

UNC Charlotte 2010 Comprehensive

with solutions

March 8, 2010

1. A cubic equation $x^3 - 4x^2 - 11x + a = 0$ has three roots, x_1, x_2, x_3 .
If $x_1 = x_2 + x_3$, what is a ?

(A) 24 (B) 28 (C) 30 (D) 32 (E) 36

Solution: C. Because the coefficient of x^2 is -4 , we have $x_1 + x_2 + x_3 = 4$. We also know that $x_1 = x_2 + x_3$, so $2x_1 = 4$ and $x_1 = 2$. Since $x = 2$ is a root, we have $2^3 - 4(2^2) - 11(2) + a = 0$ so $a = 30$.

2. For which value of a is the polynomial $P(x) = x^{1000} + ax + 9$ divisible by $x + 1$?

(A) none (B) 10 (C) -10 (D) 9 (E) -9

Solution: B. The polynomial $P(x)$ is divisible by $x + 1$ precisely when $P(-1) = 0$. This gives $a = 10$.

3. Suppose you visit Mars and meet some aliens who teach you their system of counting. You notice that they use a true place value system that is similar to ours, but the Martians use base 6 instead of base 10. Table 1 shows how to translate their characters into our digits.

#	&	<	@	/	*
0	1	2	3	4	5

Table 1: Martian characters

Convert the number 54 (base 10) to its Martian equivalent.

(A) &@ (B) &@# (C) @& (D) #@& (E) */

Solution: B. $54 = 1 \cdot 6^2 + 3 \cdot 6 + 0 \cdot 1$, so the answer is &@#.

4. Let $x = \frac{1}{2} \left(\frac{1}{2}^{-\frac{1}{2}} \right)$. To which of the following intervals does x belong?

(A) $(0, 1/8]$ (B) $(1/8, 1/4]$ (C) $(1/4, 1/2]$ (D) $(1/2, 1]$ (E) $(1, \infty)$

Solution: C. Let $y = \frac{1}{2}^{\frac{-1}{2}}$. Then $y = \sqrt{2}$. Because $1 < y < 2$ and $0 < \frac{1}{2} < 1$, we have $(\frac{1}{2})^2 < (\frac{1}{2})^y < (\frac{1}{2})^1$.

5. A room is shaped like a cube with sides of length 4 meters. Suppose that A and B denote two corners of the room that are farthest from each other. A caterpillar is crawling from A to B along the walls. What is the shortest possible length of the trip?

- (A) $4\sqrt{2}$ meters (B) $4\sqrt{3}$ meters (C) 8 meters
(D) $4\sqrt{5}$ meters (E) $4\sqrt{6}$ meters

Solution: D. A shortest path between A and B is contained in two adjacent faces of the cube and goes through the middle of their common side. The caterpillar travels x units in each of two faces where $x^2 = 4^2 + 2^2 = 20$. Therefore, the distance traveled is $2x = 4\sqrt{5}$.

6. A plane passes through the center of a cube and is perpendicular to one of the cube's diagonals. How many edges of the cube does the plane intersect? (An edge is a line connecting adjacent corners.)

- (A) 3 (B) 4 (C) 5 (D) 6 (E) 8

Solution: D. Suppose that each edge of the cube has length one and that we place the cube so that one corner is at the origin of a three dimensional coordinate system (x, y, z) so that $0 \leq x, y, z \leq 1$. Let the diagonal connect the vertices $(0, 0, 0)$ and $(1, 1, 1)$. Then the equation of the plane is $x + y + z = \frac{3}{2}$. This plane intersects edges connecting vertices whose coordinates sum to 3. (For example, it intersects the edge connecting $(1, 0, 0)$ to $(1, 1, 0)$ (because at a point halfway between these vertices, we have $x + y + z = \frac{3}{2}$). There are six such edges.

Alternate solution: The angle a diagonal of the cube makes with the corresponding diagonal of the square at the bottom is less than 45° . Thus the perpendicular plane through the midpoint of the diagonal will intersect the diagonal of the bottom inside the bottom square and the diagonal of the top inside the top square. Thus the plane must cut through two adjacent edges on the top and two adjacent edges on the bottom. It will also intersect both vertical edges that contain neither endpoint of the diagonal.

7. Four numbers are written in a row. The average of the first two numbers is 5. The average of the middle two numbers is 4 and the average of the last two numbers is 10. What is the average of the first and last numbers?

(A) 9 (B) 10 (C) 10.5 (D) 11 (E) 11.5

Solution: D. Denote the numbers by $A, B, C,$ and $D.$ Then $\frac{1}{2}(A + B) = 5,$ $\frac{1}{2}(B + C) = 4,$ and $\frac{1}{2}(C + D) = 10.$ Therefore $\frac{1}{2}(A + D) = \frac{1}{2}(A + B) - \frac{1}{2}(B + C) + \frac{1}{2}(C + D) = 11.$

8. The 8×10 grid below has numbers in half the squares. These numbers indicate the number of mines among the squares that share an edge with the given one. Squares containing numbers do not contain mines. Each square that does not have a number either has a single mine or nothing at all. How many mines are there?

