UNC Charlotte 2005 Comprehensive

March 7, 2005

1. The numbers x and y satisfy $2^x = 15$ and $15^y = 32$. What is the value xy?

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** none of **A**, **B**, **C** or **D**

   Solution: C. Note that $(2^x)^y = 15^y = 32$ so $2^{xy} = 2^5$ and $xy = 5$.

2. Suppose x, y, z, and w are real numbers satisfying $x/y = 4/7$, $y/z = 14/3$, and $z/w = 3/11$. When $(x+y+z)/w$ is written in the form m/n where m and n are positive integers with no common divisors bigger than 1, what is $m+n$?

   (A) 20 **(B)** 26 **(C)** 32 **(D)** 36 **(E)** 37

   Solution: D. Calculate x/w, y/w and compute the sum. For example, $x/w = 4/7 \cdot 14/3 \cdot 3/11 = 8/11$ and $y/w = 14/3 \cdot 3/11 = 14/11$, so $(x+y+z)/w = (8+14+3)/11$.

3. Let m be an integer such that $1 \le m \le 1000$. Find the probability of selecting at random an integer m such that the quadratic equation

   $$6x^2 - 5mx + m^2 = 0$$

   has at least one integer solution.

   (A) 0.333 **(B)** 0.5 **(C)** 0.667 **(D)** 0.778 **(E)** 0.883

   Solution: C. The quadratic equation has two solutions

   $$\begin{aligned} x &= \frac{m}{2} \\ x &= \frac{m}{3} \end{aligned}$$

   There are 500 multiples of 2, 333 multiples of 3, and 166 multiples of 6 between 1 and 1000. Therefore, the probability is

   $$p = \frac{500 + 333 - 166}{1000} = 0.667$$

4. Let $z = x + iy$ be a complex number, where x and y are real numbers. Let A and B be the sets defined by

   $$A\{z \mid |z| \le 2\} \text{ and } B = \{z|\ (1-i)z + (1+i)\bar{z} \ge 4\}.$$

   Recall that $\bar{z} = x - iy$ is the conjugate of z and that $|z| = \sqrt{x^2+y^2}$. Find the area of the region $A \cap B$.

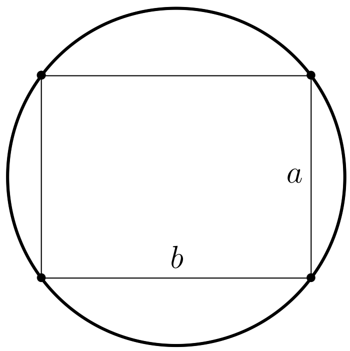
   (A) $\pi/4$ **(B)** $\pi - 2$ **(C)** $(\pi-2)/4$ **(D)** 4π **(E)** $\pi - 4$

   Solution: B. Since

   $$\begin{aligned} A &= \{(x,y)|x^2+y^2 \le 4\} \\ B &= \{(x,y)|x+y \ge 2\} \end{aligned}$$

   the area of $A \cap B$ is one-fourth the area of the circle minus the area of the triangle in the first quadrant, which is

5. The area of the $a \times b$ rectangle shown in the picture below is $\frac{6}{5\pi}$ of the area of the circle. Assuming $b > a$, what is the value of $\frac{b}{a}$?



(A) 2 **(B)** 2.5 **(C)** 3 **(D)** 3.25 **(E)** 3.5

Solution: C. By the Pythagorean Theorem, the diameter of the circle is $\sqrt{a^2+b^2}$, and the radius is $\frac{\sqrt{a^2+b^2}}{2}$. The area of the circle is $\frac{a^2+b^2}{4} \cdot \pi$, so we need to solve the equation

$$a \cdot b = \frac{a^2+b^2}{4} \cdot \pi \cdot \frac{6}{5\pi}, \quad \text{that is,} \quad a \cdot b = 0.3(a^2+b^2).$$

