# UNC Charlotte 2006 Comprehensive

March 6, 2006

1. What is the sum of the digits of the integer solution to $\sqrt{14+\sqrt{27-\sqrt{x-1}}} = 4$?

   **(A)** 5 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 11

   **Solution:** C. Square both sides to get $14+\sqrt{27-\sqrt{x-1}} = 16$, then massage it, square again, and solve $27-\sqrt{x-1} = 4$ to get $x-1 = 529$. Thus $x = 530$ and the sum of the digits is $5+3+0 = 8$.

2. In a box there are red and blue balls. If you select a handful of them with eyes closed, you have to grab at least 5 of them to make sure at least one of them is red and you have to grab at least 10 of them to make sure both colors appear among the balls selected. How many balls are there in the box?

   **(A)** 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

   **Solution:** D. If 5 balls are needed to make sure at least one of them is red, that means that there are 4 blue balls in the box. If you need 10 to make sure both colors appear then there are 9 balls of the more used color –which must be red. Therefore there are $9+4 = 13$ balls in the box.

3. Some hikers start on a walk at 9 a.m. and return at 2 p.m. One quarter of the distance walked is uphill, one half is level, and one quarter is downhill. If their speed is 4 miles per hour on level land, 2 miles per hour uphill, and 6 miles per hour downhill, approximately how far did they walk?

   **(A)** 16.4 miles **(B)** 17.1 miles **(C)** 18.9 miles

   **(D)** 20.0 miles **(E)** 21.2 miles

   **Solution:** B. Suppose that they walk $4x$ miles, so that they walk $x$ miles uphill, $x$ miles downhill, and $2x$ level miles. This takes a total time of $\frac{x}{2}+\frac{x}{6}+\frac{2x}{4}$ hours. Setting this equal to 5 hours, we find $x = 30/7$ so that the total distance is $4x = 120/7 \approx 17.1$ miles.

4. Five regular polygons, a triangle, a square, a pentagon, a hexagon, and a dodecagon (a 12-sided polygon), all have the same perimeter. Which one has the greatest area?

   **(A)** the triangle **(B)** the square **(C)** the pentagon

   **(D)** the hexagon **(E)** the dodecagon

   **Solution:** E. Suppose we have an $n$ sided regular polygon with sides of length $s$. By connecting each vertex to the center of the polygon, we can form $n$ congruent triangles. The central angle for each triangle is $360/n$ degrees. We can use trigonometry to determine that the area of each triangle is $\frac{s^2}{4\tan(180/n)}$. Setting the perimeter $p = ns$ and adding the areas of the $n$ triangles, we see that the area of the $n$ sided regular polygon is $\frac{p^2}{4n\tan(180/n)}$. This increases as $n$ increases, so $n = 12$ gives the largest area.

5. The radius of the circle given by

$$x^2 - 6x + y^2 + 4y = 12$$

   is

   **(A)** 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 36

   **Solution:** A. Complete the squares by adding 9 and 4 to both sides to get

$$x^2 - 6x + 9 + y^2 + 4y + 4 = (x-3)^2 + (y+2)^2 = 12 + 9 + 4 = 25 = 5^2.$$

   So the radius is 5.

6. What is the length of the interval of solutions to the inequality $1 \le 3-4x \le 11$?

   **(A)** 1.75 **(B)** 2.00 **(C)** 2.25 **(D)** 2.50 **(E)** 3.25

   **Solution:** D. Subtract 3 from all parts to get $-2 \le -4x \le 8$, then divide all by $-4$ to get $1/2 \ge x \ge -2$, so the length of the interval is $1/2-(-2) = 2.50$.

