UNC Charlotte 2009 Comprehensive

March 9, 2009

1. Danica was driving in a 500 mile race. After 250 miles, Danica's average speed was 150 miles per hour. Approximately how fast should Danica drive the second half of the race if she wants to attain an overall average of 180 miles per hour?

   (A) 210 **(B)** 215 **(C)** 220 **(D)** 225 **(E)** 230

2. Suppose a and b are positive numbers different from 1 satisfying $ab = a^b$ and $a/b = a^{2b}$. Then the value of $8a + 3b$ is

   (A) 26 **(B)** 27 **(C)** 28 **(D)** 29 **(E)** 30

3. Let a, b and c be positive numbers with both a and b greater than 1. Find the solution of the equation $\log_b x - \log_b(x - c) = a$.

   (A) $\dfrac{cb^a}{b^a - 1}$ **(B)** $\dfrac{ab^a}{1 - b^a}$ **(C)** $\dfrac{cb^a}{1 + b^a}$ **(D)** $\dfrac{ab^a}{1 + b^c}$ **(E)** $\dfrac{c}{b^a - 1}$

4. Let x denote the smallest positive integer satisfying $12x = 25y^2$ for some positive integer y. What is $x + y$?

   (A) 75 **(B)** 79 **(C)** 81 **(D)** 83 **(E)** 88

5. What is the area of the triangular region in the first quadrant bounded on the left by the y-axis, bounded above by the line $7x + 4y = 168$ and bounded below by the line $5x + 3y = 121$?

   (A) 16 **(B)** 50/3 **(C)** 17 **(D)** 52/3 **(E)** 53/3

6. We want to divide the L shaped region shown in Figure 1 into two pieces with equal areas by means of a line from P to Q. The point P is always in the upper left hand corner of the region and the point Q must lie along the bottom edge as shown. When this is done, which of the following numbers is closest to the distance from A to Q?

   (A) 1.2 **(B)** 1.3 **(C)** 1.4 **(D)** 1.5 **(E)** 1.6

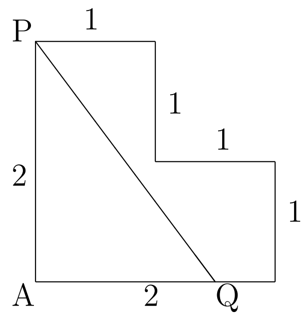

Figure 1: Illustration to Question 6.

7. Let $N = \underline{abcdef}$ be a six-digit number such that $\underline{defabc}$ is six times the value of $\underline{abcdef}$. What is the sum of the digits of N?

   (A) 27 **(B)** 29 **(C)** 31 **(D)** 33 **(E)** 35

8. Three adjacent squares rest on a line. Line L passes through a corner of each square as shown in Figure 3. The lengths of the sides of the two smaller squares are 4 cm and 6 cm. Find the length of one side of the largest square.

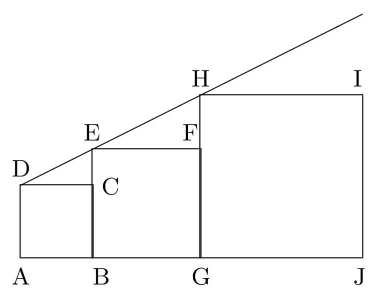

Figure 2: Illustration to Question 8.

   (A) 8 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** 14

9. Given that (x, y) satisfies $x^2 + y^2 = 9$, what is the largest possible value of $x^2 + 3y^2 + 4x$?

   (A) 22 **(B)** 24 **(C)** 36 **(D)** 27 **(E)** 29

10. Two red, two white, and two blue faces, all unit squares, are available for building a cube. How many distinguishable cubes can be built?

    (A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

11. Seven women and seven men attend a party. At this party, each man shakes hands with each other person once. Each woman shakes hands only with men. How many handshakes took place at the party?

    (A) 49 **(B)** 70 **(C)** 91 **(D)** 133 **(E)** 182

12. How many different sums can you get by adding three different numbers from the set $\{3, 6, 9, \ldots, 21, 24\}$?

    (A) 15 **(B)** 16 **(C)** 18 **(D)** 20 **(E)** 22

13. Two different unit squares are randomly selected from the 16 squares in the 4×4 grid shown in Figure 4. What is the probability that they have at least one point in common?

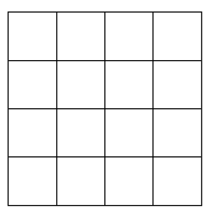

Figure 3: Illustration to Question 13.

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{7}{20}$ **(D)** $\frac{11}{30}$ **(E)** $\frac{3}{8}$

14. A doodlebug is an insect which crawls among the lattice points (points with integer coordinates) of the plane. Each move of a doodlebug is 5 units horizontally or vertically followed by 3 units in a perpendicular direction. For example, from (0,0) a doodlebug can move to any of the eight locations $(\pm5, \pm3), (\pm3, \pm5)$. What is the fewest number of moves required to get from $(0, 0)$ to $(8, 0)$?

    (A) 6 **(B)** 7 **(C)** 8 **(D)** 10 **(E)** No such sequence of moves exists

15. Charlie's current age is a prime number less than 100. The product of the digits of Charlie's age is the same number as it was seven years ago. In how many years will the product of the digits be the same again?

    (A) 7 **(B)** 8 **(C)** 9 **(D)** 11 **(E)** 13

16. Which of the following numbers is closest to the value of the continued fraction $\dfrac{2}{1+\frac{2}{1+\frac{2}{1+\dots}}}$

    (A) 1.00 **(B)** 1.02 **(C)** 1.04 **(D)** 1.06 **(E)** 1.08

17. Suppose that $f(n) = 2f(n + 1) - f(n - 1)$ for $n = 0, \pm1, \pm2, \pm3, \dots$ and $f(1) = 4$ and $f(-1) = 2$. Evaluate $f(2)$.

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

18. Each letter in the long division below stands for a single digit of a decimal number. The letter M is not zero and no leading digit is zero. Different letters may be used for the same digit. What is the value of B?

$$\begin{array}{ccccccccc}
 & & & & \mathrm{J} & \mathrm{K} & 8 & \mathrm{L} & \mathrm{M} \\ \cline{3-9}
\mathrm{A} & \mathrm{B}\,) & \mathrm{C} & \mathrm{D} & \mathrm{E} & \mathrm{F} & \mathrm{G} & \mathrm{H} & \mathrm{I} \\
 & & \mathrm{N} & \mathrm{P} & \mathrm{Q} & & & & \\ \cline{3-9}
 & & & & & \mathrm{R} & \mathrm{S} & & \\
 & & & & & \mathrm{T} & \mathrm{U} & & \\ \cline{3-9}
 & & & & & & \mathrm{V} & \mathrm{W} & \mathrm{X} \\
 & & & & & & \mathrm{Y} & \mathrm{O} & \mathrm{Z} \\ \cline{7-9}
 & & & & & & & & 1
\end{array}$$

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

19. Which of the following numbers is the sum of the squares of three consecutive odd numbers?

    (A) 1281 **(B)** 1441 **(C)** 1595 **(D)** 1693 **(E)** 1757

20. If $(2^x - 4^x)^2 + (2^x + 4^x)^2 = 144$, what is x?

    (A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{5}{4}$ **(E)** $\frac{3}{2}$

21. An urn contains marbles of four colors, red, green, blue and yellow. All but 25 are red, all but 25 are yellow, and all but 25 are blue. All but 36 are green. How many of the marbles are green?

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

22. How many two-element subsets $\{a, b\}$ of $\{1, 2, 3, \ldots, 16\}$ satisfy ab is a perfect square?

    (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8