Dividing both sides by a^2 and rearranging yields

$$0 = 0.3 \cdot \left(\frac{b}{a}\right)^2 - \frac{b}{a} + 0.3,$$

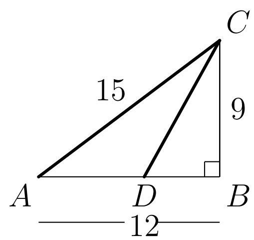
a quadratic equation in $\frac{b}{a}$. According to the quadratic formula

$$\frac{b}{a} = \frac{1 \pm \sqrt{1-0.36}}{0.6} = \frac{1 \pm \sqrt{0.64}}{0.6} = \frac{1 \pm 0.8}{0.6} = 3 \quad \text{or} \quad \frac{1}{3}.$$

Since $b > a$, the only solution is $\frac{b}{a} = 3$.

6. In the right triangle ABC the segment CD bisects angle C, $AC = 15$, and $BC = 9$. Find the length of $\overline{CD}$.

   (A) $9\sqrt{5}/2$ **(B)** 11 **(C)** $9\sqrt{6}/2$ **(D)** 12 **(E)** $15\sqrt{3}/2$



Solution: A. Use the Pythagorean Theorem to find that $AB = 12$. Then use the angle bisector theorem that says in a triangle, an angle bisector divides the opposite sides into two parts proportional to the adjacent sides. Thus $AD/15 = DB/9$. It follows from this that $DB = 9/2$. Use the Pythagorean Theorem again to see that $CD = 9\sqrt{5}/2$.

Alternate proof using trigonometry: Let z be the length of segment $\overline{CD}$ and let θ be the measure of angle $\angle DCB$. Then $\cos(\theta) = 9/z$ and $\cos(2\theta) = 9/15 = 2\cos^2(\theta) - 1 = (162/z^2) - 1$. Solve for z to get $z = 9\sqrt{5}/2$.

7. The product of the two roots of the equation $\log x + \log(x+2) = 3$ is equal to

   (A) $-\log 2$ **(B)** -10^3 **(C)** $\log 2$ **(D)** 10^3 **(E)** The equation has only one root

Solution: E. Note that $\log x + \log(x+2) = \log(x(x+2)) = 3$ which implies that $x^2 + 2x = 10^3$ from which it follows that $x^2 + 2x - 10^3 = 0$. . But one of the roots, $x = -1 - \sqrt{1 + 10^3}$ is not in the domain of $\log x$.

8. The polynomial $p(x) = 2x^4 - x^3 - 7x^2 + ax + b$ is divisible by $x^2 - 2x - 3$ for certain values of a and b. What is the sum of a and b?

   (A) -34 **(B)** -30 **(C)** -26 **(D)** -18 **(E)** 30

   Solution: A. Because $x^2 - 2x - 3 = (x-3)(x+1)$ $p(x)$ has zeros of 3 and -1, $p(3) = 72 + 3a + b = 0$ and $p(-1) = -4 - a + b = 0$ which we can solve simultaneously to get $a = -19$ and $b = -15$.

   Alternatively, simply start by dividing $2x^4 - x^3 - 7x^2 + ax + b$ by $x^2 - 2x - 3$ by long division. The "last step" is to subtract $5x^2 - 10x - 15$ from $5x^2 + (9+a)x + b$. The difference must be 0. So $b = -15$ and $9 + a = -10$. Thus $a = -19$ and $a + b + -34$. No factoring is done.

   Yet another method is to factor $x^2 - 2x - 3$ and see that it has two zeros, $x = 3$ and $x = -1$. Use synthetic division (starting with either zero–here starting with $x = -1$) to see that (1) $b - a - 4 = 0$, and (2) $2x^4 - x^3 - 7x^2 + ax + b = (x+1)(2x^3 - 3x^2 - 4x + a + 4)$. Now use synthetic division with $x = 3$ on the quotient $2x^3 - 3x^2 - 4x + a + 4$ to see that $a + 19 = 0$. Thus $a = -19$. Put this into (1) to see that $b = -15$, etc.

