UNC Charlotte 2004 Algebra with solutions

March 8, 2004

1. Let z denote the real number solution to $\sqrt{3+\sqrt{x-1}} = 5$. What is the sum of the digits of z?

   (A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

   Solution: E. Square both sides twice to get $z = (5^2 - 3)^2 + 1 = 485$, so the sum of the digits is $4 + 8 + 5 = 17$.

2. Which of the following is equivalent, where it is defined, to

   $$\frac{1 - \frac{1-x}{1+x}}{1 + \frac{x-1}{x+1}}?$$

   (A) 1 **(B)** $\dfrac{4x}{x+1}$ **(C)** $\dfrac{1}{x}$ **(D)** 0 **(E)** $\dfrac{2x}{2x-2}$

   Solution: A. Note that $x - 1 = -(1 - x)$ so the numerator and denominator are the same. Alternatively, combine the fractions and simplify.

3. A two-inch cube $(2 \times 2 \times 2)$ of silver weighs 3 pounds and is worth \$320. How much is a three-inch cube of silver worth?

   (A) \$480 **(B)** \$600 **(C)** \$800 **(D)** \$900 **(E)** \$1080

   Solution: E. The value of a $3 \times 3 \times 3$ cube is $(27/8)(320) = 27 \cdot 40 = 1080$.

4. Which of the following lines has a slope that is less than the sum of its x- and y- intercepts?

   (A) $y = 2x + 1$ **(B)** $y = 3x/2 - 1$ **(C)** $y = -4x - 1$

   (D) $y = 4x + 16/3$ **(E)** $y = 3x$

   Solution: C. Check these to see that only $y = -4x - 1$ satisfies the condition.

5. What is the remainder when $x^4 - x^2 + 1$ is divided by $x^2 + 1$?

   (A) -3 **(B)** -1 **(C)** 0 **(D)** 3 **(E)** 4

   Solution: D. By long division $x^4 - x^2 + 1 = (x^2 + 1)(x^2 - 2) + 3$, so the remainder is 3. Alternatively, since all of the exponents are even, the remainder obtained from dividing $x^4 - x^2 + 1$ by $x^2 + 1$ is the same as the remainder obtained from dividing $y^2 - y + 1$ by $y + 1$. For the latter pair, the remainder can be found easily by simply calculating $(-1)^2 - (-1) + 1 = 3$ since $y + 1 = 0$ for $y = -1$.

6. An ancient Greek problem is the following–Make a crown of gold, copper, tin, and iron weighing 60 minae: gold and copper shall be 2/3 of it; gold and tin 3/4 of it; and gold and iron 3/5 of it. How many minae of gold are in the crown?

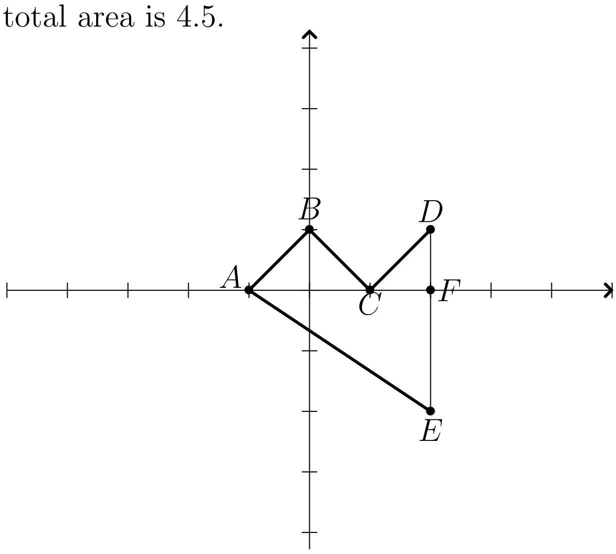
   (A) 30.5 **(B)** 31.5 **(C)** 33.5 **(D)** 35.5 **(E)** 36.5

   Solution: A. Let g, c, t, and i denote the number of minae of gold, copper, tin, and iron respectively. Then $g + c = 40$, $g + t = 45$, $g + i = 36$, and $g + (40 - g) + (45 - g) + (36 - g) = 60$ and it follows that $g = 30.5$ minae (9.5 minae of copper, 14.5 minae of tin, and 5.5 minae of iron). Alternatively, since $(g + c) + (g + t) + (g + i) = 2g + (g + c + t + i) = 121$. Thus $2g = 61$ since $g + c + t + i = 60$.