(A) 19 (B) 20 (C) 21 (D) 22 (E) 23

	1		1		2		2		1
1		2		3		2		3	
	3		2		3		2		2
1		3		2		1		3	
	3		2		1		1		2
2		4		1		1		2	
	3		3		2		3		1
1		2		2		3		2	

Solution: D. The solution is unique. Start with the 4 and build outwards.

	1		1	•	2		2	•	1
1	•	2		3	•	2	•	3	
	3	•	2	•	3		2	•	2
1	•	3		2	•	1		3	•
	3	•	2		1		1	•	2
2	•	4	•	1		1		2	
•	3	•	3		2	•	3	•	1
1		2	•	2	•	3	•	2	

Counting row by row, we see that there are 22 mines.

9. John and Bill toss a biased coin that has a 60% chance of coming up heads and a 40% chance of coming up tails. They flip the coin until either two heads or two tails in a row are observed. Bill is a winner if two heads in a row are observed first. Which of the following numbers is closest to the probability that Bill will win?
- (A) $61/95$ (B) $63/95$ (C) $64/95$ (D) $67/95$ (E) $69/95$

Solution: B. Let x be the probability that Bill wins after a head is tossed and let y be the probability that he wins after a tail is tossed. After a head is tossed, there is a 60% chance of winning on the next toss. Since a tail occurs the remaining 40% of the time and the probability of winning after that is y , we must have $x = 0.6 + 0.4y$. Similarly, if a tail was tossed, Bill loses immediately 40% of the time. The remaining 60% of the time a head was tossed and the probability of winning is now x . Thus, $y = 0.6x$. Solving these equations, we find that $x = \frac{15}{19}$ and $y = \frac{9}{19}$. Since there is a 60% chance of a head on the first toss and a 40% chance of a tail, the probability of winning is $0.6x + 0.4y = \frac{63}{95}$.

Alternate solution: The winning toss sequences for Bill are HH, HTHH, HTHTHH, HTHTHTHH, ... THH, THTHH, THTHTHH, THTHTHTHH Let h represent the probability of a head on one toss and t represent the probability of a tail. Summing the probabilities for each of these gives an “infinite” sum $h^2 + hth^2 + hthth^2 + hththth^2 + \dots + th^2 + thth^2 + ththth^2 + thththth^2 + \dots = h^2(1 + t)(1 + ht + (ht)^2 + (ht)^3 + \dots) = h^2(1 + t)\frac{1}{1 - ht}$ (since $0 < ht < 1$). For $h = 0.6 = \frac{3}{5}$ and $t = 0.4 = \frac{2}{5}$, have $\left(\frac{3}{5}\right)^2 \cdot \frac{7}{5} \cdot \frac{1}{1 - \frac{6}{25}} = \frac{63}{95}$.

10. How many times in a 24 hour period do the hour and minute hands of a clock form a right angle?
- (A) 48 (B) 44 (C) 34 (D) 24 (E) None of these.

Solution: B. The minute hand moves 360 degrees in 60 minutes, or 6 degrees per minute. The hour hand moves 360 degrees in 12 hours or 0.5 degrees per minute. Since the minute hand travels 5.5 degrees per minute faster than the hour hand, it gains 90 degrees every $90/5.5$ minutes. In 24 hours (1440 minutes) 90 degrees is gained $1440/(90/5.5) = 88$ times. For half of these times, the angular difference between the hands is 180 or zero degrees. The hands are at a right angle the remaining 44 times.

11. A pyramid has a 10 ft. by 10 ft. square base. The height of the pyramid is also 10 ft. If we want to slice the pyramid into two pieces with equal volumes by cutting it with a plane parallel to the base, how far above the base should we make the cut?
- (A) 5 ft. (B) $5 - \sqrt[3]{4}$ ft. (C) $10 - 5\sqrt[3]{4}$ ft.
(D) $\frac{10}{3}$ ft. (E) None of these.

Solution: C. The volume of the original pyramid is $V = \frac{1}{3}s^2h = \frac{1000}{3}$. Let x be the distance above the base that the cut is made. The upper half is a pyramid with height $h = 10 - x$. By similarity, the upper half has a base that is $10 - x$ by $10 - x$ and the volume of the upper half is $\frac{1}{3}(10 - x)^3$. This volume must be one half the original volume, so $(10 - x)^3 = 500$. Solving for x , we get $x = 10 - 5\sqrt[3]{4}$.

