UNC Charlotte 2011 Comprehensive

March 7, 2011

1. Three cowboys entered a saloon. The first ordered 4 sandwiches, a cup of coffee, and 10 doughnuts for \$8.45. The second ordered 3 sandwiches, a cup of coffee, and 7 doughnuts for \$6.30. How much did the third cowboy pay for a sandwich, a cup of coffee, and a doughnut?

   (A) \$2.00 **(B)** \$2.05 **(C)** \$2.10 **(D)** \$2.15 **(E)** \$2.20

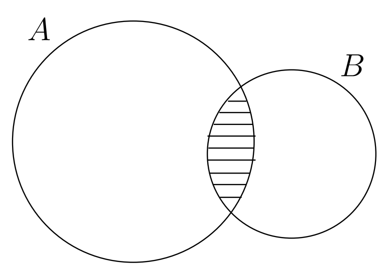
   Solution: (A). From the payment of the first cowboy we find the price of 8 sandwiches, 2 cups of coffee, and 20 doughnuts = \$16.90. From the payment of the second cowboy we calculate the price of 9 sandwiches, 3 cups of coffee, and 21 doughnuts = \$18.90. The difference of the sums $18.90 - 16.90 = 2.00$ is exactly the price of one sandwich, a cup of coffee, and a doughnut.

2. Given that $a+b+c=5$ and $ab+bc+ac=5$, what is the value of $a^2+b^2+c^2$?

   (A) 5 **(B)** 10 **(C)** 15 **(D)** 20 **(E)** 25

   Solution: (C). We use the formula $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ac$. We have $a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ac) = 5^2 - 2\cdot 5 = 15$.

3. Two circles, A and B, overlap each other as shown. The area of the common part is $2/5$ of the area of Circle A, and $5/8$ of the area of Circle B. What is the ratio of the radius of Circle A to that of Circle B?



   (A) $2:1$ **(B)** $3:2$ **(C)** $4:3$ **(D)** $5:4$ **(E)** $6:5$

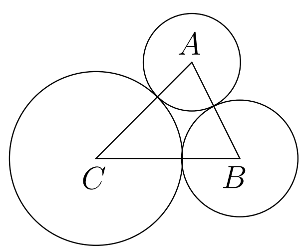
   Solution: (D). Let the radii of Circle A and B be r and R. Then we have $\frac{2}{5}\pi r^2 = \frac{5}{8}\pi R^2$. Thus, $16r^2 = 25R^2$ and $4r = 5R$ The ratio is $5:4$.

4. A pine tree is 14 yards high, and a bird is sitting on its top. The wind blows away a feather of the bird. The feather moves uniformly along a straight line at the speed 4 yards per second; it falls on the ground 4.5 seconds later at a distance D yards from the pine tree's base. Which of the following intervals contains the number D?

   (A) $(0, 10]$ **(B)** $(10, 11]$ **(C)** $(11, 12]$ **(D)** $(12, 13]$ **(E)** $(13, \inf)$

   Solution: (C). The right triangle formed by the trunk of the tree, the path of the feather and its projection on the ground, has hypotenuse 18 and legs 14 and D, so that $D^2 + 14^2 = 18^2$, or $D^2 = 324 - 196 = 128$; since $11^2 = 121 < 128 < 144 = 12^2$, we have $11 < D < 12$.

5. Three mutually tangent circles have centers A, B, C and radii a, b, and c respectively. The lengths of segments AB, BC, CA are 17, 23, and 12 respectively. Find the lengths of the radii.



   (A) $a = 13, b = 9, c = 7$ **(B)** $a = 5, b = 12, c = 6$ **(C)** $a = 6, b = 9, c = 12$
   (D) $a = 4, b = 8, c = 12$ **(E)** $a = 3, b = 14, c = 9$

   Solution: (E). We have $a + b = 17$, $b + c = 23$, and $c + a = 12$. Adding the equations gives $2(a + b + c) = 52$ or $a + b + c = 26$. Subtracting each of the original equations from this last equation gives $c = 9, a = 3$, and $b = 14$.

6. A ladder is leaning against a wall with the top of the ladder 8 feet above the ground. If the bottom of the ladder is moved 2 feet farther from the wall, the top of the ladder slides all the way down the wall and rests against the foot of the wall. How long is the ladder?

