UNC Charlotte 2012 Algebra

March 5, 2012

1. In the English alphabet of capital letters, there are 15 “stick” letters which contain no curved lines, and 11 “round” letters which contain at least some curved segment. How many different 3-letter sequences can be made of two different stick letters and one curved letter?

   Stick: A E F H I K L M N T V W X Y Z

   Round: B C D G J O P Q R S U

   (A) 2310 (B) 4620 (C) 6930 (D) 13860 (E) none of these

   Solution: (C). 6930. Without regard to order, there are $C(15, 2) = 105$ choices for the two stick letters and 11 choices for the round letter. Allowing for different orders, there are $3! = 6$ ways to order the chosen letters, so the total number with order considered is $6 \times 105 \times 11 = 6930$.

2. The area bounded by the graph of the function $y = |x|$ and by the line $y = c$ is 5. What is c equal to?

   (A) 1 (B) 5 (C) 25 (D) $\sqrt{5}$ (E) $2\sqrt{5}$

   Solution: (D). The area bounded by the two graphs is a triangle with base $2c$ and height c. Thus its area is $c^2 = 5$ hence $c = \sqrt{5}$.

3. Find the value of $\dfrac{266\ldots6}{66\ldots65}$, where both the numerator and the denominator have 2012 digits.

   (A) 13/35 (B) 3/8 (C) 2/5 (D) 3/7 (E) 16/35

   Solution: (C). The numerator is

$$2 \cdot 10^{2012} + 6 \cdot \frac{10^{2012} - 1}{9} = \frac{8}{3} \cdot 10^{2012} - \frac{2}{3} = 2\left(\frac{4}{3} \cdot 10^{2012} - \frac{1}{3}\right).$$

   The denominator is

$$60 \cdot \frac{10^{2012} - 1}{9} + 5 = \frac{20}{3} \cdot 10^{2012} - \frac{5}{3} = 5\left(\frac{4}{3} \cdot 10^{2012} - \frac{1}{3}\right).$$

After simplifying by $4/3 \cdot 10^{2012} - 1/3$ we obtain $2/5$.

Alternatively, one can simply perform a long division to see this.

4. A positive integer equals 11 times the sum of its digits. The units digit of this number is

   (A) 8 (B) 6 (C) 4 (D) 2 (E) 0

   Solution: (A). The number can not have four digits because then it is at least 1000, whereas 11 times the sum of its digits is at most $11 \cdot (9+9+9+9) = 396$. The situation gets worse if we increase the number of digits. The number can not have a single digit either, because then the sum of its digits, multiplied by 11, has at least two digits. Assume the number has two digits, i.e, it is of the form $\overline{ab}$. Then we must have $10a + b = 11a + 11b$ or, equivalently, $0 = a + 10b$. This is again impossible. We are left with the possibility that our number $\overline{abc}$ has three digits. This leads to the equation

   $$100a + 10b + c = 11a + 11b + 11c,$$

   which is equivalent to

   $$89a = b + 10c.$$

   In other words, $89a$ must equal the two digit number $\overline{cb}$. This is only possible if $a = 1$, $b = 9$, $c = 8$. The units digit of 198 is 8.

5. Let a and b be the two roots of the equation $x^2 + 3x - 3 = 0$. Evaluate the value $a^2 + b^2$.

   (A) 4 (B) 9 (C) 10 (D) 12 (E) 15

   Solution: (E). Since $a + b = -3$ and $ab = -3$ so we have $a^2 + b^2 = (a + b)^2 - 2ab = (-3)^2 - 2 \cdot (-3) = 15$.

6. Elizabeth Joy has 30 coins consisting of nickels and dimes. If her nickels were dimes and her dimes were nickels, she would have 80 cents more than she has. How many nickels does she have?

   (A) 18 (B) 19 (C) 20 (D) 21 (E) 23

   Solution: (E). Each interchange of a dime and a nickel increases her total by 5 cents. So she must have $80/5 = 16$ more dimes than nickels. Is n is the number of nickels and d the number of dimes, then $n - d = 16$ and $n + d = 30$. Solving simultaneously, we get $n = 23$.

7. What is the sum of the squares of the roots of $x^4 - 7x^2 + 10 = 0$?

   (A) 4 (B) 6 (C) 9 (D) 10 (E) 14

   Solution: (E). $(x^2 - 5)(x^2 - 2) = 0$ so $x = \pm\sqrt{2}$ and $\pm\sqrt{5}$. So the squares of the roots are 2, 2, 5 and 5. Their sum is 14.