12. How many pairs (x, y) of positive integers satisfy $2x + 7y = 1000$?
- (A) 70 (B) 71 (C) 72 (D) 73 (E) 74

Solution: B. Every even multiple of 7 between 14 and 994 = $71 \cdot 14$ works. In other words $x = 1000 - 14k$ works for $k = 1, 2, \dots, 71$.

13. Let A be the point $(7, 4)$ and D be the point $(5, 3)$. What is the length of the shortest path $ABCD$, where B is a point $(x, 2)$ and C is a point $(x, 0)$? This path consists of three connected segments, with the middle one vertical.
- (A) $2 + \sqrt{29}$ (B) $\sqrt{31}$ (C) $2 + \sqrt{31}$ (D) $2 + \sqrt{33}$ (E) $\sqrt{41}$

Solution: A. Distance = $2 + \sqrt{29}$. Reflect $(5, 3)$ about the x -axis to get $(5, -3)$. Now the shortest distance from $(7, 4)$ to $(5, -3)$ is the same as for the original problem. Imagine shrinking the vertical segment to zero by lowering the point $(7, 4)$ down to $(7, 2)$. The line between $(7, 2)$ and $(5, -3)$ goes through the x -axis at $x = \frac{31}{5}$, and the length of this segment is $\sqrt{29}$. Thus the shortest path has length $2 + \sqrt{29}$.

14. Suppose $f(0) = 3$ and $f(n) = f(n - 1) + 2$. Let $T = f(f(f(f(5))))$. What is the sum of the digits of T ?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Solution: C. In fact, $f(n) = 2n + 3$, and $T = f(f(f(f(5)))) = f(f(f(13))) = f(f(29)) = f(61) = 125$.

15. Four sets A, B, C , and D each have 500 elements. The intersection of any two of the sets has 115 elements. The intersection of any three of the sets has 52 elements. The intersection of all four sets has 30 elements. How many elements are there in the union of the four sets?

(A) 1465 (B) 1472 (C) 1482 (D) 1488 (E) 1512

Solution: D. By the inclusion-exclusion principle, the answer is $4 \cdot 500 - 6 \cdot 115 + 4 \cdot 52 - 30 = 1488$.

16. How many three digit numbers can be written as a sum of a three digit number and its (one-, two-, or three-digit) reversal?

(A) 75 (B) 80 (C) 85 (D) 90 (E) 95

Solution: C. We want three digit numbers that can be written in the form $abc + cba$ i.e. $100(a + c) + 20b + c + a = 101(a + c) + 20b$ where a, b, c are non negative integers, $a \neq 0$, and $a + c \leq 9$. When $a + c = 1$, there are 10 values of b that work. This is true when $a + c = k$ for $k = 1 \dots 8$. When $k = 9$, $101(a + c) + 20b$ will not be a three digit number unless $b \leq 4.5$. There are 5 such values of b (0, 1, 2, 3, 4). Thus, there are $8 \cdot 10 + 5 = 85$ numbers that can be written as the sum of a three digit number and its reversal.

17. If (x, y, z) satisfy the three equations below, what is $x + y + z$?

$$\left. \begin{array}{r} 2x - y + z = 7 \\ x + 2y + 3z = 1 \\ -x + y + 5z = 6 \end{array} \right\}$$

(A) -1 (B) 0 (C) 1 (D) 2 (E) 3

Solution: B. Subtracting twice the second equation from the first equation, we obtain $-5y - 5z = 5$ so that $y = -1 - z$. Adding the second and third equations yields $3y + 8z = 7$. We now have $3(-1 - z) + 8z = 7$. Therefore $z = 2, y = -3$, and $x = 1$. The sum is zero.

18. How many positive integer divisors does $N = 250 \cdot 88$ have?

- (A) 24 (B) 28 (C) 30 (D) 32 (E) 40

Solution: E. The prime factorization of N is $2^4 \cdot 5^3 \cdot 11$. Consequently there are $(4 + 1) \cdot (3 + 1)(1 + 1) = 40$ divisors.

19. A $4 \times 4 \times 4$ wooden cube is painted on *five* of its faces and is then cut into 64 unit cubes. One unit cube is randomly selected and rolled. What is the probability that the face showing is painted? Express your answer as a fraction.

- (A) $5/24$ (B) $1/4$ (C) $7/24$ (D) $1/3$ (E) $1/2$

Solution: A. There are $5 \cdot 16 = 80$ painted faces all of which are equally likely. There are $64 \cdot 6 = 384$ faces altogether. Therefore the probability is $80/384 = 5/24$.

20. How many integer triples (x, y, z) satisfy both

- $x, y,$ and z are positive integers less than 30 and
- $xy^2z^3 = 10,000$.