9. One hundred monkeys have 100 apples to divide. Each adult gets three apples while three children share one. How many adult monkeys are there?

   (A) 10 **(B)** 20 **(C)** 25 **(D)** 30 **(E)** 33

   Solution: C. Assume there are x adults and y children, then $x + y = 100$ and $3x + (1/3)y = 100$. Solving simultaneously leads to $x = 25$ and $y = 75$.

10. Let x and y be the positive integer solution to the equation

$$\frac{1}{x+1} + \frac{1}{y-1} = 5/6.$$

   Find $x + y$.

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

   Solution: D. Since x and y are natural numbers, $y \geq 2$. Let $u = x + 1$ and $v = y - 1$, u and v are natural numbers with $u \geq 2$ and $v \geq 2$. One of $1/u$ and $1/v$ is at least half of $5/6$, so either $u \leq 12/5 = 2.4$ or $v \leq 12/5 = 2.4$. Consider the two cases: case one, $u = 2$ so $x = 1$ then $v = 3$ then $y = 4$; case two, $v = 2$ so $y = 3$ then $u = 3$ then $x = 2$. In either case, $x + y = 5$.

11. Let x and y be two integers that satisfy all of the following properties:

(a) $5 < x < y$,

(b) x is a power of a prime and y is a power of a prime, and

(c) the quantities $xy + 3$ and $xy - 3$ are both primes.

Among all the solutions, let (x, y) be the one with the smallest product. Which of the following statements is true? Note the list of primes on the last page.

(A) $x + y$ is a perfect square **(B)** the number xy is prime

(C) $y = x + 3$ **(D)** $y = x + 1$ **(E)** $x + y = 17$

Solution: D. From the condition (c), we examine pairs of prime numbers that differ by 6. Listing the primes gives $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 43, 47, 53, 59, 61, \ldots$. Now by checking we can see that the pair 53 and 59 is the first pair of primes that satisfies all conditions a, b, and c; therefore, $xy = 56$ and $x = 7, y = 2^3$.

Alternate solution: (i) start a list of the integers larger than 5 that are powers of primes, 7, 8, 9, 11, 13, 16, 17,...; (ii) note that the smallest product with $x < y$ is $7 \cdot 8$ which just happens to be exactly 3 more than the prime 53 and 3 less than the prime 59.

12. In a 10-team baseball league, each team plays each of the other teams 18 times. No game ends in a tie, and, at the end of the season, each team is the same positive number of games ahead of the next best team. What is the greatest number of games that the last place team could have won.

(A) 27 **(B)** 36 **(C)** 54 **(D)** 72 **(E)** 90

Solution: D. The number of games played is $18(1 + 2 + \cdots + 9) = 810$. If n is the number of wins of the last-place team, and d is the common difference of wins between successive teams, then $n + (n + d) + (n + 2d) + \cdots + (n + 9d) = 10n + 45d = 810$ so $2n + 9d = 162$. Now n is the maximum when d is a minimum (but not zero, because there are no ties). The smallest integral value of d for which n is integral is $d = 2$. Thus $n = 72$.

13. Suppose f is a real function satisfying $f(x + f(x)) = 4f(x)$ and $f(1) = 4$. What is $f(21)$?

(A) 16 **(B)** 21 **(C)** 64 **(D)** 105 **(E)** none of **A**, **B**, **C** or **D**

Solution: C. Note that $f(1 + f(1)) = 4f(1) = 16$, so $f(5) = 16$. Next, $f(21) = f(5 + f(5)) = 4f(5) = 64$.

14. Suppose $\sin\theta + \cos\theta = 0.8$. What is the value of $\sin(2\theta)$?

    (A) -0.36 **(B)** -0.16 **(C)** 0 **(D)** 0.16 **(E)** 0.36

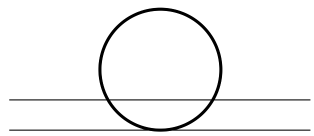
    Solution: A. Note that $0.64 = (\sin\theta + \cos\theta)^2 = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1 + 2\sin\theta\cos\theta = 1 + \sin(2\theta)$, so $\sin(2\theta) = -0.36$.