   (A) 14 feet **(B)** 15 feet **(C)** 16 feet **(D)** 17 feet **(E)** 18 feet

   Solution: (D). If the base is x feet from the wall in the initial position, then the ladder must be $x+2$ feet long. The Pythagoras theorem gives $x^2+8^2=(x+2)^2$ which yields $x=15$. The length of the ladder is 17 feet.

7. Each person in a group of five makes a statement: Amy says "At least one of us is lying." Ben says "At least two of us are lying." Carrie says "At least four of us are lying." Donna says "All of us are lying." Eddie says "None of us is lying." Based on these statements, how many are telling the truth?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

   Solution: (B). Donna's statement cannot be true so Donna and Eddie are lying. This makes Amy's statement and Ben's statement true. Therefore, Carrie is lying and the correct answer is (B).

8. The hour and minute hands of a clock are exactly 50 degrees apart at which of these times?

   (A) 2 : 20 **(B)** 5 : 36 **(C)** 7 : 50 **(D)** 9 : 30 **(E)** 11 : 10

   Solution: (A). Every hour, the hour hand covers $360/12 = 30$ degrees. At 2:20, the hour hand is $2\cdot 30+(1/3)\cdot 30 = 70$ degrees from the vertical. The minute hand is 120 degrees from the vertical, so there are 50 degrees between the two hands at 2:20.

9. The language of a pre-historic tribe is very simple and it contains only the 4 letters A, M, N, and O. Every word in this language has at most 4 letters. What is the largest number of words that can be created if every word must contain at least one vowel (A or O)?

   (A) 256 **(B)** 310 **(C)** 340 **(D)** 400 **(E)** 420

   Solution: (B). There are 4^n ways of making all n-letter words; 2^n of these contain no vowels. Adding the number of 1-letter, 2-letter, 3-letter, and 4-letter words with vowels, we obtain $4-2+4^2-2^2+4^3-2^3+4^4-2^4=310$.

10. English and American spellings are 'rigour' and 'rigor', respectively. A randomly chosen man staying at a Parisian hotel writes this word, and a letter taken at random from his spelling is found to be a vowel. If 40 percent of the English-speaking men at the hotel are English and 60 percent are Americans, what is the probability that the writer is an Englishman?

    (A) 1/4 **(B)** 1/3 **(C)** 2/5 **(D)** 5/11 **(E)** 1/2

    Solution: (D). The probability of choosing a vowel written by an Englishman is $\frac{3}{6} \cdot 0.4 = 0.2$. The probability of choosing a vowel is $\frac{3}{6} \cdot 0.4 + \frac{2}{5} \cdot 0.6 = 0.44$. Therefore, the probability that the writer was English is $0.2/0.44 = 5/11$.

11. A rational number $3.abc$ when rounded to the nearest tenth is 3.5. If $a+b+c = 8$ and $a > b > c > 0$, what is the value of $2a + bc$?

    (A) 17 **(B)** 14 **(C)** 12 **(D)** 11 **(E)** 10

    Solution: (C). Because $3.abc$ rounds to 3.5, $a = 4$ or $a = 5$. Since $b < a$, we see that $a = 5$. The conditions $a + b + c = 8$ and $a > b > c > 0$ force $b = 2$ and $c = 1$. Thus, $2a + bc = 12$.

12. A farmer has 200 yards of fencing material. What is the largest rectangular area he can enclose if he wants to use a 4 yard wide gate that does not need to be covered by the fencing material?

    (A) $2,500$ square yards **(B)** $2,601$ square yards **(C)** $2,704$ square yards **(D)** $2,809$ square yards **(E)** $2,916$ square yards

    Solution: (B). We may add the length of the gate to the length of the fencing material and then we have 204 yards of fencing material. If the width of the enclosed area is x yards then the length is $(204 - 2x)/2 = 102 - x$ yards. The enclosed area is $x(102 - x) = 102x - x^2 = 51^2 - (x - 51)^2$ square yards. The largest area of $51^2 = 2601$ is achieved when the length and the width are both 51 yards.

13. Find the value of $b^{-\log_b(a)}$.

    (A) $\dfrac{b}{a}$ **(B)** $\dfrac{a}{b}$ **(C)** b **(D)** $\dfrac{1}{a}$ **(E)** $\dfrac{1}{b}$

    Solution: (D). Let $x = b^{-\log_b(a)}$. Taking the log to the base b of both sides, we have $\log_b(x) = -\log_b(a) = \log_b(a^{-1})$. Therefore, $x = a^{-1}$.