   Alternately, solve $x^2 - 6x = 0$ and $y^2 + 4y = 0$. Obviously the solutions are $x = 0, 6$ and $y = 0, -4$. From this it follows that the center of the circle is the point $(h, k)$ where $h = (0+6)/2 = 3$ and $k = (0+(-4))/2 = -2$. One way to finish: substitute $x = 3$ (or $y = -2$) into the equation $x^2 - 6x + y^2 + 4y = 12$ and solve for $y$ (or $x$). Then $0 = y^2 + 4y - 21 = (y + 7)(y - 3)$, so $(3, -7)$ and $(3, 3)$ are on the vertical diameter. Thus the diameter is 10 and the radius is 5. OR: use that standard form for the equation is $(x-3)^2+(y+2)^2 = r^2$ and from this deduce that $r^2 = 9 + 4 + 12 = 25$.

   Alternatively, since the length is given by the difference, the shift of 3 and the negative sign on the 4 can be ignored. Simply calculate $(11 - 1)/4 = 2.5$.

7. Quadrilateral $ABCD$ with the sides $AB = 20, BC = 7, CD = 24$ and $DA = 15$ has right angles at $A$ and $C$. What is the area of $ABCD$?

   **(A)** 154 **(B)** 186 **(C)** 200 **(D)** 234 **(E)** 286

   **Solution:** D. The area of the quadrilateral is the sum of the areas of the right triangles $ABD$ and $BCD$. This sum is $\frac{1}{2}(20)(15) + \frac{1}{2}(7)(24) = 234$.

8. Benny eats a box of cereal in 14 days. He eats the same size box of cereal with his younger brother Nathan in 10 days. How many days will it take Nathan to finish the box of cereal alone?

   **(A)** 20 **(B)** 25 **(C)** 30 **(D)** 35 **(E)** 40

   **Solution:** D. Within $10 \times 14 = 140$ days Benny will eat 10 boxes of cereal alone, while together with Nathan they will eat $140 \div 10 = 14$ boxes for the same time period. That means the share of his brother is $14 - 10 = 4$ boxes for 140 days. Therefore, Nathan eats one box of cereal for $140 : 4 = 35$ days.

9. The base of a regular square pyramid is inscribed in the base of a cylinder. The height of the cylinder is triple the height of the pyramid. Find the ratio of the volume of the pyramid to the volume of the cylinder.

   **(A)** $\dfrac{2}{9\pi}$ **(B)** $\dfrac{2}{3\pi}$ **(C)** $\dfrac{\pi}{9}$ **(D)** $\dfrac{4\pi}{3}$ **(E)** $\dfrac{4\pi}{9}$

   **Solution:** A. Let $r$ be the radius of the base of the cylinder, and $s$ be the side of the square base of the pyramid. Then $s^2 = r^2 + r^2 = 2r^2$. The volume $V_p$ of the pyramid is $s^2h/3$. The volume $V_c$ of the cylinder is $\pi r^2(3h)$. The ratio is therefore $V_p/V_c = 2/9\pi$.

10. Given the following system of equations

$$\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{1}{3} \\ \frac{1}{x} + \frac{1}{z} &= \frac{1}{5} \\ \frac{1}{y} + \frac{1}{z} &= \frac{1}{7} \end{aligned}$$

   What is the value of the ratio $\frac{z}{y}$?

   **(A)** 17 **(B)** 23 **(C)** 29 **(D)** 31 **(E)** 36

   **Solution:** C. Add the three equations together to get $(*)\quad 2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{1}{3}+\frac{1}{5}+\frac{1}{7} = \frac{35+21+15}{105} = \frac{71}{105}$. Then subtract the $2\left(\frac{1}{x}+\frac{1}{y}\right) = \frac{2}{3}$ from both sides to get $\frac{2}{z} = \frac{71}{105} - \frac{70}{105} = \frac{1}{105}$, so $z = 210$. Subtracting $2\left(\frac{1}{x}+\frac{1}{z}\right) = \frac{2}{5}$ from both sides of $(*)$ yields $\frac{2}{y} = \frac{71}{105} - \frac{42}{105} = \frac{29}{105}$, so $y = \frac{210}{29}$. It follows that $\frac{z}{y} = \frac{210}{\frac{210}{29}} = 29$.