8. How many five-digit integers of the form $(a11bc)_{10} = 10^4 a + 1100 + 10b + c$ (where $a \geq 1$) are divisible by 45?

   (A) 18 (B) 19 (C) 20 (D) 21 (E) 22.

   Solution: (C). c can be either 0 or 5 and $a + b + c + 2$ is divisible by 9. Two cases:

   a) $c = 0$ and $a + b + 2$ is divisible by 9, so $a + b = 16$ (3 cases: $(7, 9)$, $(8, 8)$, $(9, 7)$) or $a + b = 7$ (7 cases: $(1, 6)$, $(2, 5)$, ..., $(7, 0)$). b) $c = 5$ and $a + b + 7$ is divisible by 9, so $a + b = 11$ (8 cases: $(2, 9)$, $(3, 8)$, ..., $(9, 2)$) or $a + b = 2$ (2 cases: $(2, 0)$ and $(1, 1)$). Thus the answer is $3 + 7 + 8 + 2 = 20$.

9. Let a and b denote any of the digits from 1 to 9. How many ordered pairs (a, b) are there with the property that both $10a + b$ and $10b + a$ are primes?

   (A) 8 (B) 9 (C) 10 (D) 11 (E) 13

   Solution: (B). If $a = b$ the only possibility is the number $(1, 1)$. If $a \neq b$, these two digits can only be $1, 3, 7, 9$. We can check that we get the pairs

   $(1, 3)$, $(3, 1)$; $(1, 7)$, $(7, 1)$; $(3, 7)$, $(7, 3)$; $(7, 9)$, $(9, 7)$. Together we get 9 pairs.

10. From the list of all natural numbers $2, 3, \ldots 999$, one deletes sublists of numbers nine times: At first, one deletes all even numbers, then all numbers divisible by 3, then all numbers divisible by 5, and so on, for the nine primes $2, 3, 5, 7, 11, 13, 17, 19, 23$. Find the sum of the composite numbers left in the remaining list.

    (A) 0 (B) 961 (C) 3062 (D) 2701 (E) 899

    Solution: (D).The composite numbers left have all prime factors at least 29. There are three of them less than 1000: $29 \cdot 29 = 841$, $29 \cdot 31 = 899$ and $31 \cdot 31 = 961$, with the sum $841 + 899 + 961 = 2701$.

11. A number c is called a multiple zero of a polynomial $P(x)$ if $P(x)$ has a factor of the form $(x - c)^m$ where $m \geq 2$. How many numbers are multiple zeros of the polynomial $P(x) = (x^6 - 1)(x - 1) - (x^3 - 1)(x^2 - 1)$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Solution: (B). One can factor the polynomial as $(x^6-1)(x-1)-(x^3-1)(x^2-1) = (x^3-1)(x^3+1)(x-1)-(x^3-1)(x-1)(x+1) = (x^3-1)(x-1)[(x^3+1)-(x+1)] = (x-1)(x^2+x+1)(x-1)x(x-1)(x+1) = x(x-1)^3(x+1)(x^2+x+1)$. Hence $x=1$ is the only multiple zero.

12. Two adjacent sides of the unit square and two sides of an equilateral triangle bisect each other. Find the area of the equilateral triangle.

    (A) $\sqrt{3}/2$ (B) 1 (C) 0.8 (D) $\sqrt{2}$ (E) $\sqrt{2}/2$

    Solution: (A). Let M and N be the two points where the two sides of the unit square and two sides of an equilateral triangle bisect each other. Their distance from each other is $|MN| = \frac{\sqrt{2}}{2}$. Hence the side of the equilateral triangle is $a=\sqrt{2}$. Its area is $\frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{2}$.

13. After a ship wreck, a surviving mouse finds himself on an uninhabited island with one kilogram of cheese and no other food. On the 1st day, he eats 1/4 of the cheese, on the 2nd, 1/9 of the remaining cheese, on the 3rd, 1/16 of the remainder, and so on. Let M be the total amount of cheese (measured in kilograms) consumed during the first 6 weeks. Which of the following statements is true?