15. The product of five consecutive positive integers divided by the sum of the five integers is a multiple of 100. What is the least possible sum of the five integers?

    (A) 605 **(B)** 615 **(C)** 620 **(D)** 625 **(E)** 645

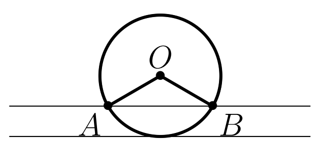
    Solution: B. If the numbers are denoted $k-2, k-1, k, k+1$ and $k+2$, then $(k-2)(k-1)k(k+1)(k+2) \div 5k = 100K$ for some integer K. This can happen only when one of the integers $k-2, k-1, k+1, k+2$ is a multiple of 125 since only one of these numbers can be a multiple of 5. Let $k+2 = 125$ to minimize the sum. Then $5 \cdot 123 = 615$.

16. Two parallel lines are one unit apart. A circle of radius 2 touches one of the lines and cuts the other line. The area of the circular cap between the two parallel lines can be written in the form $a\pi/3 - b\sqrt{3}$. Find the sum $a + b$ of the two integers a and b.



(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Solution: C. The cap is a segment minus the triangle OAB as shown below. The central angle of the segment is $120°$. The circular segment covers one third of the circle and hence has area $4\pi/3$. Hence $a = 4$. The triangle is isosceles with height 1 and two congruent sides of length 2. Its third side has length $2\sqrt{3}$ and hence its area is $\sqrt{3}$. Hence $b = 1$ and $a + b = 5$.

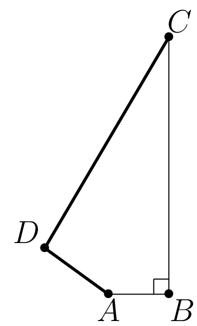


17. For each positive integer N, define $S(N)$ as the sum of the digits of N and $P(N)$ as the product of the digits. For example, $S(1234) = 10$ and $P(1234) = 24$. How many four digit numbers N satisfy $S(N) = P(N)$?

(A) 0 **(B)** 6 **(C)** 12 **(D)** 24 **(E)** None of the above

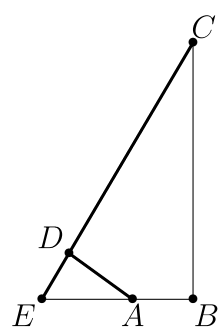
Solution: C. The only numbers satisfying the condition are those permutations of the digits $1, 1, 2, 4$. There are 12 such permutations.

18. Consider a quadrilateral $ABCD$ with $AB = 4$, $BC = 10\sqrt{3}$, and $\angle DAB = 150°$, $\angle ABC = 90°$, and $\angle BCD = 30°$. Find DC.



(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

Solution: B. Refer to the diagram below.



$EC = BC/\cos 30 = 20$, $EB = EC \sin 30 = 10$. Therefore $EA = 6$. Because $\triangle EAD$ is a right triangle with $\angle EAD = 30°$ and hypotenuse $EA = 6$, $ED = 3$. Consequently, $DC = 20 - 3 = 17$.

19. Solve the equation $\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} = 3$ for x.

    (A) 4/3 **(B)** 5/3 **(C)** 7/5 **(D)** 9/5 **(E)** none of **A**, **B**, **C** or **D**

    Solution: B. Cross multiplying and simplifying, we obtain $2\sqrt{x-1} = \sqrt{x+1}$. Squaring both sides yields $4(x-1) = x+1$ or $3x = 5$. Thus, $x = 5/3$.

20. Two rational numbers r and s are given. The numbers $r+s, r-s, rs$, and s/r are computed and arranged in order by value to get the list $1/3, 3/4, 4/3, 7/3$. What is the sum of the squares of r and s?

