UNC Charlotte 2003 Comprehensive

1. For what value of k are the lines $2x + 3y = 4k$ and $x - 2ky = 7$ perpendicular?

   (A) $-3/4$ **(B)** $1/6$ **(C)** $1/3$ **(D)** $1/2$ **(E)** $2/3$

2. Conard High School has 50 students who play on the baseball, football, and tennis teams. Some students play more than one sport. If 15 play tennis, 25 play baseball, 30 play football, 8 play tennis and baseball, 5 play tennis and football, and 10 play baseball and football, determine how many students play all three sports.

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

3. Which of the following is an equation of the line tangent to the circle $x^2+y^2 = 2$ at the point $(1, 1)$?

   (A) $x = 1$ **(B)** $y = 1$ **(C)** $x + y = 2$ **(D)** $x - y = 0$ **(E)** $x + y = 0$

4. Let r and s be the two solutions to the equation $x^2 - 3x + 1 = 0$. Find $r^3 + s^3$.

   (A) 12 **(B)** 14 **(C)** 16 **(D)** 18 **(E)** 24

5. Three points $A = (0, 1)$, $B = (2, a)$, and $C = (3, 7)$ are on a straight line. What is the value of a ?

   (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

6. A recent poll showed that nearly 30% of European school children think that

   $$\frac{1}{2} + \frac{1}{3} = \frac{2}{5}.$$

   This is wrong, of course. Is it possible that $\frac{1}{a} + \frac{1}{b} = \frac{2}{a+b}$ for some real numbers a and b?

   (A) Yes, but only if $a + b = 1$ **(B)** Yes, but only if $a + b = 2$

   (C) Yes, but only if $a^2 + b^2 = 1$ **(D)** Yes, but only if $a^2 + b^2 = 0$

   (E) No, it is not possible

7. How many ordered pairs (a, b) of integers satisfy $a^2 = b^3 + 1$?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** at least 5

8. One leg of a right triangle is two meters longer than twice the length of the other leg. The hypotenuse is four meters less than the sum of the lengths of both legs. What is the perimeter of the triangle, in meters?

   (A) 20 **(B)** 25 **(C)** 30 **(D)** 35 **(E)** 50

9. Fifteen numbers are picked from the 21-element set $\{1, 2, 3, \ldots 20, 21\}$. What is the probability that at least three of those numbers are consecutive?

   (A) 0 **(B)** 0.5 **(C)** 0.9 **(D)** 0.99 **(E)** 1

10. Margaret and Cyprian both have some nickels, dimes and quarters, at least one of each type and a different number of each type. Margaret has the same number of quarters as Cyprian has dimes, and she has the same number of dimes as Cyprian has nickels. She also has the same number of nickels as Cyprian has quarters. The value of their coins is the same. What is the smallest possible total value of Margaret's coins in cents?

    (A) 85 **(B)** 95 **(C)** 105 **(D)** 125 **(E)** 135

11. Let S denote the set $\{(-2, -2), (2, -2), (-2, 2), (2, 2)\}$. How many circles of radius 3 in the plane have exactly two points of S on them?

    (A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 16

12. The sum of the cubes of ten consecutive integers is 405. What is the sum of the ten integers?

    (A) 10 **(B)** 15 **(C)** 20 **(D)** 23 **(E)** 24

13. Let M and N denote the two integers that are respectively twice and three times the sum of their digits. What is $M + N$?

    (A) 27 **(B)** 36 **(C)** 45 **(D)** 54 **(E)** 60

14. How many positive integer triples (x, y, z) satisfy $\frac{1}{x} + \frac{y}{z} = \frac{13}{21}$, where $x < 7$ and where y and z have no common divisors?

    (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

15. Let $D(a, b, c)$ denote the number of multiples of a that are less than c and greater than b. For example, $D(2, 3, 8) = 2$ because there are two multiples of 2 between 3 and 8. What is $D(9^3, 9^4, 9^6)$?

    (A) 71 **(B)** 719 **(C)** 720 **(D)** 7200 **(E)** 72000

16. A $4 \times 4 \times 4$ wooden cube is painted on all 6 faces and then cut into 64 unit cubes. One unit cube is randomly selected and rolled. What is the probability that exactly one of the five visible faces is painted?

