UNC Charlotte Algebra Competition

March 9, 2009

1. If the operation $\oplus$ is defined by $x \oplus y = 3y + y^x$, then $2 \oplus 5 =$

   (A) 10 **(B)** -10 **(C)** 40 **(D)** -26 **(E)** None of these

   Solution: C. The solution is found by direct substitution.

2. The average of two numbers is xy. If one number is equal to x, then the other number is equal to

   (A) y **(B)** $2y$ **(C)** $xy - x$ **(D)** $2xy - x$ **(E)** None of these

   Solution: D. Let the number be N. Then we must have $\frac{x+N}{2} = xy$. Solving for N, we have $N = 2xy - x$.

3. The product of three consecutive positive integers is 33 times the sum of the three integers. What is the product?

   (A) 330 **(B)** 660 **(C)** 990 **(D)** 1120 **(E)** None of these

   Solution: C. Let x be the first integer. Then $x(x+1)(x+2) = 33(3x+3)$. Then by division, $x(x+2) = 33 \cdot 3 = 99$ since $x+1$ is not zero. Now $x^2 + 2x - 99 = 0$ has solutions $x = 9, -11$. Since x is positive, the product of the integers is $9 \cdot 10 \cdot 11 = 990$.

4. Which of the following statements is false.

   A The sum of 3 consecutive integers is always divisible by 3.
   B The sum of 4 consecutive integers is always divisible by 4.
   C The sum of 5 consecutive integers is always divisible by 5.
   D The sum of 2005 consecutive integers is always divisible by 2005.
   E None of the above.

   Solution: B. Note that A, C, and D all involve an odd number of consecutive integers. There only needs to be a single counter-example to show B is false. A counter example is found by examining the special case of the first 4 consecutive integers, which yields a sum of $\frac{4 \cdot 5}{2} = 10$.

5. The line through the points $(m, -9)$ and $(7, m)$ has slope m. What is the value of m?

   (A) 3 **(B)** $-\dfrac{7}{9}$ **(C)** 16 **(D)** 5 **(E)** None of these

   Solution: A. By the definition of slope we have $m = \frac{m+9}{7-m}$, which has solution $m = 3$.

6. Let x denote the smallest positive integer satisfying $12x = 25y^2$ for some positive integer y. What is $x + y$?

   (A) 75 **(B)** 79 **(C)** 81 **(D)** 83 **(E)** 88

   Solution: C. Note that $25y^2 = (5y)^2$ so $12x = 2^2 \cdot 3x$ must be a perfect square multiple of 5. The smallest integer $3x$ is $3^2 \cdot 5^2$, so $x = 3 \cdot 5^2$ and in this case $y = 6$. Thus $x + y = 81$.

7. College freshman Peter has an abundance of socks. He has 15 pairs, 5 each of brown, blue and black. Sadly, he does not store them in pairs, and when he reaches for socks, he grabs them sight unseen. What is the fewest number of socks Peter must choose in order to be sure to get at least two different color matching pairs?

   (A) 7 **(B)** 10 **(C)** 12 **(D)** 13 **(E)** 15

   Solution: D. The worst case is that he first chooses all 10 of one color and one each two other colors. So, on the 13^{th} pick, there must two different color pairs.

8. The ratio of $2x + y$ to $2y + x$ is 5 to 4. What is the ratio of $x + 3y$ to $3x + y$?

   (A) $3:5$ **(B)** $5:7$ **(C)** $7:9$ **(D)** $9:11$ **(E)** $11:13$

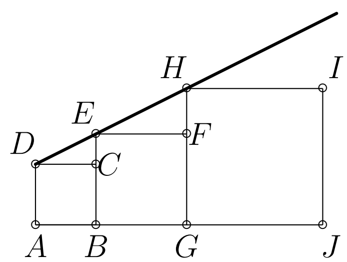
   Solution: B. Massage the equation $5(2y + x) = 4(2x + y)$ to get $x = 2y$ from which it follows that $x + 3y = 5x/2$ and $3x + y = 7x/2$.

9. Using all nine digits $1, 2, 3, 4, 5, 6, 7, 8,$ and 9, build three even integers $M, N,$ and P so that the sum of the three is as small as possible. What is that sum?

   (A) 774 **(B)** 811 **(C)** 828 **(D)** 848 **(E)** 922

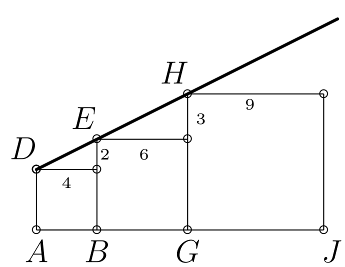
   Solution: C. We want to save the $1, 2,$ and 3 for the hundreds digits, so we have to use the $4, 6,$ and 8 as the units digits. Therefore the sum is $100+200+300+50+70+90+4+6+8 = 828$.

