1}{p} + 2\right) = 0$

(A) Note that $q = ab$ and $p = a+b$. If the new equation is $x^2 - Px + Q = 0$, we require that $P = a+b+\frac{1}{a}+\frac{1}{b} = a+b+\frac{a+b}{ab} = p+\frac{p}{q}$ and $Q = \left(a+\frac{1}{b}\right)\left(b+\frac{1}{a}\right) = ab+2+\frac{1}{ab} = q+2+\frac{1}{q}$.

30. Determine the sum of all the natural numbers less than 45 that are not divisible by 3.

(A) 600 **(B)** 625 **(C)** 650 **(D)** 675 **(E)** 700

(D) The sum S of all the natural numbers less than 45 is $S = 44\cdot(1+44)/2 = 22\cdot 45 = 990$, so the sum we want is $S - T$ where $T = 3+6+9+\cdots+42 = 3(1+2+\cdots+14) = 3(14/2)15 = 315$. Thus $S - T = 990 - 315 = 675$.

31. Some hikers start on a walk at 10 a.m. and return at 4 p.m. One third of the distance walked is uphill, one third is level, and one third is downhill. If their speed is 4 miles per hour on level land, 2 miles per hour uphill, and 6 miles per hour downhill, approximately how far did they walk?

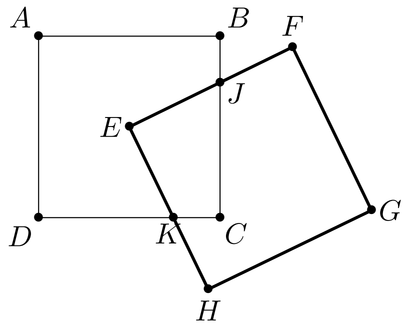
(A) 18.4 miles **(B)** 19.6 miles **(C)** 22.1 miles

(D) 24.0 miles **(E)** 26.2 miles

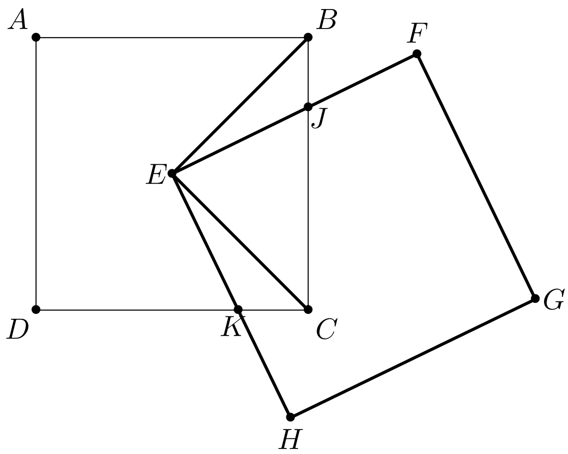
(B) Suppose that they walk $3x$ miles, so that they walk x miles uphill, x miles downhill, and x level miles. This takes a total time of $\frac{x}{2} + \frac{x}{6} + \frac{x}{4}$ hours. Setting this equal to 6 hours, we find $x = 72/11$ so that the total distance is $3x = 216/11 \approx 19.6$ miles.

32. Two squares, each with side length 12 inches, are placed so that the corner of one lies at the center of the other (see the diagram below). Suppose the length of BJ is 3. What is the area of the quadrilateral EJCK?

    (A) 25 in^2 **(B)** 30 in^2 **(C)** 36 in^2 **(D)** 40 in^2 **(E)** 49 in^2



(C) Triangle EBJ is congruent to triangle ECK (see below), hence the area of the quadrilateral EJCK is the same as the area of the triangle EBC which is $\frac{1}{2}(6)(12) = 36$ square inches.

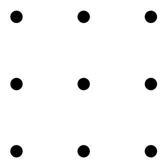


33. The natural numbers are arranged in groups as follows: {1}, {2,3}, {4,5,6}, {7,8,9,10}, etc. Note that there are always k numbers in the k^{th} group. What is the sum of the numbers in the 10^{th} group?

    (A) 260 **(B)** 369 **(C)** 452 **(D)** 505 **(E)** 638

    (D) Note that the k^{th} group ends with number $L = \frac{k(k+1)}{2}$. Since there are k numbers in the group, the group begins with number $F = \frac{k(k+1)}{2} - (k-1)$. The average value of these numbers is $\frac{F+L}{2} = \frac{k^2+1}{2}$ so the sum is $k\left(\frac{k^2+1}{2}\right)$. When $k = 10$, the sum is 505.

34. Three points are selected simultaneously and randomly from the 3 by 3 grid of lattice points shown. What is the probability that they are collinear? Express your answer as a common fraction.



    (A) 1/42 **(B)** 1/21 **(C)** 2/21 **(D)** 1/7 **(E)** 1/6

    (C) There are $\binom{9}{3} = 84$ ways to choose the three points, and exactly 8 of these result in collinear points.

35. Use each of the digits $2, 3, 4, 6, 7, 8$ exactly once to construct two three-digit numbers M and N so that $M - N$ is positive and is as small as possible. Compute $M - N$.

    (A) 33 **(B)** 35 **(C)** 39 **(D)** 41 **(E)** 47

    (C) Since the hundreds digits must be different, the smaller of the numbers N should have as large a tens digit as possible and the larger M should have as small a tens digit as possible. So we should choose 8 and 2 for these digits provided we can select the hundreds digits to be one apart. Given that the 2 and the 8 are not available, there are just two pairs that we can use, 3 and 4 and 6 and 7. Picking the tens digit of N to be as large as we can and the tens digit of M to be as small as possible leads to $M = 723$ and $N = 684$. Thus the answer we seek is $723 - 684 = 39$. Another possible pair is $M = 426$ and $N = 387$

36. What is the largest factor of 11! that is one bigger than a multiple of 6?

(A) 55 **(B)** 77 **(C)** 385 **(D)** 463 **(E)** 1925

(C) Let N denote the largest such factor of $11! = 2^8 3^4 5^2 7^1 11^1$. Then N has a factorization which can include only the primes $2, 3, 5, 7$, and 11. But N must not be even and also N cannot be divisible by 3, because otherwise it would not be one bigger than a multiple of 6. Hence $N = 5^i 7^j 11^k$ where $i = 0, 1$ or 2, $j = 0$ or 1, and $k = 0$ or 1. Examine these numbers starting with the largest $(i = 2, j = 1, k = 1)$, which leads to 1925 which is 5 larger than a multiple of 6. Next largest is the number obtained when $i = 1, j = 1$, and $k = 1$, which is 385, and this number is 1 larger than a multiple of 6.