    (A) 9/25 **(B)** 4/9 **(C)** 9/4 **(D)** 25/9 **(E)** 6

    Solution: D. Because both rs and $r+s$ are positive, both r and s are positive. Since $r-s > 0$, it follows that $s < r$ and thus $s/r < 1$. Thus either $s/r = 1/3$ or $s/r = 3/4$. If $s/r = 1/3$ then $3s = r$ and $r+s = 4s$ and $r-s = 2s$. In case $s/r = 3/4$, then $4s = 3r$ and $r+s = r+3r/4 = 7r/4$ and $r-s = r-3r/4 = r/4$ so $r+s = 7(r-s)$. It follows that $r+s = 7/3$ and $r-s = 1/3$. From this it follows that $r = 4/3$, $s = 1$ and $r^2+s^2 = 25/9$.

    An alternate solution starts the same way, with rs and $r+s$ positive, so both r and s are positive. Thus $r > s$ and $s/r < 1$. But then $r+s > r-s$, so $r+s$ must be greater than 1 since only two of the numbers are less than 1. Consider $(r+s)2 = r^2+2rs+s^2$. If $r+s = 4/3$, then s/r and $r-s$ are the two numbers less than one leaving $rs = 7/3$. But then the left side is $(r+s)^2 = 16/9$ and the right side is $r^2+2rs+s^2 = 14/3 + r^2 + s^2 > 14/3 > 16/9$, a contradiction. So we must have $r+s = 7/3$ and $49/9 = (r+s)^2 = r^2+2rs+s^2 \leq r^2+s^2+8/3$. Thus $25/9 \leq r^2+s^2 < 49/9 < 6$ and therefore the only choice is 25/9.

21. How many pairs of positive integers satisfy the equation $3x+6y = 95$?

    (A) none **(B)** one **(C)** two **(D)** three **(E)** four

    Solution: A. If x and y are integers, then $3x+6y = 3(x+2y)$ is divisible by 3. Since 95 is not divisible by 3, there is no such solution.

22. Suppose all three of the points $(-2, 10), (1, -8)$, and $(4, 10)$ lie on the graph of $y = ax^2 + bx + c$. What is abc?

(A) -24 **(B)** 0 **(C)** 12 **(D)** 24 **(E)** 48

Solution: E. By symmetry, the vertex must be the point $(x, y) = ((-2 + 4)/2, y) = (1, y) = (1, -8)$, so $-b/2a = 1$. Evaluating at 1 yields $a \cdot 1^2 - 2a \cdot 1 + c = -8$. Replacing b and c with their values in terms of a, $a(-2)^2 - 2a(-2) + a - 8 = 10$ from which it follows that $a = 2$, whence $b = -4$ and $c = -6$.Thus the product is 48.

Alternatively, write the three conditions

$$4a - 2b + c = 10 \tag{1}$$
$$a + b + c = -8 \tag{2}$$
$$16a + 4b + c = 10 \tag{3}$$

Subtracting (2) from (1), we have $3a - 3b = 18$ so $a - b = 6$. Subtracting (1) from (3) gives $12a + 6b = 0$ so $2a + b = 0$. We now find $3a = 6$ so $a = 2$, then $b = a - 6 = -4$, and $c = -8 - a - b = -6$ and their product is 48.

Alternate solution using properties of parabolas. (1) First step is the same, use symmetry to get that the x-coordinate of the vertex is $x = 1$. (2) Next note that the point $(4, 10)$ is 3 units to the right of the vertex and $18 = 2 \cdot 3^2$ units above the vertex, this implies that $a = 2$. (3) Since $-b/2a = 1$, $b = -4$. Plugging in any of the three x-coordinates and calculating to get the corresponding y-coordinate yields $c = -6$.

23. An amount of \$2000 is invested at $r\%$ interest compounded continuously. After four years, the account has grown to \$2800. Assuming that it continues to grow at this rate for 16 more years, how much will be in the account?

