UNC Charlotte 2011 Algebra

March 7, 2011

1. Find the product of all real solutions of $16^{x^2+x+4} = 32^{x^2+2x}$.

   (A) -18 **(B)** -16 **(C)** -15 **(D)** -12 **(E)** -10

   Solution: (B). Since $2^4 = 16$ and $2^5 = 32$, we have $(2^4)^{x^2+x+4} = (2^5)^{x^2+2x}$ so that $4x^2+4x+16 = 5x^2+10x$. This implies $x^2+6x-16 = 0$ or $(x+8)(x-2) = 0$. Thus, $x = -8$ or $x = 2$ and the product of the solutions is -16.

2. If $4(9a - 13b) = 6(a - 2b)$ and $b \neq 0$, what is the ratio of a to b?

   (A) $3 : 4$ **(B)** $3 : 5$ **(C)** $1 : 1$ **(D)** $5 : 3$ **(E)** $4 : 3$

   Solution: (E). Since $36a - 52b = 6a - 12b$, we find $30a = 40b$ so that $a = \frac{4}{3}b$.

3. The function f satisfies the equation $f(x) + 2f(3 - x) = 4x + 5$ for all real x. What is the value of $f(1)$?

   (A) $\dfrac{17}{3}$ **(B)** $\dfrac{16}{3}$ **(C)** 5 **(D)** $\dfrac{14}{3}$ **(E)** 4

   Solution: (A). Substitute $x = 1$ and $x = 2$ to get $f(1) + 2f(2) = 9$ and $f(2) + 2f(1) = 13$. Solving these two equations for $f(1)$ yields $f(1) = \frac{17}{3}$.

4. Each person in a group of five makes a statement: Amy says "At least one of us is lying." Ben says "At least two of us are lying." Carrie says "At least four of us are lying." Donna says "All of us are lying." Eddie says "None of us is lying." Based on these statements, how many are telling the truth?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

   Solution: (B). Donna's statement cannot be true so Donna and Eddie are lying. This makes Amy's statement and Ben's statement true. Therefore, Carrie is lying and the correct answer is (B).

5. A worm is slowly crawling to the top of a pole 70cm high. During the day it advances 7cm and during the night it slips down 4cm. When will it finally reach the top?

   (A) In 24 days **(B)** In 23 days **(C)** In 22 days

   (D) In 21 days **(E)** In 20 days

   Solution: (C). After 20 days and nights, the worm is at a height of 60 cm. On the twenty first day it reaches 67cm and then slips back to 63 cm at night. On day 22 it reaches 70cm.

6. Three cowboys entered a saloon. The first ordered 4 sandwiches, a cup of coffee, and 10 doughnuts for \$8.45. The second ordered 3 sandwiches, a cup of coffee, and 7 doughnuts for \$6.30. How much did the third cowboy pay for a sandwich, a cup of coffee, and a doughnut?

   (A) \$2.00 **(B)** \$2.05 **(C)** \$2.10 **(D)** \$2.15 **(E)** \$2.20

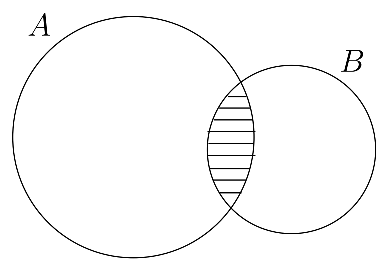
   Solution: (A). From the payment of the first cowboy we find the price of 8 sandwiches, 2 cups of coffee, and 20 doughnuts = \$16.90. From the payment of the second cowboy we calculate the price of 9 sandwiches, 3 cups of coffee, and 21 doughnuts = \$18.90. The difference of the sums $18.90 - 16.90 = 2.00$ is exactly the price of one sandwich, a cup of coffee, and a doughnut.

7. Given that $a+b+c = 5$ and $ab+bc+ac = 5$, what is the value of $a^2+b^2+c^2$?

   (A) 5 **(B)** 10 **(C)** 15 **(D)** 20 **(E)** 25

   Solution: (C). We use the formula $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ac$. We have $a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ac) = 5^2 - 2\cdot 5 = 15$.

8. Two circles, A and B, overlap each other as shown. The area of the common part is $2/5$ of the area of Circle A, and $5/8$ of the area of Circle B. What is the ratio of the radius of Circle A to that of Circle B?



(A) $2:1$ **(B)** $3:2$ **(C)** $4:3$ **(D)** $5:4$ **(E)** $6:5$

Solution: (D). Let the radii of Circle A and B be r and R. Then we have $\frac{2}{5}\pi r^2 = \frac{5}{8}\pi R^2$. Thus, $16r^2 = 25R^2$ and $4r = 5R$ The ratio is $5:4$.

9. For integers a and b, the quadratic equation $x^2 + ax + 4b = 0$ has one solution at $x = 1$ and the other solution is between 2 and 5. Find the value of ab.