7. Find the area of the polygon $ABCDE$ if the vertices are located at the following coordinates: $A = (-1, 0), B = (0, 1), C = (1, 0), D = (2, 1), E = (2, -2)$.

   (A) 3 **(B)** 3.5 **(C)** 4 **(D)** 4.5 **(E)** 5

   Solution: D. Let F be the point with coordinates $(2, 0)$. Then triangle ABC has area 1, triangle CDF has area 0.5, and triangle AFE has area 3. The total area is 4.5.



8. The mean of three numbers is ten more than the least of the numbers and fifteen less than the greatest of the three. If the median of the three numbers is 5, then the sum of the three is

   (A) 5 **(B)** 20 **(C)** 25 **(D)** 30 **(E)** 36

   Solution: D. Let a, b, and c denote the three numbers and assume $a \leq b \leq c$. Then $(a + b + c)/3 = 10 + a = c - 15$. The number b is the median, which is 5. Therefore, $a + c + 5 = 30 + 3a$ and $a + c + 5 = 3c - 45$. Thus $2a - c = -25$ and $2c - a = 50$. Solving simultaneously for a and c, we get $a = 0$ and $c = 25$. Therefore $a + b + c = 30$. Alternatively, without solving for a and c, we may suppose $a < b < c$. So $b = 5$ is the median. Avoiding fractions, the first condition is equivalent to $a + 5 + c = 3a + 30$, and the second is equivalent to $a + 5 + c = 3c - 45$. So $c = 2a + 25$ and $a = 2c - 50$. Add the two equations together to get $a + c = 2(a + c) - 25$. Thus $a + c = 25$ and $a + b + c = 30$.

9. Let N denote the smallest four-digit number with all different digits that is divisible by each of its digits. What is the sum of the digits of N?

   (A) 7 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** 16

   Solution: D. None of the four digits can be 0 because no number is divisible by 0. Thus the smallest four digit number with four different digits that we need consider is 1234. But this is not divisible by either 3 or 4; the next number 1235 is not divisible by either 2 or 3. Trying 1236, we see that it is divisible by each of its digits.

10. The polynomial $p(x) = 4x^3 - ax^2 - 41x + b$ has zeros at $x = -1/2, 3$, and $-7/2$. What is the product of a and b?

    (A) -55 **(B)** 55 **(C)** 0 **(D)** 60 **(E)** 84

    Solution: E. Recalling the relationship between the coefficients of x^{n-1} and x^0 of a degree n polynomial, and the sum and product of the zeros respectively, we find that $a/4 = -1/2 + 3 - 7/2 = -1$, so $a = -4$, and $-b/4 = (-1/2) \cdot 3 \cdot (-7/2) = 21/4$, so $b = -21$. The product is 84. Alternatively, you can find a and b by solving simultaneously any two of the equations $p(-1/2) = 0, p(3) = 0$, and $p(-7/2) = 0$. Yet another possible solution comes by taking the product $4 \cdot (x + \frac{1}{2})(x - 3)(x + \frac{7}{2})$ and finding a and b from what is obtained.

11. Suppose a and b are positive integers for which $(2a+b)^2-(a+2b)^2=9$. What is ab?

    (A) 2 **(B)** 6 **(C)** 9 **(D)** 12 **(E)** 24

    Solution: A. Expand and cancel to get $4a^2+4ab+b^2-a^2-4ab-4b^2=9$, so $3a^2-3b^2=9$. This can happen only if $(a-b)(a+b)=3$, which is true in case $a-b=1$ and $a+b=3$ since $a+b$ cannot be negative. It follows that $a=2$ and $b=1$, so $ab=2$.

