

UNC Charlotte 2013 Algebra Exam

SOLUTIONS

March 4, 2013

1. Mystery gifts are offered for sale in boxes, bags, and bundles. If 3 boxes and 2 bundles cost \$46, while 4 bags and 2 boxes cost \$44, and 1 box and 3 bags cost \$28, what is the total cost of 4 boxes, 5 bags and 3 bundles?

(A) \$91 (B) \$94 (C) \$97 (D) \$99 (E) \$99.20

Solution (B): Have three equations: $3x + 2n = 46$, $2x + 4g = 44$ and $x + 3g = 28$ where x is the price of a box, g the price of a bag and n the price of a bundle, all in dollars. Solve the system to obtain $x = 10$, $g = 6$ and $n = 8$. So total cost is $4 \cdot 10 + 5 \cdot 6 + 3 \cdot 8 = 94$.

2. The distance a falling object travels in the first t seconds is $16t^2$ feet. In exhibition diving, one person dives from a platform 20 feet above the water and a second person dives from a platform 10 feet above the water. How much later must the person on the lower platform dive after the person on the upper platform does so that they hit the water at the same time?

(A) $(\sqrt{20} - \sqrt{10})/4$ (B) $\sqrt{5}/4$ (C) $\sqrt{8} - \sqrt{5}$ (D) $5/8$ (E) $\sqrt{20} - \sqrt{10}$

Solution (A): The first diver will hit the water at $t = \sqrt{20}/4$ and the second at $t = \sqrt{10}/4$. The difference in times is $(\sqrt{20} - \sqrt{10})/4$.

3. Let $T = \left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{4}\right)\left(1 + \frac{1}{5}\right) \cdots \left(1 + \frac{1}{k}\right)$. For how many choices of k in the range $3 \leq k \leq 100$ is T an integer?

(A) 0 (B) 22 (C) 32 (D) 33 (E) more than 33

Solution (C): For each positive integer k , $1 + \frac{1}{k} = (k+1)/k$. So $T_k = \frac{4}{3} \frac{5}{4} \cdots \frac{k}{k-1} \frac{k+1}{k} = (k+1)/3$. This is an integer for $k = 5, 8, \dots, 95, 98$. Thus there are 32 integers where T_k is an integer.

4. Sally is trying to determine the number of seats in her schools auditorium. She notices that the first row contains 10 seats, the second row contains 12 seats, the third row contains 14 seats, and the fourth row contains 16 seats. The pattern continues for each row – the next one up always has 2 more seats. If there are 21 rows in the auditorium, how many seats does the auditorium contain?

(A) 610 (B) 620 (C) 630 (D) 640 (E) 650

Solution (C): Simply calculate the sum $10 + (10 + 2) + (10 + 4) + \cdots + (10 + 2 \cdot 20)$. There are twenty one terms, so the total is $21(10 + 50)/2 = 630$.

5. The number $N = 4ab3 + 3b95$ is a multiple of 99. What is the product of the two digits a and b ?

(A) 0 (B) 6 (C) 12 (D) 16 (E) 24

Solution (D): Since N is a multiple of 9, the sum of the digits of the two numbers must be an integer multiple of 9. So $6 + a + 2b$ is a multiple of 9. Since N is also a multiple of 11, $3 - b + a - 4 + 5 - 9 + b - 3 = a - 8$ is an integer multiple of 11 which implies $a = 8$ and from this we get $b = 2$.

6. Arthur takes a walk every day. On the first day of each month, he walks one mile. Each day afterward in the same month, he walks half the total distance he covered so far in the previous days of that month. On what day (after his walk that day) will Arthur's total distance exceed 10 miles for the month?

- (A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Solution (C): The second day he walks $1/2$ mile for a total of $3/2$ for the two days. The next he walks $3/4$ mile for a three day total of $9/4$. On the fourth day he walks $9/8$, total of $27/8$. The total on day $n + 1$ is $3^n/2^n$ (use recursion: the previous day the total was $3^{n-1}/2^{n-1}$ so he walked $3^{n-1}/2^n$ on day $n + 1$, with sum $(2 \cdot 3^{n-1} + 3^{n-1})/2^n = 3^n/2^n$). For $n = 5$ and $n = 6$, $243/32 < 10 < 729/64$.

7. If each root of the quadratic equation $x^2 + px + q = 0$ is a positive real number and p is at least -6 , then q is at most which of the following values?

