Test with Solutions for Sponsors

1. Find the minimum value of the function $f(x) = x^2 + \dfrac{1}{x^2+1}, \quad x \in \mathbb{R}$.

   (A) $\frac{1}{16}$ (B) $\frac{3}{16}$ (C) $\frac{1}{4}$ (D) $\frac{3}{4}$ (E) 1

   Answer: E.
   Solution. We have $f(x) = x^2 + 1 + \dfrac{1}{x^2+1} - 1 \geqslant 2\sqrt{(x^2+1)\dfrac{1}{x^2+1}} - 1 = 1$. The equality occurs when $x^2 + 1 = \dfrac{1}{x^2+1}$, which is equivalent to $x = 0$.

2. How many solutions does the equation $\sqrt{x+1} + 2\exp(x^3+1) = 2019$ have?

   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

   Answer: B.
   Solution. The function $f(x) = \sqrt{x+1} + 2\exp(x^3+1) - 1$ is an increasing function on $[-1, \infty)$ with range $[1, \infty)$. Hence the given equation has a unique solution.

3. What is the remainder when $x^{2019} + 2019x - 2018$ is divided by $x - 1$?

   (A) 1 (B) 2 (C) 2017 (D) 2019 (E) 2020

   Answer: B.
   Solution. The remainder is the value of the polynomial at $x = 1$.

4. Let $n \geqslant 2$. Assume that $(x - a_1)(x - a_2)\ldots(x - a_n) = x^n + P(x)$ for all $x \in \mathbb{R}$, where $P(x)$ is a polynomial of degree $n - 2$. Find the value of the sum $a_1 + a_2 + \cdots + a_n$. (A polynomial of degree k is a function of the form $\alpha_k x^k + \alpha_{k-1}x^{k-1} + \cdots + \alpha_0$.)

   (A) 1 (B) -1 (C) n (D) $-n$ (E) 0

   Answer: E.
   Solution. This sum equals the negative of the coefficient of x^{n-1}.

5. Let a be a real number. The system of equations $3x + 2y = 8$ and $ax - 8y = 9$ has no solutions (x, y). What is the value of a?

   (A) 0 (B) 1 (C) 3 (D) -8 (E) -12

   Answer: E.
   Solution. The lines must be parallel, i.e., the answer is E.
   Alternatively, the first equation yields $y = 4 - (3/2)x$. Substitute into the second equation to get $ax - 32 + 12x = 9$. This becomes $(a + 12)x = 41$. If $a = -12$, we get $0 = 41$, so there is no solution (and if $a \neq -12$, there is a solution).

6. How many real numbers x with $0 < x \leqslant 10$ are solutions to $\log_{10}(x) = \sin(x)$, where x in $\sin(x)$ is in radians and $\log_{10}(x)$ is the logarithm of x to base 10?

   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

   Answer: D.
   Solution. Draw the graphs of $y = \log_{10}(x)$ and $y = \sin(x)$. The graphs cross once between 0 and π, and twice between 2π and 3π.

7. Positive integer numbers a and b satisfy the equation $\sqrt{3 + 2\sqrt{2}} = a + b\sqrt{2}$. What is the value of $a + b$?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

   Answer: B.
   Solution. From $3 + 2\sqrt{2} = a^2 + 2ab\sqrt{2} + 2b$, we have $a^2 + 2b = 3$ and $ab = 1$. Hence either $a = b = 1$ or $a = b = -1$, and only in the former case we have $a^2 + 2b = 3$.

8. Let $x^2 + y^2 = 10$. What is the biggest value for xy?

   (A) 10 (B) 20 (C) 5 (D) $10\sqrt{5}$ (E) 6

   Answer: C.
   Solution. Since $0 \leqslant (x - y)^2 = x^2 - 2xy + y^2 = 10 - 2xy$, we have $xy \leqslant 5$. The equality is attained when $x = y$.

9. If $n! = 7!\,6!$ then what is n?

   (A) 8 (B) 9 (C) 10 (D) 13 (E) Such n does not exist

   Answer: C.
   Solution. Since $n > 7$, the problem is equivalent to finding $n \geqslant 8$ such that $8 \cdot \ldots \cdot n = 6! = 720$. Trying successively $n = 8, 9, \ldots$, we find: $n = 10$.