   Alternatively, change variables to $u = 1/x$, $v = 1/y$ and $w = 1/z$. Then (using the new forms) subtract the second equation from the first to get $v - w = 2/15$. Add this to the third to get $v = 29/210$. Substitute into the third to get $w = 1/210$. Thus $z/y = v/w = 29$.

11. Suppose $a, b, c$ are integers such that

   **1.** $0 < a < b$,

   **2.** The polynomial $x(x-a)(x-b) - 17$ is divisible by $(x-c)$.

What is $a + b + c$ ?

**(A)** 14 **(B)** 17 **(C)** 21 **(D)** 24 **(E)** 27

**Solution:** C. Since it is divisible by $(x - c)$, we have

$$c(c - a)(c - b) = 17.$$

Since $c(c - a)(c - b) = 17 > 0$, it follows that $c > 0$ and we have the following two cases:

**Case 1:** $0 < (c - b) < (c - a) < c$

**Case 2:** $(c - b) < (c - a) < 0 < c$.

Since 17 is a prime number, case 1 does not occur. In case 2, $c = 1, c - a = -1, c - b = -17$. Hence $a = 2, b = 18, c = 1$. Thus, $a + b + c = 21$.

12. Let $x, y$ be positive integers with $x > y$. If $1/(x + y) + 1/(x - y) = 1/3$, find $x^2 + y^2$.

**(A)** 52 **(B)** 58 **(C)** 65 **(D)** 73 **(E)** 80

**Solution:** E. Write $u = x+y$ and $v = x-y$, then $u$ and $v$ are positive integers and $v < u$. Now $1/u + 1/v = 1/3$ so $3 < v < 6$, so $v = 5$ or $v = 4$. If $v = 5$ then $u = 7.5$ which is not an integer. If $v = 4$ then $u = 12$, so we will have $x = 8, y = 4$. Therefore, $x^2 + y^2 = 64 + 16 = 80$.

13. A ball is dropped onto a floor from a height of 1 meter. Each time that the ball hits the floor it rebounds to half its previous height. (After falling one meter it rebounds to a height of $1/2$ meter. The next time it hits the floor, it rebounds to a height of $1/4$ meter, etc.). How far has the ball traveled when it hits the floor for the 40th time?

    **(A)** $T = 2 + (2^{38} - 1)/2^{38}$ **(B)** $T = 1 + (2^{38} - 1)/2^{39}$ **(C)** $T = 2$

    **(D)** $T = 3$ **(E)** $T > 3$

    **Solution:** A. When it hits the floor the first time, it has traveled 1 meter. Then it rebounds to a height of $1/2$ meter and falls back to the floor to hit the floor for the second time. This adds 1 more meter distance so it now has traveled 2 meters. After the second hit, the ball rebounds to a height of $(1/4)$ meter and the total distance traveled when the ball hits the floor the third time is $1+2(1/2)+2(1/4) = 1+1+(1/2)$. The total distance traveled when the ball hits the floor for the 40th time is $1+1+...+(1/2)^{38} = 1+\frac{1-(1/2)^{39}}{1-1/2} = 3-(1/2)^{38}$.

14. Let $a$ and $b$ be two positive integers such that $b$ is a multiple of $a$. If $\log_{10}(b/a)^{b/2}+\log_{10}\left(\sqrt{a/b}\right)^{9a} = 1$, then $b^2 - a^2 =$

    **(A)** 357 **(B)** 396 **(C)** 1600 **(D)** 5967 **(E)** 8436

    **Solution:** B. The equation could be written as $\frac{b}{2}\log(\frac{b}{a}) - \frac{9a}{2}\log(\frac{b}{a}) = 1$. Since $b = ka$, then $\frac{ka}{2}\log(k) - \frac{9a}{2}\log(k) = 1$ . So $\log k = \frac{2}{a(k-9)}$ and $a = \frac{2}{(k-9)\log k}$. Since $a$ is an integer, the only solution is when $k = 10$, so $a = 2$ and $b = 20$ and $b^2 - a^2 = 396$.