- (A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Solution: A. The prime factorization of 10,000 is $2^4 \cdot 5^4$. Writing $x = 2^a \cdot 5^b, y = 2^c \cdot 5^d$, and $z = 2^e \cdot 5^f$, we have $a + 2c + 3e = 4$ and $b + 2d + 3f = 4$. Because $0 \leq a, b, c, d, e, f$ we can only have $(a, c, e) = (4, 0, 0), (2, 1, 0), (1, 0, 1), (0, 2, 0)$. The same choices are available for (b, d, f) . We do not get 16 possibilities, however. The restriction $x, y, z < 30$ implies that if $a = 4$, then $b = 0$. If $a = 2$ or $a = 1$, then $b \leq 1$, if $a = 0$, then $b \leq 2$. Similar restrictions are true for c, d and e, f . There are six possibilities for (x, y, z) : $(2, 25, 2), (5, 4, 5), (10, 1, 10), (16, 25, 1), (20, 2, 5), (25, 20, 1)$.

21. What is the sum of the three positive integers $a, b,$ and c that satisfy

$$a + \frac{1}{b + \frac{1}{c}} = 5.4?$$

- (A) 8 (B) 9 (C) 10 (D) 11 (E) 12

Solution: B. First note that $a = 5$ and that $\frac{1}{b + \frac{1}{c}} = 0.4$. So $b + \frac{1}{c} = 2.5$ and it follows that $b = 2$ and $c = 2$. Thus $a + b + c = 9$.

22. Use each of the digits 2, 3, 4, 6, 7, 8 exactly once to construct two three-digit numbers M and N so that $M - N$ is positive and is as small as possible. Compute $M - N$.
- (A) 19 (B) 29 (C) 39 (D) 49 (E) 59

Solution: C. Since the hundreds digits must be different, the smaller of the numbers N should have as large a tens digit as possible and the larger M should have as small a tens digit as possible. So we should choose 8 and 2 for these digits provided we can select the hundreds digits to be one apart. Given that the 2 and the 8 are not available, there are just two pairs that we can use, 3 and 4 and 6 and 7. Picking the tens digit of N to be as large as we can and the tens digit of M to be as small as possible leads to $M = 723$ and $N = 684$. Thus the answer we seek is $723 - 684 = 39$. Another possible pair is $M = 426$ and $N = 387$. In each case, $M - N = 39$.

23. How many integers x with $1 \leq x \leq 100$ satisfy the equation $x^2 + x^3 = y^2$ for some integer y ?
- (A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Solution: D. We know x and y are integers and $x^2(x + 1) = y^2$. Using the prime factorization of x and y , we see that $x + 1$ must be a perfect square. Since $1 \leq x \leq 100$, x can be 3, 8, 15, 24, 35, 48, 63, 80, or 99. There are nine possible solutions.

24. Consider the 4×4 Kenken® puzzle below. The solution uses the numbers 1 to 4 exactly once in each row and each column. The sum of the digits in each *cage* is the number given in the upper left corner of one of the squares. What digit goes in the square with the ‘?’ ?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) The given puzzle has no solution.

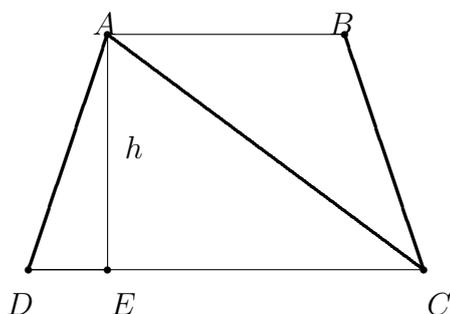
5		7	
	11		?
5			
12			

Solution: D. The solution is unique. The square with the ‘?’ must be a 4. Begin in the cage in the upper left hand corner. (Note that the numbers 3 and 4 must occur in the cage in the upper right hand corner.) Then fill in the cage below it with sum 5. Eventually, you find the following solution:

⁵ 1	2	⁷ 4	3
2	¹¹ 3	1	4
⁵ 4	1	3	2
¹² 3	4	2	1

25. $ABCD$ is an isosceles trapezoid with \overline{AB} parallel to \overline{DC} , $AC = DC$, and $AD = BC$. If the height h of the trapezoid is equal to AB , find the ratio $AB : DC$.

- (A) 2 : 3 (B) 3 : 5 (C) 4 : 5 (D) 5 : 7 (E) 5 : 9



Solution: B. Let E denote the intersection between the altitude h and DC and let $x = DE$. Since $AB = h$ we have $DC = 2x + h$. Noting that $AC = DC$ and applying the Pythagorean theorem to triangle ACE , we have $h^2 + (h + x)^2 = (h + 2x)^2$. This is equivalent to $(h - 3x)(h + x) = 0$. Therefore $h = 3x$ and $AB : DC = h : (h + 2x) = 3 : 5$.