- (A) 3 (B) 5 (C) 7 (D) 9 (E) 10

Solution (D): To have only positive roots, we must have $-6 \leq p < 0$, $p^2 - 4q \geq 0$ and $q > 0$. Thus $4q \leq 36$, so q is at most 9. In addition, $(x - 3)^2 = x^2 - 6x + 9 = 0$ has only positive real root(s).

8. In an interstellar store, a customer is buying construction materials for his granddaughter who is going to build a 6-dimensional cube. The customer only needs the edges for the cube. A big sign says edges are on sale for 2 ISD (interstellar dollar) each. The grandfather (using a cheat sheet provided by the granddaughter) asks for the correct number of edges, but is surprised at the total price. The clerk explains that even though he is buying only edges, he is required by law to also pay for the vertices necessary to build the cube. If each vertex costs 1 ISD, what did the grandfather pay?

- (A) 288 ISD (B) 448 ISD (C) 576 ISD (D) 768 ISD (E) 832 ISD

Solution (B): Just as the points $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ and $(1, 1, 1)$ are the vertices of a three-dimensional cube, the points of the form $(a_1, a_2, a_3, a_4, a_5, a_6)$ where each $a_i \in \{0, 1\}$ are the vertices of a six-dimensional cube (in six-dimensional space). There are $2^6 = 64$ vertices. An edge connects a pair of vertices if and only if the vertices have exactly one coordinate that is different. So each vertex is connected by an edge to six other vertices. Thus the total number of edges is $6 \cdot 2^6/2 = 192$. The total price is $2 \cdot 192 + 64$.

9. Let $T = \{0, 1, 2, 3, 5, 7, 11\}$. How many different numbers can be obtained as the product of three different members of T ?

- (A) 10 (B) 12 (C) 15 (D) 20 (E) 21

Solution (E): For the nonzero numbers, no three give the same product. There are 20 such products. Any product which includes 0 is 0. So there are 21 different products.

10. How many points in the plane satisfy both $x^2y - y^3 = 0$ and $y = x^2 - 3$?

- (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Solution (E): The first equation factors as $y(x - y)(x + y) = 0$, so $y = 0$, $x = y$ and $x = -y$ are possible and each of these is an equation for a line. The graph of the second equation is a parabola which intersects each of the lines in two points, none on more than one of the lines. So there are 6 solutions.

11. The diagonals of the sides of a rectangular box have lengths $\sqrt{13}$, $\sqrt{10}$ and $\sqrt{5}$. What is the volume?

- (A) 5 (B) $\sqrt{30}$ (C) 6 (D) $\sqrt{50}$ (E) $\sqrt{65}$

Solution (C): Let x , y and z be the side lengths with $x^2 + y^2 = 13$, $x^2 + z^2 = 10$ and $y^2 + z^2 = 5$. Solving for the squares yields $x^2 = 9$, $y^2 = 4$ and $z^2 = 1$. So $x = 3$, $y = 2$ and $z = 1$.

12. Mr Green sells apples for \$1.50 each at the local Farmers Market and Ms Blue sells slightly smaller apples for \$1 each. One day Ms Blue had to leave early so she asked Mr Green to manage her stall as the two were side-by-side. To make calculations easier, Mr Green mixed the apples together and changed the signs to read "5 apples for \$6". At that point they had the same number of apples left. By the end of the day he had sold all the apples, but oddly (to him) when he compared how much each would have made by selling separately and how much he had in the till, he found he was 80 dollars short. He had no clue what the problem was, so he split the money evenly and apologized to Ms Blue for messing things up. Certainly, at least one of them lost money. Did both lose money on the deal, or did one come out ahead, and how much did each lose/gain?

- (A) Blue lost \$120, Green made \$40 extra (B) Blue lost \$32 and Green lost \$48
 (C) Both lost \$40 (D) Blue made \$80 extra and Green lost \$160
 (E) Blue made \$160 extra, Green lost \$240

Solution (E): Let x denote the number of apples each has for sale. Then Ms Blue should receive x dollars and Mr Green $1.5x$ dollars. By mixing and selling $2x$ apples at 5 for \$6, Mr Green is effectively charging \$1.20 per apple. At this rate he is losing \$0.30 an apple and Ms Blue is receiving an extra \$0.20 per apple. As he is \$80 short, $80 = 0.3x - 0.2x = 0.1x$ and we have $x = 800$. At the end of the day, Ms Blue made an extra \$160 and Mr Green lost \$240.