12. For which of the following values of a does the line $y=a(x-3)$ and the circle $(x-3)^2+y^2=25$ have two points of intersection, one in the 1^{st} quadrant and one in the 4^{th} quadrant?

    (A) -1 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** None of A,B,C, and D

    Solution: D. The line goes through the center of the circle, $(3,0)$ so it must intersect the circle twice. For $a=-1$ the intersection includes a point in the second quadrant. For $a=2$, the line is $y=2x-6$, so $(x-3)^2+(2x-6)^2=25$, which becomes $(x-3)^2=5$, which has two positive solutions. For $a=0$ the intersection includes only points on the x-axis. For $a=1$, the intersection includes a point of the third quadrant.

    Alternate Solution: The intersection of the circle with the y-axis is a pair of points $(0,4)$ and $(0,-4)$. The slope of the line through the center and $(0,4)$ is $-4/3$ and the slope of the line thought the center and the point $(0,-4)$ is $4/3$. The lines with slopes 1, 0 and -1 cross the y-axis between $(0,4)$ and $(0,-4)$, so cannot intersect the circle in both the first and fourth quadrants. On the other hand, a line through the center whose slope has absolute values greater than $4/3$ intersects the circle in both of these quadrants. Thus $a=2$ works.

13. Which of the following is not an asymptote of the function

$$R(x)=\frac{|x|(x-2)(x+3)}{x(x+2)(x-3)}?$$

    (A) $x=0$ **(B)** $x=-2$ **(C)** $x=3$ **(D)** $y=1$ **(E)** $y=-1$

    Solution: A. The numbers $x=-2$ and $x=3$ are zeros of the denominator but not the numerator. Also, $\lim_{x\to\infty} R(x)=1$ and $\lim_{x\to-\infty} R(x)=-1$, so the function has both horizontal asymptotes. But the line $x=0$ is not a vertical asymptote.

14. The non-zero real numbers a, b, c, d have the property that $\frac{ax+b}{cx+d} = 1$ has no solution in x. What is the value of $\frac{a^2}{a^2+c^2}$?

    (A) 0 **(B)** 1/2 **(C)** 1 **(D)** 2 **(E)** an irrational number

    Solution: B. The equation is solvable if $ax + b = cx + d$ which is equivalent to $(a - c)x = d - b$. This has a solution unless $a = c$. Therefore $a = c$ and it follows that $\frac{a^2}{a^2+c^2} = \frac{1}{2}$.

15. What is the sum of the coefficients of the expanded form of $(2x - 3y + 3)^4$?

    (A) 0 **(B)** 16 **(C)** 81 **(D)** 625 **(E)** 1000

    Solution: B. Let $x = y = 1$. Then each term of the expansion is its coefficient, and the sum of these coefficients is $(2 - 3 + 3)^4 = 16$. Alternatively, expand the expression and add the coefficients.

16. For what value of k do the graphs $f(x) = |x + 1|$ and $g(x) = |x/2 - 3| + k$ intersect in exactly one point?

    (A) -3.5 **(B)** -2.5 **(C)** -1.5 **(D)** 0.5 **(E)** 2.5

    Solution: A. Since $f(-1) = 0$, to satisfy the hypothesis, the point $(-1, 0)$ must belong to the graph of g. Thus $g(-1) = |-1/2 - 3| + k = 0$, from which it follows that $k = -3.5$.

17. Let the function f be defined by $f(x) = x^2 + 40$. If m is a positive number such that $f(2m) = 2f(m)$ which of the following is true?

    (A) $0 < m \leq 4$ **(B)** $4 < m \leq 8$ **(C)** $8 < m \leq 12$ **(D)** $12 < m \leq 16$ **(E)** $16 < m$

    Solution: B. Note that $f(2m) = (2m)^2 + 40 = 4m^2 + 40$ and $2f(m) = 2(m^2 + 40) = 2m^2 + 80$. It follows that $2m^2 = 40$, so $4 < m \leq 8$.