14. Find the real numbers m, n such that $m + n = 3$, and $m^3 + n^3 = 117$. What is the value of $m^2 + n^2$?

    (A) 29 **(B)** 3 **(C)** 9 **(D)** 17 **(E)** 45

    Solution: (A). Factor the second equation and use the first equation. We have $m^3 + n^3 = (m+n)(m^2 - mn + n^2) = 3(m^2 - mn + n^2) = 117$. Therefore $39 = m^2 - mn + n^2 = (m+n)^2 - 3mn = 9 - 3mn$. Thus, $mn = -10$. Substituting this into $m^2 - mn + n^2 = 39$ we find that $m^2 + n^2 = 29$.

15. A taxicab driver charges \$3.00 for the first half mile or less and \$0.75 for each quarter mile after that. If f represents the fare in dollars, which of the following functions models the fare for a ride of x miles, where x is a positive integer?

    (A) $f(x) = 3.00 + 0.75(x - 1)$ **(B)** $f(x) = 3.00 + 0.75(x/4 - 1)$

    (C) $f(x) = 4.50 + 3(x - 1)$ **(D)** $f(x) = 3.00 + 3(x - 1)$

    (E) $f(x) = 3.00(x + 1)$

    Solution: (C). The first mile costs \$4.50, and every mile thereafter costs \$3.00, so item C. is the right choice.

16. The cells of an infinite chess board are labeled with two integers each (the number of the column and that of the row). A child rolls a very small ball that starts from the center of the cell (8,11). There are tiny bugs at the centers of some cells. The ball rolls along a straight line; it hits a bug if and only if it rolls exactly through the center of a cell containing a bug. This does not affect the subsequent motion of the ball. It is known that the ball has hit a bug at the center of the cell (20,24). The other bugs are at the centers of the cells listed below. Which of them will be hit?

    (A) $(68, 75)$ **(B)** $(69, 76)$ **(C)** $(67, 75)$ **(D)** $(69, 77)$ **(E)** $(68, 76)$

    Solution: (E). The bug at the center of the cell (a, b) is hit if and only if the centers of the cells (8,11), (20,24) and (a, b) are on the same line, that is, $\frac{a-8}{20-8} = \frac{b-11}{24-11}$. This is true only for the last pair ($a = 68$, $b = 76$).

17. Football teams score $1, 2, 3$,or 6 points at a time. They can score 1 point (point-after-touchdown) only immediately after scoring 6 points (a touchdown). A scoring sequence is a sequence of numbers $1, 2, 3, 6$, where all the 1's are immediately preceded by 6. Both $2, 6, 1, 3, 2$ and $3, 3, 3, 3, 2$ are scoring sequences with *value* 14. How many scoring sequences have value 10?

(A) 12 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 18

Solution: (A). There is either one touchdown or none. With 1 touchdown the scoring sequences with value 10 are $613, 361, 622, 262, 226$. With no touchdowns there is 3322 and its permutations, of which there are 6 total, and 22222 So the total number of sequences with value 10 is $5 + 6 + 1 = 12$.

18. Let $r = 11 \cdot \sqrt{10!}, s = 10 \cdot \sqrt{11!}, t = \sqrt{12!}$. Rank the numbers r, s, t from smallest to largest.

(A) $r < t < s$ **(B)** $r < s < t$ **(C)** $t < r < s$ **(D)** $t < s < r$ **(E)** $s < t < r$

Solution: (A). $r = 11\sqrt{10!}$, $t = \sqrt{12 \cdot 11}\sqrt{10!}$ and $s = 10\sqrt{11}\sqrt{10!}$, so $r < t < s$.

19. The integers $3, 4, 5, 6, 12$, and 13 are arranged, without repetition, in a horizontal row so that the sum of any two numbers in adjoining positions is a perfect square (the square of an integer). Find the sum of the first and last.

(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 17

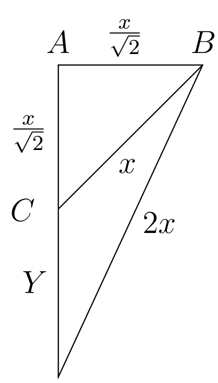
Solution: (C). Build a graph with the six vertices labeled $3, 4, 5, 6, 12$, and 13. Join two when their sum is a perfect square. Note that 5 and 6 are related to only 4 and 3 respectively, so they must be at the ends. The sequence 6 3 13 12 4 5 works.