10. Three adjacent squares rest on a line. Line L passes through a corner of each square as shown. The lengths of the sides of the two smaller squares are 4 cm and 6 cm. Find the length of one side of the largest square.



(A) 8 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** 14

Solution: B. The slope of the line is 1/2, so the largest square is $9 \cdot 9$.



11. What is the area of the triangular region in the first quadrant bounded on the left by the y-axis, bounded above by the line $7x + 4y = 168$ and bounded below by the line $5x + 3y = 121$?

    (A) 16 **(B)** 50/3 **(C)** 17 **(D)** 52/3 **(E)** 53/3

    Solution: B. The triangle has a base along the y-axis of $42 - 121/3 = 5/3$ and an altitude of 20 (the lines intersect at $(20, 7)$. So the area is $\frac{1}{2}\frac{5}{3}20 = 50/3$.

12. Given that (x, y) satisfies $x^2 + y^2 = 9$, what is the largest value of $x^2 + 3y^2 + 4x$?

    (A) 22 **(B)** 24 **(C)** 36 **(D)** 27 **(E)** 29

    Solution: E. Replace y^2 with $9 - x^2$ to get $x^2 - 3x^2 + 27 + 4x = 29 - 2x^2 + 4x - 2 = 29 - 2(x - 1)^2$, which must be at least 29. It's value at $x = 1$ is 29.

13. Find the sum $12 + 17 + 22 + 27 + 32 + \ldots + 97$ of all the two-digit numbers whose units digit is either 2 or 7.

    (A) 972 **(B)** 981 **(C)** 990 **(D)** 999 **(E)** 1008

    Solution: B. Subtract 7 from each entry, then divide the resulting terms by 5 to get $1, 2, 3, \ldots 18$, so there are 18 numbers in the sum. The average value is $(12 + 97) \div 2$, so the total sum is $18 \cdot (12 + 97) \div 2 = 9(109) = 981$.

14. If x and y are two-digit positive integers with $xy = 555$, what is $x + y$?

    (A) 116 **(B)** 188 **(C)** 52 **(D)** 45 **(E)** None of these

    Solution: C. By prime factorization, we have $555 = 3 \cdot 5 \cdot 37$ and since x and y and are required to be two-digit integers, the only pair of integers that satisfy the problem conditions is 15 and 37.

15. Find the sum of the x-intercepts of the function

$$g(x) = 3(2x+7)^2(x-1)^2 - (2x+7)(x-1)^3.$$

    (A) $-69/10$ **(B)** $-67/10$ **(C)** $-33/5$ **(D)** $-5/2$ **(E)** $5/2$

    Solution: A. Factor out the common terms to get $g(x) = (2x+7)(x-1)^2[3(2x+7-(x-1)] = (2x+7)(x-1)^2[5x+22]$. Setting each factor equal to zero, we find the zeros are $x = -7/2, x = 1$, and $x = -22/5$. So the sum is $-69/10$.

16. An urn contains marbles of four colors, red, yellow, blue and green. All but 25 are red, all but 25 are yellow, and all but 25 are blue. All but 36 are green. How many of the marbles are green?

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

    Solution: A. Let r, g, y, b denote the number of red, green, yellow and blue marbles respectively. Then the equations translate into $g+b+y = 25, r+g+y = 25, r+b+g = 25$ and $r+b+y = 36$. Adding all four equations together yields $3(r+g+y+b) = 111$, which means that $r+g+y+b = 37$. Subtract the fourth equation from this one to get $g = 1$.

17. Each of four cubes is a different size, such that the side of the largest is twice the side of the smallest, and each adjacent size has an equal increment of length. Which of the following statements are true?

    I. The sum of the volumes of the three smallest cubes equals the volume of the largest cube.

    II. The total length of the largest cube's edges is half the sum of the other three cubes' edges.

III. The sum of the areas of one face on the two middle cubes equals the sum of the areas of one face on the other two cubes.

(A) I only **(B)** II only **(C)** both I and II

(D) III only **(E)** none of these

Solution: C. The cubes must satisfy the edge ratio $3:4:5:6$. Therefore their volumes must have the ratio $3^3:4^3:5^3:6^3$ or $27:64:125:216$. Since $27+64+125=216$, I is true. Every cube has 12 equal edges, so that the sums of their edges are in the same ratio $3:4:5:6$. Clearly 6 is half the sum of $3+4+5=12$, so II is true. The middle two cubes faces areas correspond to 4^2 and 5^2 whose sum is $16+25=41$. The other two cubes face areas correspond to 3^2 and 6^2 whose sum is $9+36=45$. Since $41 \neq 45$, III is false.

18. Define a "prime time" to be when a digital display indicates both hours and minutes as a prime number on a twelve hour clock. Which of the following is closest to the percent of the time is the time displayed "prime time"?

(A) 11.8 **(B)** 17.0 **(C)** 28.3 **(D)** 41.7 **(E)** none of these

Solution: A. The prime hours are 2, 3, 5, 7, and 11. both a.m. and p.m. which is 10 hours. The prime minutes are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59$. There are 17 such minute values. The total duration of 17 minutes in each of 10 hours gives us 170 minutes. One day has $24 \cdot 60 = 1440$ minutes. So $170/1440 \approx 0.118$ or 11.8 percent.