    (A) \$8976.47 **(B)** \$9874.23 **(C)** \$10001.99

    (D) \$10756.48 **(E)** \$2004.35

    Solution: D. Use the formula $A = Pe^{rt}$ where r is the annual rate of interest, t is the time in years, P is the principle, and A is the amount in the account at time t. Then $2800 = 2000e^{4r}$, which implies that $r = \log 1.4/4$. So the amount in the account after $4 + 16 = 20$ years is $A = 2000e^{20r} = 2000 \cdot (e^{\ln 1.4/4})^{20} = 2000(1.4)^5 = 10756.48$.

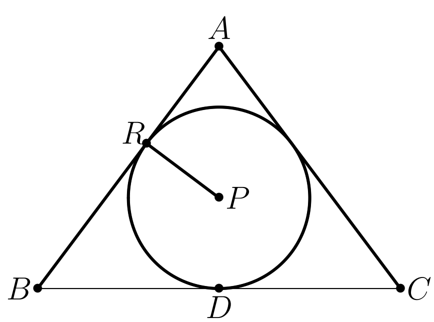
    Alternate solution: Since 16 is an integer multiple of 4, all that matters is that we start with \$2000 and have \$2800 after four years–the type of compounding doesn't matter. Based on compounding on a four year cycle, the (decimal) rate is $800/2000 = .4$. So we simply use 5 "compoundings" at this rate to see that the amount in the account will be $A = 2000 \cdot (1.4)^5 = 10,756.48$.

24. Two sides of an isosceles triangle have length 5 and the third has length 6. What is the radius of the inscribed circle?

    (A) 1 **(B)** 1.25 **(C)** 1.5 **(D)** 1.75 **(E)** 2

Solution: C. The altitude associated to the side of length 6 cuts the isosceles triangle into two congruent right triangles. The hypotenuse in these right triangles is 5, and one of the legs has length 3. By the Pythagorean Theorem, the height of the triangle is 4. Hence the area is $A = (6 \times 4)/2 = 12$ square units, while the semiperimeter is $s = (5 + 5 + 6)/2 = 8$ units. The radius ρ of the inscribed circle satisfies $A = s \cdot \rho^*$

Alternate solution. Let $\overline{AB}$ and $\overline{AC}$ be the sides of length 5 in the triangle $\triangle ABC$ and let P be the center of the inscribed circle. Let R be the point where the circle is tangent to side $\overline{AB}$ and let D be the foot of the perpendicular from A to the side $\overline{BC}$. By the Pythagorean Theorem, the length of $\overline{AD}$ is 4. Thus the length of segment $\overline{AP}$ is $4 - r$. Finally, let θ be the measure of angle $\angle BAD = \angle RAP$. Then $\sin(\theta) = 3/5 = r/(4 - r)$, the first value based on $\triangle ADB$ and the second based on $\triangle APR$. Solving yields $r = 1.5$.



*If P is the incenter of $\triangle ABC$, the triangles APB, BPC, CPA have areas given by $\frac{1}{2}\rho \cdot AB$, $\frac{1}{2}\rho \cdot BC$, $\frac{1}{2}\rho \cdot CA$ so the area of $\triangle ABC$ is $\frac{1}{2}\rho(AB + BC + CA) = \rho \cdot s$, and so we get $\rho = A/s = 12/8 = 3/2$.

25. Consider the circle $x^2 - 14x + y^2 - 4y = -49$. Let L_1 and L_2 be lines through the origin O that are tangent to the circle at points A and B. Which of the following is closest to the measure of the angle AOB?

(A) $35.1°$ **(B)** $34.8°$ **(C)** $33.6°$ **(D)** $32.8°$ **(E)** $31.9°$

Solution: E. One of the tangent lines is the x-axis since the radius of the circle is 2 and the center is at $(7, 2)$. Therefore the angle is given by $\angle AOB = 2\tan^{-1}\frac{2}{7} \approx 31.9°$.

26. Three digits a, b, and c are selected from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, one at a time, with repetition allowed. What is the probability that $a > b > c$?