13. Consider the following equation where m and n are real numbers:

$$(x^2 - 2x + m)(x^2 - 2x + n) = 0.$$

Suppose the four roots of the equation form an arithmetic sequence with the first (and smallest) term being $1/4$. What is the value of $|m - n|$?

- (A) $3/8$ (B) $1/2$ (C) $5/8$ (D) $3/4$ (E) 1

Solution (B): The four roots are 0.25 , $k + 0.25$, $2k + 0.25$, $3k + 0.25$. Since the coefficient on x is 2 in both factors, one of the two quadratic expressions factors as $(x - 0.25)(x - (3k + 0.25))$ and the other as $(x - (k + 0.25))(x - (2k + 0.25))$. In addition, $3k + 0.5 = 2$ so $k = 1/2$. If we let $m = (k + 0.25)(2k + 0.25)$ and $n = 0.25(3k + 0.25)$, we have $m - n = 15/16 - 7/16 = 1/2$.

14. What is the units digit of 3^{2013} ?

- (A) 1 (B) 3 (C) 5 (D) 7 (E) 9

Solution (B): Note that $3^4 = 81$, so the units digit of 3^{4k} is a 1 for each positive integer k . We can rewrite the product as $3 \cdot 3^{2012} = 3 \cdot (3^4)^{503}$. Thus the units digit is a 3.

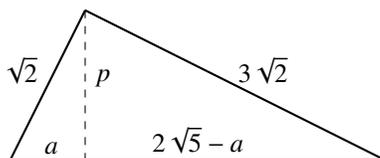
15. Front tires of a car wear out after 45,000 miles, while the rear tires wear out after 75,000 miles. After how many miles should you rotate the tires (front↔back) to drive the maximal distance for the set?

- (A) 27,125 (B) 28,125 (C) 29,125 (D) 30,125 (E) 31,125

Solution (B): Let x be the number of miles the car is driven before rotating the tires (not necessarily at the optimal point). Then the front tires have used $x/45,000$ fraction of their useful tread life and the back tires $x/75000$ of their useful tread life. After the tires are rotated, the old front tires will last for $(1 - x/45000) \cdot 75000$ miles on the rear, and the old back tires will last $(1 - x/75000) \cdot 45000$ miles on the front. To get the most miles out of the set, all four tires should wear out at the same time. So we simply need to solve $(1 - x/45000) \cdot 75000 = (1 - x/75000) \cdot 45000$ to get $x = 30000 \cdot 15/16 = 28,125$.

16. Calculate the area of a triangle whose side lengths are $\sqrt{2}$, $3\sqrt{2}$ and $2\sqrt{5}$.

- (A) 2
- (B) $\sqrt{5}$
- (C) 3
- (D) 3.5
- (E) $\sqrt{15}$



Solution (C): One way to calculate the area is by Heron's formula, simply calculate the value of:

$$\frac{\sqrt{(\sqrt{2} + 3\sqrt{2} + 2\sqrt{5})(\sqrt{2} + 3\sqrt{2} - 2\sqrt{5})(3\sqrt{2} + 2\sqrt{5} - \sqrt{2})(2\sqrt{5} + \sqrt{2} - 3\sqrt{2})}}{4}.$$

Multiplying the pairs: $(4\sqrt{2} + 2\sqrt{5})(4\sqrt{2} - 2\sqrt{5}) = 12 = (2\sqrt{5} + 2\sqrt{2})(2\sqrt{5} - 2\sqrt{2})$, so the area is $12/4 = 3$. Alternately, choose a vertex and find the length of the perpendicular from the vertex to the opposite side (using Theorem of Pythagoras). Then calculate the area in the usual way. Using vertex opposite the side of length $2\sqrt{5}$: $p^2 + a^2 = 2$, $p^2 + 20 - 4a\sqrt{5} + a^2 = 18$, solve to get $a = 1/\sqrt{5}$ and then $p = 3/\sqrt{5}$. But the easiest way for this particular problem is to simply realize that $(\sqrt{2})^2 + (3\sqrt{2})^2 = 20 = (2\sqrt{5})^2$, so the triangle is a right triangle.