    (A) $0.4878 < M \le 0.4881$ (B) $0.4881 < M \le 0.4884$
    (C) $0.4884 < M \le 0.4887$ (D) $0.4887 < M \le 0.4890$
    (E) $0.4890 < M \le 0.4893$

    Solution: (B). The initial amount of cheese is 1kg; since on the kth day the mouse eats $\frac{1}{(k+1)^2}$ of the remaining cheese, the amount remaining after n days is

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(n-1)^2}\right)\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right)$$

$$= \frac{1\cdot 3}{2\cdot 2}\cdot\frac{2\cdot 4}{3\cdot 3}\cdot\frac{3\cdot 5}{4\cdot 4}\cdots\frac{(n-2)\cdot n}{(n-1)\cdot(n-1)}\cdot\frac{(n-1)\cdot(n+1)}{n\cdot n}\cdot\frac{n\cdot(n+2)}{(n+1)\cdot(n+1)}$$

$$= \frac{1}{2}\cdot\frac{n+2}{n+1},$$

so that the consumed amount is

$$1-\frac{1}{2}\cdot\frac{n+2}{n+1} = \frac{1}{2}\cdot\frac{n}{n+1}.$$

For $n = 42$, this is

$$\frac{1}{2} \cdot \frac{42}{43} \approx 0.48837.$$

14. Three crazy painters painted the floor in three different colors. One painted 75% of the floor in red on Monday, the second painted 70% of the floor in green on Tuesday, and the third painted 65% of the floor in blue on Wednesday. At least what percent of the floor must be painted in all the three colors?

    (A) 10% (B) 25% (C) 30% (D) 35% (E) 65%

    Solution: (A). 25% of the floor is NOT painted in red. 30% of the floor is NOT painted in green. 35% of the floor is NOT painted in blue. $25\% + 30\% + 35\% = 90\%$ That means at least 10% are not painted in the three color. One can show that such coloring exists.

15. After each mile on a highway from Charlotte (CLT) to the City of Mathematics Students (CMS) there is a two sided distance marker. One side of the marker shows the distance from CLT while the opposite side shows the distance to CMS. A curious student noticed that the sum of the digits from the two sides of each marker stays a constant 13. Find the distance d between the cities.

    (A) $10 \le d < 20$ (B) $20 \le d < 30$ (C) $30 \le d < 40$ (D) $40 \le d < 50$
    (E) $50 \le d < 60$

    Solution: (D). The distance cannot exceed 49 miles since otherwise the marker at 49 miles will have 49 on one side and a positive number on the other side and the sum of the digits would be greater than 13. Now look at the markers at one mile and ten miles. On the other side of these markers, we must have one of the numbers 48 and 39. Thus the first one must be 48 and the second one 39. Thus, the distance can only be 49 miles. On the other hand, one can check that the distance 49 does give a solution.

16. When an orchestra plays a national anthem, its musicians are ordered in a square. When the orchestra plays any other song, the musicians are ordered in a rectangle such that the number of rows increases by five. What is the number m of musicians in the orchestra?

    (A) $333 \le m < 444$ (B) $222 \le m < 333$ (C) $111 \le m < 222$
    (D) $50 \le m < 111$ (E) $m < 50$

    Solution: (A). Let the number of musicians in the orchestra be $m = x \times x$. If we increase the number of rows by five, we must decrease the number of columns by, say, c. The positive integers x and c must satisfy the equation

$$x^2 = (x+5)(x-c),$$

which is equivalent to

$$5c = (5-c)x.$$

The left hand side ($5c$) is positive, hence the right hand side is also positive. Thus $5-c>0$ and c is at most 4. Trying $c=1,2,3,4$ we see that only $c=4$ yields an integer x. Thus we must have $c=4$, $x=20$, and $x^2=400$.

17. Find the number of solutions (m,n) to the equation $1/n - 1/m = 1/12$ where m and n are both natural numbers.

    (A) 8 (B) 7 (C) 6 (D) 5 (E) 4

    Solution: (B). Since $1/n = 1/12 + 1/m > 1/12$, $n \leq 11$. Now, $1/m = 1/n - 1/12$ so $m = 12n/(12-n)$ is a natural number. Plug $n = 1, 2, ..., 11$ in the above, we have seven solutions for the (n,m) pair: $(3,4)$, $(4,6)$, $(6,12)$, $(8,24)$, $(9,36)$, $(10,60)$ and $(11,132)$.

18. Infinitely many empty boxes, each capable of holding six dots are lined up from right to left. Each minute a new dot appears in the rightmost box. Whenever six dots appear in the same box, they **fuse** together to form one dot in the next box to the left. How many dots are there after 2012 minutes? For example, after seven minutes we have just two dots, one in the rightmost box and one in the next box over.

    (A) 8 (B) 9 (C) 10 (D) 11 (E) 12

    Solution: (E). This is just base 6 enumeration in disguise. The base 6 representation of 2012 is 13152, so there are $1+3+1+5+2=12$ dots in the boxes after 2012 minutes.

19. The Fibonacci numbers $1, 1, 2, 3, 5, 8, \ldots$ form a sequence where each term, after the first two, is the sum of the two previous terms. How many of the first 1000 terms are even?