15. A triangle with sides $a = 15$, $b = 28$ and $c = 41$ has an altitude of integer length. What is the length of this altitude?

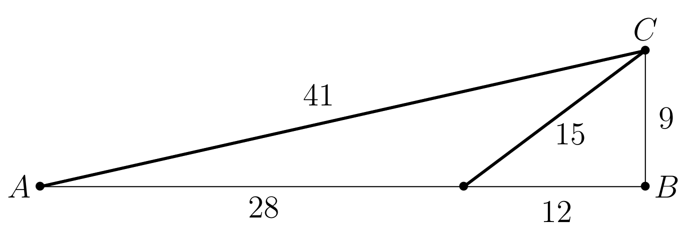
    **(A)** 6 **(B)** 7 **(C)** 9 **(D)** 16 **(E)** 17

**Solution:** C. With $s = (a + b + c)/2 = 42$, use Heron's formula for the area

$$F = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{42 \cdot 27 \cdot 14 \cdot 1} = \sqrt{2^2 \cdot 3^4 \cdot 7^2} = 2 \cdot 3^2 \cdot 7.$$

Since the three heights are $h_a = 2F/a = 4 \cdot 9 \cdot 7/(3 \cdot 5)$; $h_b = 2F/b = 4 \cdot 9 \cdot 7/(4 \cdot 7) = 9$; and $h_c = 2F/c = 4 \cdot 9 \cdot 7/41$, so only $h_b = 9$ is an integer.

Alternatively, notice that to the triangle with sides 15, 28, and 41, we can append a triangle with sides 9, 12 and 15 to get a right triangle with sides 9, 40, and 41 as shown in the diagram.




Alternatively, let $\theta$ be the angle measure of the angle opposite the side of length 41 and let $\phi$ be its supplement. Note that $41^2 > 28^2 + 15^2$, so the angle is obtuse. Next use the law of cosines to get the cosine of $\theta$ and thus $-\cos(\phi)$, $-\cos(\phi) = \cos(\theta) = (28^2 + 15^2 - 41^2)/(2 \cdot 28 \cdot 15) = -4/5$. Thus $\sin(\phi) = 3/5$, and $9 = 15 \cdot \sin(\phi)$ is the altitude for the side of length 28 ($84/5 = 28 \cdot \sin(\phi)$ is the altitude for the side of length 15, and $9 \cdot 28/41$ is the altitude for the side of length 41).

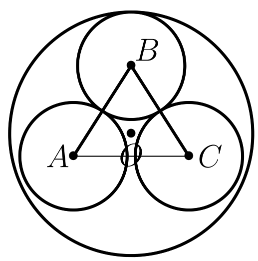
16. Inside the unit circle $D = \{(x, y) \mid x^2 + y^2 = 1\}$ there are three smaller circles of equal radius $a$, tangent to each other and to $D$. If $a = p\sqrt{3} - q$, find the sum of the integers $p + q$.

    **(A)** 4 **(B)** 5 **(C)** 7 **(D)** 10 **(E)** 12

    **Solution:** B. The centers of the three smaller circles form an equilateral triangle with side $2a$ and height $h = \sqrt{3} \cdot a$. The center $O$ of this triangle satisfies $OA = OB = OC = 2h/3 = 2a\sqrt{3}/3$. The radius of the unit circle satisfies $1 = OA + a = \frac{2\sqrt{3}+3}{3}a$. Hence

$$a = \frac{3}{2\sqrt{3}+3} = \frac{3(2\sqrt{3}-3)}{(2\sqrt{3}-3)(2\sqrt{3}+3)} = \frac{3(2\sqrt{3}-3)}{12-9} = 2\sqrt{3} - 3.$$

    We see that $p = 2$ and $q = 3$.




17. Let $N$ denote a six-digit integer whose 6 digits are 1,2,3,4,5,6 in random order. What is the probability that $N$ is divisible by 6?