17. How many ordered pairs of positive integers (x, y) satisfy the equation $\frac{1}{x} - \frac{2}{y} = \frac{1}{6}$?

- (A) 1
- (B) 2
- (C) 3
- (D) 4
- (E) 5

Solution (D): Manipulate the equation to have equivalent forms (for nonzero integers) $6y - 12x = xy$ and $y(6 - x) = 12x$. Since both x and y are positive integers, $y > 2x \geq 2$ and $6 > x \geq 1$. Also xy is an integer multiple of 6 and $12x/(6 - x)$ is a positive integer. The latter restriction implies $x \neq 1$. But integer solutions for y exist when $x = 2, 3, 4, 5$: $x = 2, y = 6$; $x = 3, y = 12$; $x = 4, y = 24$; and $x = 5, y = 60$.

18. On a recent trip from Northburg to Southtown, Jill decided to make a detour so she could pass through Center City. Forty minutes after she left Northburg, she noted that the remaining distance to Center City was twice as much as what she had traveled so far. After traveling another twenty one miles, she calculated that the remaining distance to Southtown was twice as much as what she had left to get to Center City. She arrived in Southtown an hour and a half later. Assuming she traveled at a constant speed, how long was this trip from Northburg to Southtown?

- (A) 99 miles
- (B) 108 miles
- (C) 112 miles
- (D) 127 miles
- (E) 142 miles

Solution (A): Let x denote the number of miles Jill had traveled in the first 40 minutes. Since this was one third of the way to Center City, it took her a total of 2 hours to get to Center City. Next let y be the number of miles she had left to get to Center City after she had gone the additional 21 miles. We have $2x = 21 + y$, and the total trip is $3x + y = x + 2y + 21$. She can travel $1.5x = 4y/3$ miles in 1 hour. Thus $9x = 8y$. Solve to get $x = 24$ and $y = 27$. The total distance is 99 miles.

19. What is the sum of all two-digit numbers whose tens digit and units digit differ by exactly one?

- (A) 878 (B) 890 (C) 900 (D) 990 (E) 991

Solution (B): The numbers are 10, 12, 21, 23, 32, 34, ..., 78, 87, 89, 98. So the sum of all of them can be split into the sum $10 + 21 + 32 + \cdots + 98 = 9(108)/2$ plus the sum $12 + 23 + \cdots + 89 = 8(101)/2$.

20. How many real numbers differ by exactly 2 from their reciprocals?

- (A) 0 (B) 2 (C) 3 (D) 4 (E) more than 4

Solution (D): There are two possibilities $x - 1/x = 2$ and $x - 1/x = -2$. Each of these has two solutions for a total of four solutions.

21. Which of the following describes the set of points in the plane which satisfy the equation $y^2 - y - x^2y + x^2 = 0$?

- (A) the union of a parabola and a line (B) the union of two lines (C) a parabola
(D) a hyperbola (E) the union of a hyperbola and a line

Solution (A): The left side of the equation factors as $(y - x^2)(y - 1)$. So either $y - x^2 = 0$ or $y - 1 = 0$. The first is the equation for a parabola and the second an equation for a (horizontal) line.

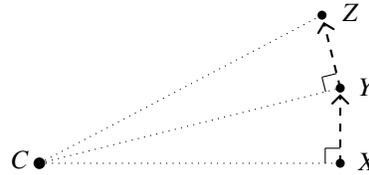
22. Every Monday, Harvey puts a new puzzle on his blog. The puzzle for today goes like this: "My two sisters, all of my children and my younger brother and I were born between Jan. 1, 1901 and Dec. 31, 1999, each of us in a different year. Oddly, we all satisfy a very peculiar property. Each of us turned yx in some year $19xy$ where $0 \leq y < x \leq 9$ (a different year for each of us of course). And odder still, if someone born in $19ab$ satisfies this peculiar property, then one of us was born in that year. My sisters, my brother and I take care of the first four such years $19ab$, so how many pairs $0 \leq y < x \leq 9$ are there where my oldest child turned yx in the year $19xy$?"

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution (E): Consider all years $19xy$ with $0 \leq y < x \leq 9$ and subtract yx to see what year $19ab$ would be the birth year. Starting with $x = 1, y = 0$, a person who turned 01 in 1910 was born in 1909. That person will turn 12 in 1921, 23 in 1932, etc. To be 02 in 1920 means a birth year of 1918. The person who turned 03 in 1930 was born in 1927. Next is the one who turned 04 in 1940 and was born 1936. So the oldest child was born in 1945 and turned 05 in 1950. This child turned 16 in 1961, 27 in 1972, 38 in 1983 and 49 in 1994 for a total of five years where the child's age was yx in $19xy$.