18. Suppose that $\log_b 7 = 1.209062$. Which of the following values is closest to b?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Solution: C. The given equation is equivalent to $b^{1.209062} = 7$. Taking the common log of both sides we obtain $1.209062 \log b = \log 7$. Therefore $\log b = \frac{\log 7}{1.209062} \approx 0.6987$ and $b = 10^{0.6987} \approx 5$. Alternatively, simply calculate $b^{1.209062}$ for the five numbers given. Start in the middle, that way if $5^{1.209062}$ isn't equal to 7, which side of 7 it is on tells you whether to try larger values for b or smaller ones. In this case a "miracle" happens and it turns out that 5 is the correct choice.

19. Suppose the number a satisfies $a \log a + \log \log a - \log(\log \log 2 - \log \log a) = 0$. What is the value of $a^{\left(a^{(a^a)}\right)}$?

(A) 1 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** 16

Solution: B. Use the laws of logarithms to get $a^a \cdot \log a = \log \log 2 - \log \log a$, which is equivalent to $\log a^{(a^a)} = \log \left(\frac{\log 2}{\log a}\right)$. Since the two logs are equal, their arguments are also equal, so $a^{(a^a)} = \frac{\log 2}{\log a}$. But this implies that $\log a \cdot a^{(a^a)} = \log a^{a^{(a^a)}} = \log 2$, and from this it follows that $a^{\left(a^{(a^a)}\right)} = 2$.

20. Find an ordered pair (n, m) of positive integers satisfying

$$\frac{1}{n} - \frac{1}{m} + \frac{1}{mn} = \frac{2}{5}.$$

What is mn?

(A) 5 **(B)** 10 **(C)** 15 **(D)** 20 **(E)** 45

Solution: B. By multiplying each fraction by $5mn$, we can transform the equation into the equivalent $2mn = 5(m - n + 1)$. Notice that this implies that either m or n is a multiple of 5. This leads us to try various multiples of 5 for m. Alternatively, transform the equation further to get

$$2mn - 5m + 5n - 5 = 0.$$

Subtract $-15/2$ from both sides so that we can factor to get

$$m(2n - 5) + 5(2n - 5)/2 = -15/2.$$

Massage this to get $(2n-5)(2m+5) = -15$. Since m is positive, $2m+5 > 0$. Therefore, $2n - 5 < 0$. The only values of n making $2n - 5 < 0$ are $n = 1$ and $n = 2$. The choice $n = 1$ requires $m = 0$. So the solution $m = 5, n = 2$ is unique. Alternatively, clear the fractions to get $5(m+1)-5n = 5m-5n+5 = 2mn$. Since m and n are positive integers, the same is true for $2mn$. Thus $m + 1 > n$ and $5 - 5n \leq 0$. If $n = 1$, we would have $5m = 2m$ which is impossible. Thus $n > 1$ and we have $5m > 2mn$ so $5 > 2n$. Therefore the only possible value for n is 2. Now simply solve for m: $5m - 10 + 5 = 4m$, so $m = 5$ and $mn = 10$.

21. A standard deck of playing cards with 26 red and 26 black cards is split into two non-empty piles. In pile A there are four times as many black cards as red cards. In pile B, the number of red cards is an integer multiple of the number of black cards. How many red cards are in Pile B?