    **(A)** 1/6 **(B)** 1/3 **(C)** 2/5 **(D)** 1/2 **(E)** 3/5

    **Solution:** D. Let $N$ denote the number. Since the sum $1+2+3+4+5+6 = 21$ of the digits of $N$ is a multiple of 3, $N$ must be a multiple of 3. Exactly half the numbers formed from the digits 1 through 6 without replacement are even. A number is a multiple of 6 if it is both even and a multiple of 3. Thus, the probability that $N$ is a multiple of 6 is 1/2.

18. Three fair dice are rolled. What is the probability that the product of the three outcomes is a prime number? Recall that 1 is not considered to be prime.

    **(A)** 0 **(B)** 1/72 **(C)** 1/36 **(D)** 1/24 **(E)** 1/8

    **Solution:** D. There are $6^3 = 216$ ordered triples of dice rolls. The product is prime precisely when two rolls are 1 and the third is a prime number. Since the prime (2,3, or 5) can appear in any of the three positions, there are 9 such triples, so the probability is $9/216 = 1/24$.

19. The odd numbers from 1 to 17 can be used to build a $3 \times 3$ magic square (the rows and columns have the same sum). If the $1, 5$, and 13 are as shown, what is $x$?

    **(A)** 7 **(B)** 9 **(C)** 11 **(D)** 15 **(E)** 17

| | | |
|---|---|---|
| | 1 | |
| 5 | | 13 |
| $x$ | | |

**Solution:** A. The magic sum is 27 because the sum of the first nine odd positive integers is $9^2 = 81$. This means that we can fill in the 9 and the 17 as shown.

| | | |
|---|---|---|
| | 1 | |
| 5 | 9 | 13 |
| $x$ | 17 | |

We can quickly eliminate $x = 11$ and $x = 15$. If $x = 3$, then we would have to use the 7 in the bottom right square, which would require another 7 in the top right square. Thus $x = 7$. The complete square is shown below.

| | | |
|---|---|---|
| 15 | 1 | 11 |
| 5 | 9 | 13 |
| 7 | 17 | 3 |

20. Let $N$ denote the two-digit number whose cube root is the square root of the sum of its digits. How many positive divisors does $N$ have?

    **(A)** 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

    **Solution:** C. There are just two two-digit cubes, 27 and 64. Checking each one shows that 27 satisfies the requirements and 64 does not. The number 27 has 4 divisors.

21. At a picnic there were $c$ children, $f$ adult females, and $m$ adult males, where $2 \leq c < f < m$. Every person shook hands with every other person. The sum of the number of handshakes between children, the number of handshakes between adult females and the number of handshakes between adult males is 57. How many handshakes were there altogether?

    **(A)** 153  **(B)** 171  **(C)** 190  **(D)** 210  **(E)** 231

    **Solution:** B. The number of handshakes among children is $T_{c-1} = C_2^c = \binom{c}{2} = \frac{c!}{(c-2)!2!}$, and similarly for the women and the men. The only way to write 57 as a sum of three *different* triangular numbers is $6 + 15 + 36$, so there must have been 4 children, 6 women, and 9 men, and $\binom{19}{2} = 171$ handshakes.

22. Two consecutive positive integers $n$ and $n+1$, both with exactly four divisors, have the same sum of divisors. What is the number of divisors of their product?

    **(A)** 6  **(B)** 12  **(C)** 16  **(D)** 20  **(E)** 24

    **Solution:** C. To say the numbers have exactly four divisors means that they are each a product of two distinct prime numbers, say $pq$ and $rs$. To say the sum of their divisors is the same means $1 + p + q + pq = 1 + r + s + rs$, and to say they are consecutive means $pq + 1 = rs$. Putting the two equations together gives $p+q = 1+r+s$, which can happen only when one of the primes is even (hence 2). A little trial and error produces some primes that work: $p = 2, q = 7, r = 3$, and $s = 5$. So the two numbers could be 14 and 15, both with sum of factors equal to 24. The number of factors of $14 \cdot 15 = 2 \cdot 3 \cdot 5 \cdot 7$ is $2^4 = 16$.