23. Surveyors are laying out a rather unusual road through a park. The park is completely flat and forms a disc of radius 20 miles. From the center C , the road is to go exactly two miles north to a point B_0 , then make a 90° left turn and go another two miles to a point B_1 . At B_1 the road turns left again (not nearly as sharply), this time perpendicular to $\overline{CB_1}$. As before, it goes exactly two miles in this direction to a point B_2 . This pattern is followed for the entire road – at B_k , the road makes a left turn that is perpendicular to $\overline{CB_k}$ and goes exactly two miles in this direction to B_{k+1} . Eventually the road reaches a point A (one of the B_j s) that is exactly 10 miles from C . Starting from A , how many **more** two-mile segments will be needed before the road gets out of the park? [In the figure, two consecutive segments are shown starting from an arbitrary point X to the point Y and then from Y to Z .]

- (A) fewer than 30
- (B) between 30 and 49
- (C) between 50 and 69
- (D) between 70 and 89
- (E) at least 90



Solution (D): If the square of the length of $\overline{CB_k}$ is q , then the square of the length of $\overline{CB_{k+1}}$ is $q + 4$. To have B_{k+1} outside the circle of radius 20 with center C , we need $q + 4$ to be greater than 400. The sequence of distances from $A = B_j$ on runs as follows: $\overline{CB_j} = \sqrt{100}$, $\overline{CB_{j+1}} = \sqrt{104}$, $\overline{CB_{j+2}} = \sqrt{108}$. In general, $\overline{CB_{j+i}} = \sqrt{100 + 4i}$. To have B_{j+i} outside we need $100 + 4i > 400$, so $i > 75$ (if $i = 75$, B_{j+i} is on the park boundary).

24. Consider the equation $\sqrt{x + 3 - 4\sqrt{x - 1}} + \sqrt{x + 8 - 6\sqrt{x - 1}} = 1$ where x represents a real number. How many solutions are there?

- (A) Exactly one solution
- (B) Exactly two solutions
- (C) Exactly three solutions
- (D) Exactly four solutions
- (E) Infinitely many solutions

Solution (E): Make the substitution $z = \sqrt{x - 1}$. Then the equation becomes $\sqrt{z^2 + 4 - 4z} + \sqrt{z^2 + 9 - 6z} = 1$. Next note that $z^2 - 4z + 4 = (z - 2)^2$ and $z^2 - 6z + 9 = (z - 3)^2$. Thus the equation reduces to $|z - 2| + |z - 3| = 1$. If $2 \leq z \leq 3$, then $|z - 2| + |z - 3| = z - 2 + 3 - z = 1$. So all x in the interval $5 \leq x \leq 10$ are solutions.

25. The Mainter brothers, Abe, Ben and Cal paint houses. They have been in the business so long that each knows exactly how many square feet he (and each of his brothers) paints in one hour and these rates never change. For their latest job, they calculated that if Abe and Ben did the job together, it would take exactly 11 hours. On the other hand, if Abe and Cal did the job together, it would take exactly 9 hours. Finally, Ben and Cal could do it in exactly 9.9 hours (or if you prefer, 9 hours and 54 minutes). They decided all three would paint this particular house. Ben and Cal started the job at 8 AM and Abe joined them at 9:00. Cal left at 1:30 and so Ben and Abe finished the job. After deducting the supply costs (paint etc.), the brothers split the net profit based on what percentage of the total square footage each painted. Who earned the most and who earned the least for this job?

- (A) Abe the most, Ben the least
- (B) Abe the most, Cal the least
- (C) Ben the most, Abe the least
- (D) Cal the most, Ben the least
- (E) Cal the most, Abe the least

Solution (A): Let a be the rate at which Abe paints, b the rate Ben paints and c the rate Cal paints. Also let H be the total number of square feet of painting to do in the house. Then $11a + 11b = H$, $9a + 9c = H$ and $9.9b + 9.9c = H$. An equivalent system is $a + b = H/11$, $a + c = H/9$ and $b + c = H/9.9$. Solving this systems yields $a = H(1/9 + 1/11 - 1/9.9) = 5H/99$, $b = H(1/11 + 1/9.9 - 1/9) = 4H/99$ and $c = H(1/9 + 1/9.9 - 1/11) = 6H/99$. Cal paints for 5.5 hours, so he paints exactly 1/3 of the house. Abe paints in 4 hours what Ben paints in 5. So by 1:30, Abe has painted more of the house than Ben. Since Abe paints faster than Ben and Cal painted exactly one third of the house, Abe painted the most and Ben the least.