10. What is the value of $\sqrt{1 + 2 + 4 + 8 + 16 + \cdots + 2^{2019}}$, rounded up to the nearest whole number?

    (A) $2^{1010} - 1$ (B) 2^{1010} (C) $2^{1010} + 1$ (D) $2^{2019} - 1$ (E) $2^{2019} + 1$

    Answer: B.
    Solution. The formula for the sum of a geometric series implies that

    $$1 + 2 + 4 + 8 + 16 + \cdots + 2^{2019} = 2^{2020} - 1.$$

    Now $\sqrt{2^{2020}} = 2^{\frac{1}{2}2020} = 2^{1010}$, so $\sqrt{1 + 2 + 4 + 8 + \cdots + 2^{2019}} = \sqrt{2^{2020} - 1}$. Then, putting $m = 2^{1010}$ and $s = \sqrt{2^{2020} - 1}$, we have $m - s = (m^2 - s^2)/(m + s) = 1/(m + s) < 1/m$ so that $m - m^{-1} < s < m$.

11. The numbers x and y satisfy $2^x = 9$ and $3^y = 16$. What is the value of xy?

    (A) 7 (B) 8 (C) $\frac{64}{9}$ (D) $\frac{69}{8}$ (E) $\frac{25}{3}$

    Answer: B.
    Solution. $x = \log 9/\log 2 = (2\log 3)/\log 2$ and $y = \log 16/\log 3 = (4\log 2)/\log 3$, hence $xy = 8$.

12. Let $f(x) = \dfrac{x-1}{x+1}$ and let $f^{(n)}(x)$ denote the n-fold composition of $f(x)$ with itself. That is, $f^{(1)}(x) = f(x)$ and $f^{(n)}(x) = f(f^{(n-1)}(x))$. Which of the following is $f^{(2019)}(x)$?

(A) $-\dfrac{x+1}{x-1}$ (B) $-\dfrac{1}{x}$ (C) $\dfrac{x-1}{x+1}$ (D) x (E) $-\dfrac{x-1}{x+1}$

Answer: A.
Solution. It is convenient to set $f^{(0)}(x) = x$. Then we have $f^{(1)}(x) = f(x)$, $f^{(2)}(x) = -\dfrac{1}{x}$, $f^{(3)}(x) = -\dfrac{x+1}{x-1}$, and $f^{(4)}(x) = x = f^{(0)}(x)$. After that, the sequence $f^{(n)}(x)$ repeats periodically with period 4 so that for any $n \geqslant 0$ we have $f^{(n)}(x) = f^{(r_n)}(x)$, where r_n is the remainder of the division of n by 4. Since $2019 = 4 \cdot 504 + 3$, we have $f^{(2019)}(x) = f^{(3)}(x) = -\dfrac{x+1}{x-1}$.

13. It is known that $a + b + c = 5$ and $ab + bc + ac = 5$. What could be the value of $a^2 + b^2 + c^2$?

(A) 10 (B) 15 (C) 20 (D) 25 (E) 30

Answer: B.
Solution. $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ac) = 5^2 - 2 \cdot 5 = 15$.

14. For which value of a does the straight line $y = 6x$ intersect the parabola $y = x^2 + a$ at exactly one point?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Answer: E.
Solution. This will happen when equation $6x = x^2 + a$ has only one solution, i.e. its discriminant $6^2 - 4a = 0$. Hence, $a = 9$.

15. The solutions of the quadratic equation $x^2 + px + q = 0$ are obtained by adding 5 to each of the solutions of $x^2 - 4x + 2 = 0$. Find the value of $3p + q$.

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Answer: A.
Solution. If x is a root of the equation $x^2 + px + q = 0$, then $x - 5$ is a root of $x^2 - 4x + 2 = 0$, that is,

$$0 = (x-5)^2 - 4 \cdot (x-5) + 2 = x^2 - 14x + 47.$$

Therefore, the quadratic polynomials $x^2 + px + q$ and $x^2 - 14x + 47$, having the same roots and the same leading term x^2, are identical so that $3p + q = 3 \cdot (-14) + 47 = 5$.

16. How many solutions (a, b, c) does the following system have?

$$1 + a + b = ab,$$
$$2 + a + c = ac,$$
$$5 + b + c = bc.$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) Infinitely many

Answer: C.
Solution. The given equations can be rewritten in the form:

$$(a-1)(b-1) = 2,$$
$$(a-1)(c-1) = 3,$$
$$(b-1)(c-1) = 6,$$

or, setting $x = a - 1,\ y = b - 1,\ z = c - 1$, in the form

$$xy = 2,$$
$$xz = 3,$$
$$yz = 6.$$

Multiplying these equations, we get $(xyz)^2 = 36$, so that either $xyz = 6$ or $xyz = -6$. In the former case we have $z = (xyz)/(xy) = 6/2 = 3,\ y = 6/3 = 2,\ x = 6/6 = 1$; in the latter case, similarly, $z = -3,\ y = -2,\ x = -1$. Hence there are two solutions.