    (A) 16 **(B)** 18 **(C)** 20 **(D)** 22 **(E)** 24

    Solution: C. Let r_1 and b_1 denote the number of red and black cards in pile A, so the numbers in pile B must be $r_2 = 26 - r_1$ and $b_2 = 26 - b_1$. Also, the numbers r_1 and b_1 satisfy $b_1 = 4r_1$ and $26 - r_1 = k(26 - b_1)$ for some integer k. Combining these equations and solving for r_1, we find that

$$r_1 = \frac{26(k-1)}{4k-1}.$$

Trying values of k starting at 2, we finally find that when $k = 10$, r_1 is an integer. Solving this we get $r_1 = \frac{9 \cdot 26}{39} = 6$ and thus $r_2 = 20$. Alternate Solution: Let r be the number of red cards in stack A and let b be the number of black cards in stack A. Then $4r = b > 0$ and $26 - r = p(26 - b)$ for some integer p. Since b is even and p is an integer, r must be even. Thus b is a positive integer multiple of 8. The only such numbers less than or equal to 26 are 8, 16 and 24. Having $b = 8$ requires $r = 2$. For this pair $24 = 26 - 2$ is not an integer multiple of $18 = 26 - 8$. Having $b = 16$ requires $r = 4$. For this pair $22 = 26 - 4$ is not an integer multiple of $10 = 26 - 16$. But for $b = 24$ we have $r = 6$ and $20 = 26 - 6 = 10(2) = 10(26 - 24)$.

For yet another approach, substitute $4r$ for b into the second equation and solve for r to get $r = 26(p-1)/(4p-1)$. If the denominator is not an integer multiple of 13, then r will be. But that is impossible since $4r = b \leq 26$. Thus $4p - 1$ must be an odd integer multiple of 13. The smallest positive integer multiple of 13 that is one less than a multiple of 4 is 39. This occurs when $p = 10$. The corresponding value for r is 6. This yields $b = 24$. [Here is a formal proof that this is the only positive integer solution: If $0 < r < s$ are integer multiples of 13 that are both one less than a multiple of 4, then $s - r$ is both an integer multiple of 13 and an integer multiple of 4. Thus every positive integer multiple of 13 that is both one less than a multiple of 4 and strictly larger than 39 is of the form $39 + 4 \cdot 13k = 39 + 52k$ for some positive integer k. Let $q = 10 + 13k$. Then $4q - 1 = 39 + 52k$. Substitute q in for p in the fraction $26(p-1)/(4p-1)$ and then replace q by $10 + 13k$. The resulting fraction is $26(9+13k)/(39+52k)$ which reduces to $(18 + 26k)/(3 + 4k) = 6 + (2k/(3 + 4k)) > 6$ since $k > 0$.

Obviously, $2k/(3+4k)$ is never an integer when k is a positive. So the only solution is when $p = 10$.]

22. The graphs of $x^2 + y^2 = 24x + 10y - 120$ and $x^2 + y^2 = k^2$ intersect when k satisfies $0 \le a \le k \le b$, and for no other positive values of k. Find $b - a$.

    (A) 10 **(B)** 14 **(C)** 26 **(D)** 34 **(E)** 144

    Solution: B. The first circle $(x-12)^2 + (y-5)^2 = 49$ is centered at $(12, 5)$ and has radius 7, while the second is centered at $(0, 0)$ and has radius k. The two circle intersect when $6 \le k \le 20$, so $b - a = 14$.

    Alternatively, let C be the circle given by the equation $x^2+y^2 = 24x+10y-120$. An equivalent equation for C is $(x - 12)^2 + (y - 5)^2 = 49$. So the radius of C is 7 and the center is $(12, 7)$. Let L be the line through the origin and the point $(12, 5)$, the center of C. Since the radius of C is less than 12, the line L intersects C at two points in the first quadrant, call them P and Q and assume the x-coordinate of P is smaller than the x-coordinate of Q. The point P must be on the circle with center at $(0, 0)$ and radius a and the point Q must be on the circle with center at $(0, 0)$ and radius b. Since P and Q are on the circle C, on a line through the center of C and on the same line through the center of the other two circles, the distance between P and Q is $b - a = 14$.

23. The product of three consecutive non-zero integers is 33 times the sum of the three integers. What is the sum of the digits of this product?