(A) 3/25 **(B)** 4/25 **(C)** 1/5 **(D)** 6/25 **(E)** 7/25

Solution: A. There are 10^3 ways to pick three digits, and $6\binom{10}{3} = 720$ ways pick three *different* digits. In exactly one case out of six, the digits are in order, so the probability is $120/1000 = 3/25$.

27. Let p denote the smallest prime number greater than 200 for which there are positive integers a and b satisfying

$$a^2 + b^2 = p.$$

What is $a + b$? Note the list of primes on the last page.

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

Solution: B. Note that $a^2 + b^2$ is even if both a and b have the same parity. Since $a^2 + b^2$ is odd, one of a and b is odd and the other is even. Suppose a is odd. Then $a^2 = (2k + 1)^2 = 4k^2 + 4k + 1$ is one bigger than a multiple of 4. Also $b^2 = (2l)^2 = 4l^2$ is a multiple of 4.. Thus $a^2 + b^2$ is one bigger than a multiple of 4. Checking $201, 205, 209, 213, 217, 221$ and 225, we find that none are prime ($11|209$, $7|217$ and $13|221$). Therefore 229 is the first viable candidate. And $229 = 2^2 + 15^2$, so $a = 2$ and $b = 15$ and $a + b = 17$.

28. Four positive integers a, b, c and d satisfy $abcd = 10!$. What is the smallest possible sum $a + b + c + d$?

    (A) 170 **(B)** 175 **(C)** 178 **(D)** 183 **(E)** 185

    Solution: B. Note that $10! = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 2^8 3^4 5^2 7$. The sum $a+b+c+d$ is smallest if all four numbers are equal, in which case $a+b+c+d = 4\sqrt[4]{10!} = 2^2 \cdot 3 \cdot \sqrt{5} \cdot \sqrt[4]{7} \approx 174.58$. Hence $a + b + c + d \geq 175$. This optimum is obtained for $a = 40, b = 42, c = 45$, and $d = 48$.

29. The faces of a cube are colored red and blue, one at a time, with equal probability. What is the probability that the resulting cube has at least one vertex P such that all three faces containing P are colored red?

    (A) 1/4 **(B)** 5/16 **(C)** 27/64 **(D)** 1/2 **(E)** None of the above

    Solution: C. The expansion $(r+b)^6 = 1r^6+6r^5b+15r^4b^2+20r^3b^3+15r^2b^4+6rb^5+1b^6$ provides an inventory of the 64 possible colorings of the cube. For example the coefficient 15 of the term r^4b^2 can be interpreted as saying that there are 15 ways to build the cube with four red faces and two blue faces. Among all 64 possible cubes, those with 6 red (only one of these), or with 5 red faces (there are 6 of these) must have such a vertex. Among the 15 with 4 red faces, 12 have such a vertex (they correspond to the 12 edges) and among the 20 with 3 red faces, exactly 8 (these correspond to the eight vertices) have such a vertex. So the number of 'good' cubes is 27, and the probability is 27/64.

    Or, there are six sides, so the number of ways the cube can be painted is $2^6 = 64$. To have no such vertex P is the same as having some pair of opposite sides both blue. Thus there are at most three patterns with more than three red sides but with no vertex P. There are exactly 20 ways to have exactly three red sides. So $(64-20)/2$ with more than three red sides. Thus $19 = 22-3$ is the number with at least one vertex P and more than three red sides. To have exactly three red sides and no vertex P, exactly one pair of opposite sides is red. There are three such pairs, and for each of these there are four choices for the other red side. So there are $8 = 20 - 12$ with a vertex P. Alternately, simply pick one of the eight vertices to be the one at the intersection of the three red sides. The total number with at least one vertex P is 27. So the probability is 27/64.