    (A) 222 (B) 333 (C) 499 (D) 500 (E) 501

    Solution: (B). The sequence of remainders, when we divide by 2, is $1, 1, 0, 1, 1, 0 \ldots$, i.e., a periodic sequence that repeats itself after each third entry. Hence the number of even entries among the first thousand Fibonacci numbers is the same as the number of positive integers up to 1000 that are multiples of 3.

20. Let $\dfrac{m}{n}$ denote the fraction, in lowest terms, that is caught between $\frac{4}{7}$ and $\frac{5}{8}$ and has the smallest denominator of any such fraction. What is $m+n$?

    (A) 5 (B) 6 (C) 7 (D) 8 (E) 11

    Solution: (D). One possible fraction is the mediant of $\frac{4}{7}$ and $\frac{5}{8}$, namely $\frac{4+5}{7+8} = \frac{9}{15} = \frac{3}{5}$. To see that no fraction between 4/7 and 5/8 has a smaller denominator, note that $1/3 < 1/2 < 4/7 < 5/8 < 2/3 < 3/4$.

21. Find the remainder of 3^{2012} when it is divided by 7.

    (A) 0 (B) 1 (C) 2 (D) 3 (E) 5

    Solution: (C). $3^3 = 27 \equiv -1 \pmod 7$. $2012 = 3 \times 670 + 2$ so $3^{2011} = 3^2 \cdot 27^{670} \equiv 2 \pmod 7$.

22. How many two-digit numbers are divisible by each of their digits?

    (A) 9 (B) 10 (C) 11 (D) 13 (E) 14

    Solution: (E). We are looking for two-digit numbers $10a+b$ with the property that a and b both divide $10a+b$. Equivalently, a divides b and b divides $10a$. If $a=1$ then b divides 10, thus $b \in \{1,2,5\}$ (three values). If $a=2$ then b is even and divides 20, thus $b \in \{2,4\}$ (two values). If $a=3$ then b is a multiple of 3 and divides 30, thus $b \in \{3,6\}$ (two values). If $a=4$ then $b \in \{4,8\}$ (two values). If a is at least 5, i.e., $a \in \{5,6,7,8,9\}$ then the only way for b be a multiple of a and a digit is $b=a$ (five values). Altogether we get fourteen solutions.

23. Let C be a circle that intersects each of the circles $(x+2)^2+y^2=2^2$, $(x-4)^2+(y-2)^2=2^2$ and $(x-4)^2+(y+2)^2=2^2$ in exactly one point and does not contain any of these circles inside it. If the radius r of C is of the form $r=p/q$ where p and q are relatively prime integers, what is $p+q$?

    (A) 5 (B) 7 (C) 9 (D) 11 (E) 13

    Solution: (B). The center of C is of the form $(r,0)$ by symmetry. Finding its distances to the center $(4,2)$ of $(x-4)^2+(y-2)^2=2^2$ and to the center $(-2,0)$ of $(x+2)^2+y^2=2^2$ lead to $(r+2)^2=(r-4)^2+(0-2)^2$. Solving for r, we get $r=\frac{4}{3}$.

24. What is the sum of the digits for a 5 digit number that has the property that if we put a 1 at the end of the number (making a 6 digit number ending in 1)

the value is three times what we would get if we put the number 1 in front of our five digit number (this gives a six digit number starting with 1)?

(A) 18 (B) 20 (C) 24 (D) 26 (E) 30

Solution: (D). Let x be the five digit number. Then the value when we put one in front is $100000 + x$ and the value when we put the one at the end is $10x + 1$. We get $10x + 1 = 3(100000 + x)$ so $7x = 299999$ and $x = 42857$.

25. Find $\sqrt{\dfrac{100}{2\cdot 3} + \dfrac{100}{3\cdot 4} + \cdots + \dfrac{100}{2011\cdot 2012}}$. Round your answer to the nearest decimal digit.

(A) 3.3 (B) 5.3 (C) 7.1 (D) 8.1 (E) 9.9

Solution: (C). Using the equality $1/k(k+1) = 1/k - 1/(k+1)$ repeatedly, we get

$$\frac{100}{2\cdot 3}+\frac{100}{3\cdot 4}+\cdots+\frac{100}{2011\cdot 2012} = \frac{100}{2}-\frac{100}{3}+\frac{100}{3}-\frac{100}{4}+\cdots+\frac{100}{2011}-\frac{100}{2012} = \frac{100}{2}-\frac{100}{2012}.$$

Thus what we are looking for is $\sqrt{50 - 100/2012} \approx 7.1$.