    (A) 5 **(B)** 6 **(C)** 12 **(D)** 16 **(E)** 18

    Solution: E. If the three integers are denoted $n - 1, n$, and $n + 1$, then their sum is $3n$ and their product is $n(n^2 - 1)$. Thus $n(n^2 - 1) = 33 \cdot 3n = 99n$, from which it follows that $n^2 - 1 = 99$ and so $n = 10$. The product is 990 and the sum of the digits is $9 + 9 + 0 = 18$.

24. It is possible that the difference of two cubes is a perfect square. For example, $28^2 = a^3 - b^3$ for certain positive integers, a and b. In this example, what is $a + b$?

    (A) 12 **(B)** 14 **(C)** 16 **(D)** 18 **(E)** 20

    Solution: C. Since $28^2 = a^3 - b^3$ is an even number, a and b must have the same parity (either both are odd or both are even). In either case, $a-b$ is even. Next note that $b \neq 1$ since 28^2+1^3 is not a perfect cube. Therefore $b \geq 2$. Note that $28^2 = a^3-b^3 = (a-b)(a^2+ab+b^2)$. Now $a-b < 8$ because $a-b \geq 8$ implies that $a^2 + ab + b^2 \leq 28^2/8 = 98$, but $a \geq b + 8 \geq 10$ implies $a^2 + ab + b^2 \geq 100$. Thus $a - b$ is an even factor of 28^2 that is less than 8. The only possibilities are 2 and 4. Trying $a - b = 2$ yields the quadratic $a^2 + ab + b^2 = 196$ which reduces to $(b+2)^2+(b+2)b+b^2 = 196$, which has no integer solutions. Trying $a - b = 4$ yields $a = b + 4$ and so $(b + 4)^2 + (4 + b)b + b^2 = 28^2 \div 4 = 196$. This reduces to $3b^2 + 12b - 180 = 0$, which can be factored to yield roots of $b = -10$ and $b = 6$. It follows that $a = 10$ and $a + b = 16$.

    Alternatively, one could build a table with values $784 + n^3$ for integer values of n and see when a perfect cube comes up. Yet another alternative is to rewrite the equation as $28^2 + b^3 = a^3$. Since a and b are positive integers, $a^3 > 28^2 = 784$. Note that $1000 = 10^3$ is larger than 784 and $729 = 9^3$ is smaller. Thus the smallest integer that might possibly work for a is $a = 10$. This actually works since $1000 - 784 = 216 = 6^3$. So if there is a correct single answer, it must be that $a + b = 16$.

25. The three faces of a rectangular box have areas of 40, 45, and 72 square inches. What is the volume, in cubic inches, of the box?

    (A) 300 **(B)** 330 **(C)** 360 **(D)** 400 **(E)** 450

    Solution: C. Let x, y, and z denote the dimensions of the box. Then, $x^2y^2z^2 = xy \cdot xz \cdot yz = 40 \cdot 45 \cdot 72 = 129,600$. Therefore, $xyz = \sqrt{129,600} = 360$. Alternatively, let x, y and z be the dimensions with $xy = 40$, $xz = 45$ and $yz = 72$. Solve directly for x by solving $yz = 72$ for z and then substituting into $xz = 45$ to get $72x/y = 45$. Thus $y = 72x/45$. Substitute this into $xy = 40$ to obtain $72x^2/45 = 40$. This yields $x^2 = 25$, so $x = 5$. Thus the volume is $x(yz) = 5(72) = 360$. [Or solve for y or z.]

26. Two women and three girls wish to cross a river. Their small rowboat will carry the weight of only one woman or two girls. What is the minimum number of times the boat must cross the river in order to get all five females to the opposite side? At least one person must be in the boat each time it crosses the river.

    (A) 9 **(B)** 10 **(C)** 11 **(D)** 13 **(E)** 15

Solution: C. It takes 11 trips back and forth. We can write this as follows: gg g w g gg g w g gg g gg where the first, third, fifth, etc symbol represents a trip from the starting bank to the destination bank, and the symbols in the even positions represent the return trips.