(A) -5 **(B)** -3 **(C)** -1 **(D)** 4 **(E)** 6

Solution: (A). Since one solution is $x = 1$, we get $1 + a + 4b = 0$, and we can rewrite the given equation as $(x-1)(x-4b) = 0$. So the other solution is $4b$, and we obtain $2 < 4b < 5$, that is, $\frac{1}{2} < b < \frac{5}{4}$ for integer b. Thus, $b = 1$, and $a = -5$.

10. A pine tree is 14 yards high, and a bird is sitting on its top. The wind blows away a feather of the bird. The feather moves uniformly along a straight line at the speed 4 yards per second; it falls on the ground 4.5 seconds later at a distance D yards from the pine tree's base. Which of the following intervals contains the number D?

(A) $(0, 10]$ **(B)** $(10, 11]$ **(C)** $(11, 12]$ **(D)** $(12, 13]$ **(E)** $(13, infinity)$

Solution: (C). The right triangle formed by the trunk of the tree, the path of the feather and its projection on the ground, has hypotenuse 18 and legs 14 and D, so that $D^2 + 14^2 = 18^2$, or $D^2 = 324 - 196 = 128$; since $11^2 = 121 < 128 < 144 = 12^2$, we have $11 < D < 12$.

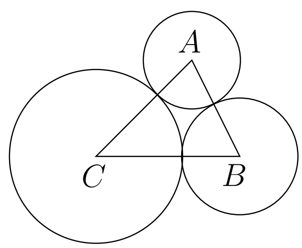
11. If Bob can beat Jim by one-tenth of a mile in a two-mile race and Jim can beat Henry by one-fifth of a mile in a two mile race, by what distance could Bob beat Henry in a two-mile race?

(A) 0.13 miles **(B)** 0.20 miles **(C)** 0.29 miles

(D) 0.37 miles **(E)** 0.53 miles

Solution: (C). Bob runs $\frac{2}{1.9}$ times as fast as Jim. Jim runs $\frac{2}{1.8}$ times as fast as Henry, so Bob runs $\frac{4}{(1.9)(1.8)} = \frac{2}{1.71}$ times as fast as Henry. Thus, Bob beats Henry by $2 - 1.71 = 0.29$ miles.

12. Three mutually tangent circles have centers A, B, C and radii a, b, and c respectively. The lengths of segments AB, BC, CA are 17, 23, and 12 respectively. Find the lengths of the radii.



(A) $a = 13, b = 9, c = 7$ **(B)** $a = 5, b = 12, c = 6$ **(C)** $a = 6, b = 9, c = 12$
(D) $a = 4, b = 8, c = 12$ **(E)** $a = 3, b = 14, c = 9$

Solution: (E). We have $a + b = 17$, $b + c = 23$, and $c + a = 12$. Adding the equations gives $2(a + b + c) = 52$ or $a + b + c = 26$. Subtracting each of the original equations from this last equation gives $c = 9, a = 3$, and $b = 14$.

13. A ladder is leaning against a wall with the top of the ladder 8 feet above the ground. If the bottom of the ladder is moved 2 feet farther from the wall, the top of the ladder slides all the way down the wall and rests against the foot of the wall. How long is the ladder?

(A) 14 feet **(B)** 15 feet **(C)** 16 feet **(D)** 17 feet **(E)** 18 feet

Solution: (D). If the base is x feet from the wall in the initial position, then the ladder must be $x + 2$ feet long. The Pythagoras theorem gives $x^2 + 8^2 = (x + 2)^2$ which yields $x = 15$. The length of the ladder is 17 feet.

14. Find the real numbers m, n such that $m+n=3$, and $m^3+n^3=117$. What is the value of m^2+n^2?

    (A) 29 **(B)** 3 **(C)** 9 **(D)** 17 **(E)** 45

    Solution: (A). Factor the second equation and use the first equation. We have $m^3+n^3=(m+n)(m^2-mn+n^2)=3(m^2-mn+n^2)=117$. Therefore $39=m^2-mn+n^2=(m+n)^2-3mn=9-3mn$. Thus, $mn=-10$. Substituting this into $m^2-mn+n^2=39$ we find that $m^2+n^2=29$.

15. A right triangle has legs of length 8 and 15. What is the radius of the inscribed circle?

    (A) 1 **(B)** 1.5 **(C)** 2 **(D)** 2.5 **(E)** 3

    Solution: (E). By Pythagoras' theorem, the hypotenuse of the triangle is $\sqrt{15^2+8^2}=17$, thus half of the perimeter is $s=(15+8+17)/2=20$. The area of the triangle is $A=(15\times 8)/2=60$ and this also equals $s\cdot\rho$ where ρ is the radius of the inscribed circle. Therefore $\rho=A/s=60/20=3$.