30. Let a, b, c, and d denote four digits, not all of which are the same, and suppose that $a \leq b \leq c \leq d$. Let n denote any four digit integer that can be built using these digits. Define $K(n) = \underline{dcba} - \underline{abcd}$. The function K is called the *Kaprekar* function. For example $K(1243) = 4321 - 1234 = 3087$. A four-digit integer M is called a Kaprekar number if there is a four-digit integer N such that $K(N) = M$. Which of the following is not a Kaprekar number?

    (A) 2936 **(B)** 7263 **(C)** 5265 **(D)** 3996 **(E)** 6264

    Solution: A. Each Kaprekar number is the difference of two numbers with the same digit sum. These two numbers, $\underline{dcba}$ and $\underline{abcd}$ yield the same remainder when divided by 9. Hence their difference is a multiple of 9. Only 2936 is not a multiple of 9. On the other hand, $K(A) = B, K(B) = C, K(C) = D$, and $K(D) = E$.

    Rather involved alternate solution: Write the numbers as $1000d + 100c + 10b + a$ and $1000a + 100b + 10c + d$. Then the difference is $1000(d-a) + 100(c-b) + 10(b-c) + (a-d)$. $M = 1000w + 100x + 10y + z$ has four digits, so it must be that $a < d$. Thus $a - d < 0$. This means the units digit of M must be $z = 10 + a - d > 0$, the 10 must be deducted from $10(b-c)$.

    If $b = c$, then the "tens" digit of M will be a 9, as will the "hundreds" digit and the "thousands" digit will be $d - a - 1$. The possibilities are $z = 1$, $y = x = 9$ and $w = 8$ (example: $a = 0 \leq b = c \leq d = 9$); $z = 2$, $y = x = 9$ and $w = 7$ $(a = 1 \leq b \leq c \leq d = 9)$; $z = 3$, $y = x = 9$ and $w = 6$ $(a = 2 \leq b = c \leq d = 9)$; $z = 4$, $y = x = 9$ and $w = 5$ $(a = 2 \leq b = c \leq d = 8)$; $z = 5$, $y = x = 9$ and $w = 4$ $(a = 1 \leq b = c \leq d = 6)$; $z = 6$, $y = x = 9$ and $w = 3$ $(a = 2 \leq b = c \leq d = 6)$; $z = 7$, $y = x = 9$ and $w = 2$ $(a = 4 \leq b = c \leq d = 7)$; $z = 8$, $y = x = 9$ and $w = 1$ $(a = 3 \leq b = c \leq d = 5)$.

    If $b < c$, then the 10 needed for $z = 10 + a - d > 0$, can come from $100 + 10(b-c)$ and the 100 needed for $100 + 10(b-c)$ can come from $100(c-b)$ without changing the thousands digit which will be $1000(d-a)$. Thus $y = 10 + b - c - 1$ and $x = c - b - 1$. In this case the possible values of the pairs (w, z) and (x, y) are related. For (w, z) the possible pairs are $(1, 9)$, $(2, 8)$, $(3, 7)$, $\ldots$, $(8, 2)$, and $(9, 1)$. For (x, y) the possible pairs are $(0, 8)$, $(1, 7)$, $(2, 6)$, $\ldots$, $(7, 1)$ and $(8, 0)$. However, since $w = d - a$ and $a \leq b < c \leq d$, $x = c - b - 1 < c - b \leq d - a = w$. For a given w, any $x < w$ is OK, but no $x \geq w$ can occur. So $M = 3267$ can occur, but $M = 1269$ cannot.

    As in the case that $x = y = 9$, we can use the above information to construct a number N where $K(N) = M$ with M a given number fitting the pattern (with $(x, y) \neq (9, 9)$). Start with $w \geq 1$ and $0 \leq x < w$. Next choose a pair $d > a$

with $d - a = w$ and set $c = d$ and $b = c - x - 1$. For $M = 6264$, take (for example) $d = 8$, $c = 8$, $b = 5$ and $a = 2$ [or $d = 9$ $c = 9$, $b = 6$, and $a = 3$]. Then $8852 - 2588 = 6264[= 9963 - 3699]$, as desired.

List of Primes between 1 and 500:

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499					