    Actually, we really only need to know that the conditions say that the four primes $p, q, r, s$ are **distinct**. Suppose a factor in the first number is the same as a factor in the second number, say $p = r$. Then $rs - pq = 1 \Rightarrow r = \frac{1}{s-q}$ which will not give a prime number (the only positive integer solution is $r = 1$). This means that the product of the two numbers is $pqrs$ with each prime distinct. The number of divisors is always 16.

23. Suppose $a, b, c$, and $d$ are positive integers satisfying

$$ab + cd = 38$$

$$ac + bd = 34$$

$$ad + bc = 43$$

What is $a + b + c + d$?

**(A)** 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 20

**Solution:** D. Add the 2nd and 3rd equations together and factor to get

$$\begin{aligned} ac + ad + bc + bd &= a(c + d) + b(c + d) \\ &= (a + b)(c + d) \\ &= 77 = 7 \cdot 11 \end{aligned}$$

It follows that $\{a + b, c + d\} = \{7, 11\}$ and $a + b + c + d = 18$. The equations are satisfied if $a = 7, b = 4, c = 2$ and $d = 5$.

24. Let $a_0 = 2$ and $a_1 = 3$ and let $a_{n+2} = |a_n| - a_{n+1}$ for all $n \geq 0$. What is the smallest $n$ such that $a_n \geq 100$?

**(A)** 10 **(B)** 11 **(C)** 13 **(D)** 15 **(E)** 17

**Solution:** E. Computing the first few terms, we see that the odd terms are positive, the even terms are negative, and the odd terms satisfy a Fibonacci-type relation $a_{2n+3} = a_{2n+1} + a_{2n-1}$ for all $n \geq 1$. Computing successive values, we get $a_3 = 4, a_5 = 7, a_7 = 11, a_9 = 18, a_{11} = 29, a_{13} = 47, a_{15} = 76$, and finally, $a_{17} = 123$.

25. Let $S(n) = n$ in case $n$ is a single digit integer. If $n \geq 10$ is an integer, $S(n)$ is the sum of the digits of $n$. Let $N$ denote the smallest positive integer such that $N + S(N) + S(S(N)) = 99$. What is $S(N)$?

    **(A)** 9 **(B)** 10 **(C)** 12 **(D)** 15 **(E)** 18

    **Solution:** D. If $n < 75$ then $S(n) \leq S(69) = 15$ and $S(S(n)) \leq 9$, so $n + S(n) + S(S(n)) < 99$. On the other hand, $75 + S(75) + S(S(75)) = 90$, $76 + S(76) + S(S(76)) = 93$, $77 + S(77) + S(S(77)) = 96$, and $78 + S(78) + S(S(78)) = 99$. So 78 is the least integer with the property and $S(78) = 15$.

    Alternatively, let $T(N) = N + S(N) + S(S(N))$. Since neither $S(N)$ nor $S(S(N))$ can be zero, we must have $N \leq 97$. Thus $S(N) \leq 17$ (using $N = 89$) and $S(S(N)) \leq 9$. This gives a crude lower bound for $N$ of $99-(17+9) = 73$. If $S(N+1)$ has the same number of digits as $S(N)$, then $T(N+1) = T(N)+3$ since $S(N+1)$ will be one more than $S(N)$ and $S(S(N+1))$ will be one more than $S(N)$. Moreover, as long as $S(N+k)$ continues to have the same number of digits as $S(N)$, we have $T(N+k) = T(N) + 3k$. The first integer $M$ greater than 73 such that $S(M)$ has only one digit is $M = 80$. Thus the formula will work from 73 up to 79, so for $1 \leq k \leq 6$. We have $T(73) = 73 + S(73) + S(S(73)) = 73 + 10 + 1 = 84$, and $99 - 84 = 15 = 3 \cdot 5$. Thus $99 = T(73) + 3 \cdot 5 = T(78)$. Therefore $N = 78$ is the smallest integer with $T(N) = 99$.