19. If $(mx+7)(5x+n) = px^2 + 15x + 14$, what is $m(n+p)$

(A) -10 **(B)** -520 **(C)** 480 **(D)** 2 **(E)** 520

Solution: C. Since $(mx+7)(5x+n) = 5mx^2 + 35x + mnx + 7n$, it follows that $n = 2$ and $35x + 2m = 15$. Thus $m = -10$ and $5m = p$ so $p = -50$. Hence $m(n+p) = -10(2 + -50) = 480$.

20. Solve for x: $\sqrt{1+\sqrt{3-\sqrt{1+\sqrt{2+\sqrt{x}}}}} = 1$.

    (A) 3844 **(B)** 62 **(C)** $\sqrt{62}$ **(D)** 64 **(E)** 4096

    Solution: A. Square both sides and get $1+\sqrt{3-\sqrt{1+\sqrt{2+\sqrt{x}}}} = 1$ so $\sqrt{3-\sqrt{1+\sqrt{2+\sqrt{x}}}} = 0$ so again square both sides and get $3-\sqrt{1+\sqrt{2+\sqrt{x}}} = 0$ so $\sqrt{1+\sqrt{2+\sqrt{x}}} = 3$ so square both side and get $1+\sqrt{2+\sqrt{x}} = 9$ then $\sqrt{2+\sqrt{x}} = 8$ so again square both sides and have $2+\sqrt{x} = 64$ so $\sqrt{x} = 62$ so square both sides on last time to get $x = 3844$.

21. Solve for x: $8^{3x+1} - 8^{3x} = 448$.

    (A) 2/3 **(B)** 3/2 **(C)** 2 **(D)** 2.93578 **(E)** none of the above

    Solution: A. Factor out 8^{3x} to get $8^{3x} \cdot (8-1) = 448$ so $8^{3x} \cdot 7 = 448$ and $8^{3x} = 64$. Thus $8^{3x} = 8^2$ so $3x = 2$ and $x = 2/3$.

22. What is the product of all the even divisors of 1000?

    (A) 32×10^{12} **(B)** 64×10^{14} **(C)** 128×10^{16}

    (D) 64×10^{18} **(E)** 10^{24}

    Solution: D. Since $1000 = 2^3 5^3$ it has only four odd divisors, $1, 5, 25$, and 125. The product of all 16 divisors is 1000^8 since the 16 divisors form pairs of numbers that have 1000 as their product. Thus the product P of the even divisors is $P = (2^3 \cdot 5^3)^8 \div (5 \cdot 5^2 \cdot 5^3) = 2^{24} \cdot 5^{24} \div 5^6 = 2^{24}5^{18} = 2^6(2\cdot 5)^{18} = 64 \cdot 10^{18}$.

23. Let N denote the largest number satisfying all three of the properties

    (a) N is a product of three consecutive integers,

    (b) N is a sum of three consecutive integers, and

    (c) $N < 1000$.

What is the sum of the digits of N?

(A) 6 **(B)** 12 **(C)** 18 **(D)** 24 **(E)** 36

Solution: C. Every multiple of 3 is the sum of three consecutive integers. Every product of three consecutive integers is a multiple of 3 because on of the integers is itself a multiple of 3. So condition (a) does not add any constraint. The largest number less than 1000 that can be written as a product of three consecutive integers is $9 \cdot 10 \cdot 11 = 990$, and the sum of its digits is 18.

24. How many two-element subsets $\{a, b\}$ of $\{1, 2, 3, \ldots, 16\}$ satisfy ab is a perfect square?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Solution: E. Any two-element subset of $\{1, 4, 9, 16\}$ satisfies the condition. There are just two other sets, $\{2, 8\}$ and $\{3, 12\}$. A four element set has $\binom{4}{2} = 6$ two-element subsets, so there are $6 + 2 = 8$ (unordered) pairs whose product is a perfect square.

25. A rectangular box with integral sides has a volume of 72 cubic units. What is the least possible surface area.

(A) 100 **(B)** 108 **(C)** 114 **(D)** 120 **(E)** 290

Solution: B. There are 12 different shapes. If the dimensions are $a \leq b \leq c$, then the possible triplets are $(1, 1, 72), (1, 2, 36), (1, 3, 24), (1, 4, 18), (1, 6, 12),$ $(1, 8, 9), (2, 2, 18), (2, 3, 12), (2, 4, 9), (2, 6, 6), (3, 3, 8)$, and $(3, 4, 6)$. The surface area S is given by $S = 2(ab + ac + bc)$, which is minimized when the values of a, b, and c are as close together as possible. That happens for the last of the triplets listed, so $S = 2(3 \cdot 4 + 3 \cdot 6 + 4 \cdot 6) = 108$.