20. You are walking across a very large field with no obstructions when you see your friend, Sandy, Y yards to your north. Sandy is walking at 1 yard per second toward the north-east. If you walk at a steady speed of 2 yards per second and Sandy does not alter her speed or direction, what is the shortest distance (measured in yards) you must walk before you catch up with her?

(A) $2Y\left(\frac{\sqrt{3}}{2}\right)$ **(B)** $2Y\left(\frac{\sqrt{3}+\sqrt{8}}{6}\right)$ **(C)** $2Y\left(\frac{1+\sqrt{3}}{3}\right)$

(D) $2Y\left(\frac{\sqrt{2}+\sqrt{14}}{6}\right)$ **(E)** $2Y\left(\frac{\sqrt{2}+\sqrt{15}}{6}\right)$

Solution: (D). To keep the distance as short as possible, you must walk in a straight line. Because of your relative speeds, when you intercept Sandy, you must have walked twice as far as she has. In the diagram below, you walk $2x$ yards while Sandy walks x yards.



Triangle ABC is an isosceles right triangle with hypotenuse of length x. Using the Pythagorean theorem we find that $\left(Y+\frac{x}{\sqrt{2}}\right)^2+\left(\frac{x}{\sqrt{2}}\right)^2=(2x)^2$. This is equivalent to $3x^2-\sqrt{2}xY-Y^2=0$. Using the quadratic formula and discarding the negative root, we find that $x=Y\left(\frac{\sqrt{2}+\sqrt{14}}{6}\right)$. This problem can also be solved using the law of cosines.

21. Let n be the product of four consecutive positive integers. How many of the numbers 4, 8, 10, 12, and 15 must be divisors of n?

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

    Solution: (C). Two of the four consecutive integers must be even and at least one of these must be divisible by 4. Also, one of the four must be divisible by 3. Therefore, 4, 8, and 12 must be divisors of the product. Since 10 and 15 are not divisors of $1 \cdot 2 \cdot 3 \cdot 4$, the correct answer is 3.

22. The inequality $|x+3| \leq 2$ is equivalent to the inequality $a \leq \frac{6}{x+7} \leq b$. Find the value of $a^2 + b$.

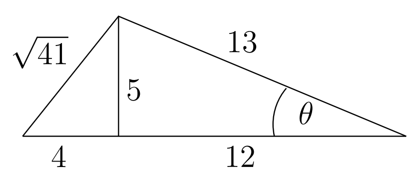
    (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

    Solution: (B). $|x+3| \leq 2$ is equivalent to $-2 \leq x+3 \leq 2$. Adding 4 to each part of this inequality gives $2 \leq x+7 \leq 6$. Therefore, $\frac{1}{6} \leq \frac{1}{x+7} \leq \frac{1}{2}$ or $1 \leq \frac{6}{x+7} \leq 3$. Thus, $a = 1$ and $b = 3$.

23. What is the area of a triangle with sides of length 13, 16, and $\sqrt{41}$?

    (A) 36 **(B)** 38 **(C)** 40 **(D)** 42 **(E)** 44

    Solution: (C).



    Use the diagram above or use the law of cosines: $41 = 16^2 + 13^2 - 2 \cdot 16 \cdot 13 \cos(\theta)$ to find $\cos(\theta) = \frac{12}{13}$. Since $\sin(\theta) = \sqrt{1 - \cos^2(\theta)} = \frac{5}{13}$. the area is $\frac{1}{2} \cdot 16 \cdot 5 = 40$.

24. Suppose that $f(x) = ax + b$, where a and b are real numbers. Given that $f(f(f(x))) = 8x + 21$, what is the value of $a + b$?

    (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

    Solution: (D). $f(f(f(x))) = a^3x + a^2b + ab + b = 8x + 21$ so $a = 2$ and $7b = 21$. Therefore $a + b = 5$.

25. For how many integers n is the value of $\frac{n}{50-n}$ the square of an integer?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Solution: (E). Let $\frac{n}{50-n} = k^2$. Then $n = \frac{50k^2}{k^2+1}$. Because n is an integer and k^2+1 does not divide k^2 unless $k=0$, we see that k^2+1 must divide $50 = 5 \cdot 5 \cdot 2$. This happens when $k = 0, 1, 2, 3$ or 7.