16. A farmer has 200 yards of fencing material. What is the largest rectangular area he can enclose if he wants to use a 4 yard wide gate that does not need to be covered by the fencing material?

    (A) $2,500$ square yards **(B)** $2,601$ square yards **(C)** $2,704$ square yards **(D)** $2,809$ square yards **(E)** $2,916$ square yards

    Solution: (B). We may add the length of the gate to the length of the fencing material and then we have 204 yards of fencing material. If the width of the enclosed area is x yards then the length is $(204-2x)/2=102-x$ yards. The enclosed area is $x(102-x)=102x-x^2=51^2-(x-51)^2$ square yards. The largest area of $51^2=2601$ is achieved when the length and the width are both 51 yards.

17. Suppose that $f(x)=ax+b$, where a and b are real numbers. Given that $f(f(f(x)))=8x+21$, what is the value of $a+b$?

    (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

    Solution: (D). $f(f(f(x)))=a^3x+a^2b+ab+b=8x+21$ so $a=2$ and $7b=21$. Therefore $a+b=5$.

18. The points $(2, k)$ and $(5, 5)$ belong to the line perpendicular to the line $3x - 2y = 7$. Find the value of k.

    (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

    Solution: (E). The given line has slope $3/2$ so the one perpendicular has slope $-2/3$. Hence $\frac{k-5}{2-5} = -2/3$. Solving, we get $k = 7$.

19. What is the product of all solutions of $||2x - 5| - 3| = 2$?

    (A) -12 **(B)** 0 **(C)** 6 **(D)** 12 **(E)** 30

    Solution: (B). First note that $|2x - 5| - 3$ could be either 2 or -2. That is, $|2x - 5| - 3 = 2$ or $|2x - 5| - 3 = -2$. Thus either $|2x - 5| = 5$ or $|2x - 5| = 1$. Each of these has two solutions. The former, $x = 0$ and $x = 5$ and the later, $x = 2$ and $x = 3$.

20. What is the radius of the circle with equation $x^2 - 4x + y^2 + 6y = 3$?

    (A) $\sqrt{3}$ **(B)** 2 **(C)** $\sqrt{5}$ **(D)** 3 **(E)** 4

    Solution: (E). In order to complete the squares we add $4 + 9 = 13$ and obtain $x^2 - 4x + 4 + y^2 + 6y + 9 = 16$ or $(x - 2)^2 + (y + 3)^2 = 4^2$. The radius is 4.

21. Suppose $2 \log x + \log 4 - \log 2x = 1$. Find the value of $x^2 + x$.

    (A) 12 **(B)** 20 **(C)** 30 **(D)** 42 **(E)** 56

    Solution: (C). Combining the left-hand side into one logarithm we find $\log\left(\frac{4x^2}{2x}\right) = 1$ or $2x = 10^1$. Therefore $x = 5$ and $x^2 + x = 30$.

22. What is the coefficient of x^5 in the polynomial $(x + 2)^8$?

    (A) 112 **(B)** 224 **(C)** 448 **(D)** 560 **(E)** 896

    Solution: (C). Using Pascal's triangle or the binomial theorem, we see that the coefficient of x^5 is $\frac{8!}{5! \cdot 3!} \cdot 2^3 = 448$.

23. Given that $x - \frac{1}{x} = 2\sqrt{3}$ for some $x \neq 0$. find $(x^2 - \frac{1}{x^2})^2$.

    (A) 75 **(B)** 108 **(C)** 147 **(D)** 192 **(E)** 243

    Solution: (D). $12 = (x - \frac{1}{x})^2 = x^2 - 2 + \frac{1}{x^2}$. Therefore $(x + \frac{1}{x})^2 = x^2 + 2 + \frac{1}{x^2} = 16$ and $(x^2 - \frac{1}{x^2})^2 = (x - \frac{1}{x})^2 (x + \frac{1}{x})^2 = 12 \cdot 16$.

24. Which of the following numbers is a perfect square?

    (A) $23! \cdot 24!$ **(B)** $24! \cdot 25!$ **(C)** $25! \cdot 26!$ **(D)** $26! \cdot 27!$ **(E)** $27! \cdot 28!$

    Solution: (B). $24! \cdot 25! = 24! \cdot 25 \cdot 24! = (5 \cdot 24!)^2$.

25. Find the sum of all the zeros of the function $F(x) = 18x^3 - 9x^2 - 5x + 2$.

    (A) 0.25 **(B)** 0.5 **(C)** 0.75 **(D)** 1.00 **(E)** 1.25

    Solution: (B). Suppose the zeros are x_1, x_2 and x_3. Then $F(x) = 18(x - x_1)(x - x_2)(x - x_3)$ so that the coefficient of x^2 is $-18(x_1 + x_2 + x_3) = -9$. Consequently, the sum is 0.5.