26. Find the number of odd divisors of 7!.

    **(A)** 4 **(B)** 6 **(C)** 10 **(D)** 12 **(E)** 24

    **Solution:** D. The prime factorization of 7! is $2^4 \cdot 3^2 \cdot 5 \cdot 7$. Each odd factor of 7! is a product of odd prime factors, and there are $3 \cdot 2 \cdot 2 = 12$ ways to choose the three exponents.

27. John was contracted to work $A$ days. For each of these $A$ days that John actually worked, he received $B$ dollars. For each of these $A$ days that John didn't work, he had to pay a penalty of $C$ dollars. After the $A$ days of contracted work was over, John received a net amount of $D$ dollars for his work. How many of the $A$ days of contracted work did John not work?

    **(A)** $(AB - D)/(B + C)$ **(B)** $(AB + D)/(B + C)$ **(C)** $(AB - D)/(B - C)$
    **(D)** $(AB + D)/(B - C)$ **(E)** $(AC - B)/(D - C)$

    **Solution:** A. Let $x$ denote the number of days John did not work. Then he worked $A - x$ days and so earned $B(A - x) - Cx = D$ dollars. Solving this for $x$, we get $-(B + C)x = D - AB$ and so $x = (AB - D)/(B + C)$. Taken from Horatio Nelson Robinson's 1859 book 'A Theoretical and Practical Treatise on Algebra', with thanks to Dave Renfro.

28. Let $\Gamma$ be a plane containing three points $A(1,0,0), B(0,2,0), C(0,0,1)$. Find the distance from the origin $(0,0,0)$ to the plane $\Gamma$.

    **(A)** $1/3$ **(B)** $2/3$ **(C)** $1$ **(D)** $\pi/3$ **(E)** $2\pi/3$

    **Solution:** B. Note that the volume of the tetrahedron $OABC$ is $= \frac{1}{3}(\frac{1}{2} \times 2) = 1/3$. Since $AB = \sqrt{5} = BC$ and $AC = \sqrt{2}$, the area of the triangle $CAB$ is $= (1/2) \times \sqrt{2}\sqrt{(\sqrt{5})^2 - (\sqrt{2}/2)^2} = 3/2$. If the distance from $O$ to $\Gamma$ is $h$, then the volume can be written as $(1/3)(h)(3/2)$. Thus, the volume $(1/3)(h)(3/2)$ should be $1/3$. Hence $h = 2/3$.

    Alternatively, an equation of the plane containing $A$, $B$, and $C$ is $2x + y + 2z = 2$. Therefore the square of the distance from the origin to the point $(x, y, z)$ on the plane is $d^2 = x^2 + y^2 + z^2 = x^2 + (2 - 2x - 2z)^2 + z^2$. By symmetry, the point on the plane that is closest to the origin must satisfy $x = z$. Then $d^2 = 18x^2 - 16x + 4$. The $x$-coordinate of the vertex of this parabola is $x = \frac{16}{36} = \frac{4}{9}$. It will minimize $d^2$ and we get $d^2 = 18(\frac{4}{9})^2 - 16(\frac{4}{9}) + 4 = \frac{4}{9}$, so that $d = \frac{2}{3}$.

29. Let $N$ denote the 180-digit number obtained by listing the 90 two-digit numbers from 10 to 99 in order. Thus $N = 10111213\ldots99$. What is the remainder when $N$ is divided by 99?