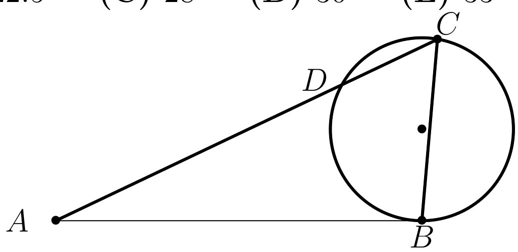
    (A) 5/16 **(B)** 7/16 **(C)** 15/31 **(D)** 31/64 **(E)** 1/2

17. How many non-empty sets T of natural numbers satisfy the property: $s \in T$ implies $\frac{8}{s} \in T$. ?

    (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

18. An interesting fact about circles is that if a line AB is tangent to a circle at the point B and a different line through A intersects the circle in points C and D as in the diagram below, then $AB^2 = AC \cdot AD$. If $AC = 4$, $AD = 3$ and $BC = 2$, what is the measure of the angle DAB to the nearest degree? Note that the chord $\overline{BC}$ does not pass through the center of the circle.

    (A) 15 **(B)** 22.5 **(C)** 28 **(D)** 30 **(E)** 33



19. What is the smallest value of the positive integer n for which

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)}$$

is at least 1? *Hint:* Rewrite each $\frac{1}{k \cdot (k+1)}$ in the from $\frac{1}{a} - \frac{1}{b}$.

(A) 10 **(B)** 100 **(C)** 1000 **(D)** 2002 **(E)** there is no such value of n

20. Consider the number x defined by the periodic continued fraction

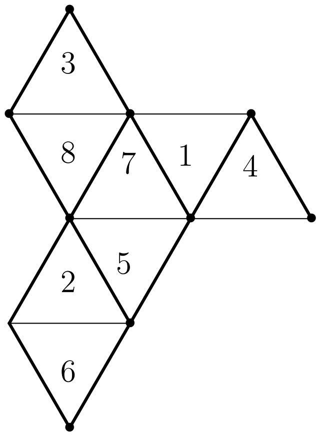
$$x = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \cdots}}}}.$$

Then $x =$

(A) $\frac{1}{3}$ **(B)** $\frac{3}{7}$ **(C)** $\frac{-3-\sqrt{15}}{2}$ **(D)** $\frac{-3+\sqrt{15}}{2}$ **(E)** None of these

21. An octahedral net is a collection of adjoining triangles that can be folded into a regular octahedron. When the net below is folded to form an octahedron, what is the sum of the numbers on the faces adjacent to one marked with a 3?

(A) 13 **(B)** 15 **(C)** 17 **(D)** 18 **(E)** 19



22. The stairs leading up to to the entrance of Joe's house have 5 steps. Playful Joe is able to go up 1 step or 2 steps at a time. How many ways are there for him to reach the top of the stairs? In other words, how many ways are there to write 5 as a sum of 1's and 2's if the order of the summands matters?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 10

23. How many of the 1024 integers in the set $1024, 1025, 1026, \ldots, 2047$ have more 1's than 0's in their binary representation?

(A) 252 **(B)** 512 **(C)** 638 **(D)** 768 **(E)** 772

24. Suppose x and y are integers satisfying both $x^2 + y = 62$ and $y^2 + x = 176$. What is $x + y$?

 (A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

25. Find all solutions of the equation $\log_2 x + \log_2(x + 2) = 3$.

 (A) There is one solution x, and $x \geq 3$.

 (B) There is one solution x, and $1 < x < 3$.

 (C) There are two positive solutions.

 (D) There is one positive solution and one negative solution.

 (E) There is are no solutions.

26. Find the area of the polygon $ABCDEF$ whose vertices are $A = (-1, 0), B = (0, 2), C = (1, 1), D = (2, 2), E = (2, 0)$, and $F = (0, -1)$.

 (A) 4 **(B)** 4.5 **(C)** 5 **(D)** 5.5 **(E)** 6

27. Let $f(a, b, c) = \frac{a+b}{c} + \frac{b+c}{a} + \frac{a+c}{b}$. An integer u exists such that $f(u-1, u, 2u) = 8$. What is the value of $f(u^2 - 3, 2u - 3, 2u - 4)$?

 (A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

28. Consider the trapezoid $ABCD$ (see the diagram). Suppose $\overline{MN}$ is parallel to $\overline{DC}$, $AB = a, DC = b$, and $MN = x$. If the area of $ABNM$ is half the area of $ABCD$, express x as a function of a and b.

 (A) $x = \dfrac{a+b}{2}$ **(B)** $x = \dfrac{b-a}{2}$ **(C)** $x = \dfrac{3b-a}{2}$

 (D) $x = \sqrt{a^2+b^2}$ **(E)** $x = \dfrac{\sqrt{a^2+b^2}}{2}$