17. Find the value of the product $P = \left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{10^2}\right)$.

(A) 0.25 (B) 0.33 (C) 0.44 (D) 0.55 (E) 0.66

Answer: D.
Solution. Use the formula for the difference of two squares:

$$P = \left(1 - \frac{1}{2}\right) \cdot \left(1 + \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) \cdot \left(1 + \frac{1}{3}\right) \cdot \ldots \cdot \left(1 - \frac{1}{9}\right) \cdot \left(1 + \frac{1}{9}\right) \cdot \left(1 - \frac{1}{10}\right) \cdot \left(1 + \frac{1}{10}\right)$$
$$= \frac{1}{2} \cdot \frac{\cancel{3}}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{4}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{4}} \cdot \ldots \cdot \frac{\cancel{9}}{\cancel{8}} \cdot \frac{\cancel{8}}{\cancel{9}} \cdot \frac{\cancel{10}}{\cancel{9}} \cdot \frac{\cancel{9}}{\cancel{10}} \cdot \frac{11}{10} = \frac{1}{2} \cdot \frac{11}{10} = \frac{11}{20} = 0.55.$$

18. The sequence a_n is defined by $a_n = 1 + \sqrt{\frac{1}{n}} - \sqrt{\frac{1}{n+1}} - \sqrt{\frac{1}{n} - \frac{1}{n+1}}$. What is the value of the product $a_1 a_2 \cdots a_{99}$?

(A) $\frac{1}{55}$ (B) $\frac{1}{110}$ (C) $\frac{1}{99}$ (D) $\frac{2}{99}$ (E) $\frac{1}{100}$

Answer: A.
Solution. We start with converting a_n into a product:

$$\begin{aligned} a_n &= 1 + \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}\frac{1}{\sqrt{n+1}} = \left(1 + \frac{1}{\sqrt{n}}\right)\left(1 - \frac{1}{\sqrt{n+1}}\right) \\ &= \frac{\sqrt{n}+1}{\sqrt{n}} \cdot \frac{\sqrt{n+1}-1}{\sqrt{n+1}} = \frac{\sqrt{n}+1}{\sqrt{n}} \cdot \frac{(n+1)-1}{\sqrt{n+1}+1} \cdot \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n}+1}{\sqrt{n+1}+1} \cdot \frac{\sqrt{n}}{\sqrt{n+1}}. \end{aligned}$$

Then

$$\begin{aligned} a_1 \cdot a_2 \cdots a_{99} &= \left(\frac{\sqrt{1}+1}{\cancel{\sqrt{2}+1}} \cdot \frac{\sqrt{1}}{\cancel{\sqrt{2}}}\right) \cdot \left(\frac{\cancel{\sqrt{2}+1}}{\cancel{\sqrt{3}+1}} \cdot \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{3}}}\right) \cdot \left(\frac{\cancel{\sqrt{3}+1}}{\cancel{\sqrt{4}+1}} \cdot \frac{\cancel{\sqrt{3}}}{\cancel{\sqrt{4}}}\right) \cdot \ldots \cdot \\ &= \left(\frac{\cancel{\sqrt{99}+1}}{\sqrt{100}+1} \cdot \frac{\cancel{\sqrt{99}}}{\sqrt{100}}\right) = \frac{2 \cdot 1}{(10+1) \cdot 10} = \frac{1}{55}. \end{aligned}$$

19. The graph of the function $y = \frac{x-3}{x^2 - x + 6}$ is obtained from the graph of $y = \frac{1}{x+2}$ by deleting a single point (u, v). What is the value of $u \cdot v$?

(A) $-\frac{3}{5}$ (B) $-\frac{1}{5}$ (C) 0 (D) $\frac{1}{5}$ (E) $\frac{3}{5}$

Answer: E.
Solution. We may rewrite $y = \frac{x-3}{x^2 - x + 6}$ as $y = \frac{x-3}{(x-3)\cdot(x+2)}$. For $x \neq 3$ we may simplify and get $y = \frac{1}{x+2}$, hence the point to be deleted is $(3, 1/5)$.

20. Find the value of the expression $S = 1! \cdot 3 - 2! \cdot 4 + 3! \cdot 5 - 4! \cdot 6 + \ldots - 2016! \cdot 2018 + 2017!$.

(A) 1 (B) -1 (C) -2018 (D) 2018 (E) 2017

Answer: A.
Solution. Observe that

$$\begin{aligned} S &= 1! \cdot (1+2) - 2! \cdot (1+3) + 3! \cdot (1+4) - 4! \cdot (1+5) + \ldots \\ &\qquad + 2015! \cdot (1+2016) - 2016! \cdot (1+2017) + 2017! \\ &= 1! + 2! - 2! - 3! + 3! + 4! - 4! - 5! + \ldots \\ &\qquad + 2015! + 2016! - 2016! - 2017! + 2017! \\ &= 1. \end{aligned}$$