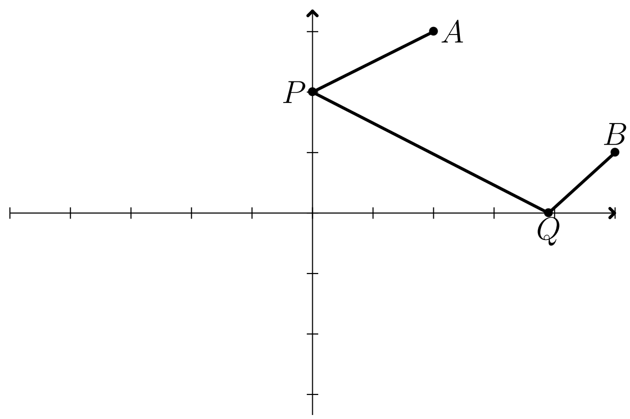
**(A)** 0 **(B)** 10 **(C)** 45 **(D)** 54 **(E)** 90

**Solution:** D. Let $U$ denote the sum of the 90 units digits and $T$ the sum of the 90 tens digits. Thus $U = 0+1+2+\cdots+9+\cdots+9 = 405$ and $T = 1+1+\cdots+1+2+2+\cdots+9+9+\cdots+9 = 450 = U+45$. Since $99 = 9\cdot 11$, we ask the two questions 'What is the remainder when $N$ is divided by 11' and 'What is the remainder when $N$ is divided by 9'. The divisibility test for 11 tells us that the remainder when a number $N$ is divided by 11 is the same as when the alternating sum of the digits of $N$ is divided by 11. That alternating sum is $U - T = -45$, and the remainder when $-45 = -5(11)+10$ is divided by 11 is 10. The remainder when $N$ is divided by 9 is the same as the remainder when the sum of digits of $N$ is divided by 9. This sum of digits is $U + T = 9\cdot 45 + 9\cdot 50 = 9\cdot 95$ which is a multiple of 9. So far we know that there are integers $p$ and $q$ such that $N = 9p$ ($N$ is a multiple of 9) and $N = 11q + 10$ (when $N$ is divided by 11, the remainder is 10). Let $r$ be the remainder when $N$ is divided by 99. Then there is an integer $k$ such that $N = 99k + r$. Now $N/9 = 11k + r/9 = p$ shows $r$ is a multiple of 9. Furthermore $N/11 = 9k + r/11 = q + 10/11$ shows $r = 11(q-9k)+10$ shows $r$ is 10 more than a multiple of 11. The only number in the range 0 to 98 that is both a multiple of 9 and 10 bigger than a multiple of 11 is 54.

Alternatively, consider a base 100 expansion of a positive integer $A$ with an even number of digits base 10, say $2n$ digits – $A = a_n\cdot 100^n + a_{n-1}\cdot 100^{n-1} + \cdots + a_1\cdot 100 + a_0$. Since $100 = 9\cdot 11 + 1 = 99 + 1$, the remainder on dividing $A$ by 99 is the same as the remainder on dividing the sum $a_n + a_{n-1} + \cdots + a_1 + a_0$ by 99. For the given integer, the corresponding sum is $10 + 11 + \cdots + 99 = (109\cdot 90)/2$ (90 terms in an arithmetic sequence where the sum of the first and last terms is $109 = 10 + 99$). Now simply divide by 99 to get the remainder of 54 for both the sum and the given integer $101112\ldots99$.

30. What is the length of the shortest path $APQB$ in the plane, where $A = (2, 3)$, $B = (5, 1)$, $P$ lies on the $y$-axis, and $Q$ lies on the $x$-axis.

    **(A)** 7 **(B)** 8 **(C)** $\sqrt{65}$ **(D)** $5\sqrt{3}$ **(E)** 9




**Solution:** C. The solution uses the *reflection principle*. Let $A' = (-2, 3)$ and $B' = (5, -1)$ represent the reflections of $A$ and $B$ across the $y$-axis and $x$-axis, respectively. The sum $A'P + PQ + QB'$ is the same as the sum $AP + PQ + QB$ for any points $P$ on the $y$-axis and $Q$ on the $x$-axis. But the shortest path from $(-2, 3)$ to $(5, -1)$ is a straight line that hits both the $y$ and $x$ axes. The distance between $(-2, 3)$ and $(5, -1)$ is $\sqrt{65}$, so this is the length of the shortest path.