1. Suppose f, g and h are polynomials of degrees $7, 8$ and 9 respectively. Let d be the degree of the quotient $(f \circ g) \cdot h \div (f + g + h)$, where $\circ$ means composition. Which of the following statements is true?

   (A) $d \leq 40$ (B) $40 < d \leq 50$ (C) $50 < d \leq 60$ (D) $60 < d \leq 70$ (E) $d > 70$

   Answer: C.
   Solution. The degree of the numerator is $7 \cdot 8 + 9$ and the degree of the denominator is 9, so the quotient has degree $65 - 9 = 56$.

2. A ball of radius 1 is rolling on the floor $Q: -3 \leq x \leq 3,\ -3 \leq y \leq 3$ of a square room, getting reflected elastically off its walls. Otherwise, it moves straight. Initially, it touches the Eastern wall of the room at the point $(3, 0)$, then moves in the North-West direction and hits the Northern wall at the point $(0, 3)$, then gets reflected and moves South-West, etc. Let L be the distance the ball travels before returning to the initial position. Which of the following statements is correct?

   (A) $11 < L \leq 12$ (B) $12 < L \leq 14$ (C) $14 < L \leq 15$ (D) $15 < L \leq 16$ (E) $16 < L \leq 18$

   Answer: A.
   Solution. Since the radius of the ball is 1, its center is moving in a smaller square $S: -2 \leq x \leq 2,\ -2 \leq y \leq 2$ (lifted to the height of 1 above the floor), getting reflected off its sides. Upon leaving the point $(2, 0)$, it attains the boundary of the square S successively at the points $(0, 2),\ (-2, 0), (0, -2)$, and then returns to the initial point $(2, 0)$ having traveled a distance of $L = (4)2\sqrt{2} = 8\sqrt{2}$. We have $11 < L < 12$ since $L^2 = 128$ is between 11^2 and 12^2.

3. There are exactly 2 positive values of r for which the system of equations

$$\begin{cases} x^2 + y^2 & = 9, \\ (x-8)^2 + (y-6)^2 & = r^2 \end{cases}$$

   has a unique solution (x, y). Let us denote these values of r by r_1 and r_2, and put $s = r_1 + r_2$. Which of the following statements is true?

   (A) $s \leq 10$ (B) $10 < s \leq 17$ (C) $17 < s \leq 21$ (D) $21 < s \leq 26$ (E) $s > 26$

   Answer: C.
   Solution. The first equation of the system represents the circle U of radius 3 centered at the origin $(0, 0)$. The second equation represents the circle V_r of radius r centered at the point $(8, 6)$ whose distance from the origin equals 10. The uniqueness of the solution of the system means that the two circles are tangent to one another. This occurs for two values of r: $r = 10 - 3 = 7$ (V_r is tangent to U from outside) and $r = 10 + 3 = 13$ (U is tangent to V_r from inside). Hence $s = 7 + 13 = 20$.

4. What is the area of the region determined by $|x - 1| + |y - 1| \leq 2$?

   (A) 2 (B) 4 (C) $4\sqrt{2}$ (D) 8 (E) $8\sqrt{2}$

   Answer: D.
   Solution. Translate the center of the region from $(1, 1)$ to the origin and note that the resulting region described by $|x| + |y| \leq 2$ is a square with vertices $(0, \pm 2), (\pm 2, 0)$, so its area is 8.

5. Six points are distributed around a circle. In how many ways can you build two disjoint (i.e, non-intersecting) triangles using the six points as vertices?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) more than 4

   Answer: C.
   Solution. Number the points from 1 to 6 in clockwise fashion. Once 1's two vertex partners are selected, the figure is determined. But note that 1's partners can only be 2,3; 6,2; or 5,6.

6. The top of a rectangular box has area 40 square inches, the front has area 48 square inches, and the side has area 30 square inches. How high is the box?

   (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

   Answer: E.
   Solution. Let x, y, z be the width, depth, and height of the box, respectively. Then $xy = 40$, $xz = 48$ and $yz = 30$. Hence

   $$x^2y^2z^2 = (xy)(xz)(yz) = 40 \cdot 48 \cdot 30 = 57600.$$

   Therefore, $xyz = 240$, and $z = (xyz)/(xy) = 240/40 = 6$.

7. The lower two vertices of a square lie on the x-axis and the upper two vertices of the square lie on the parabola $y = 15 - x^2$. What is the area of the square?

   (A) 9 (B) $10\sqrt{2}$ (C) 16 (D) 25 (E) 36

   Answer: E.
   Solution. The parabola, and hence the whole picture, is symmetric with respect to the y-axis. Therefore, the midpoint of the bottom side of the square is at the origin, so that the upper right vertex (x, y) of the square satisfies two equations $y = 2x$ and $y = 15 - x^2$. They imply that $2x = 15 - x^2$. This quadratic equation has a unique nonnegative root $x = 3$. Therefore, the sides of the square have length $2 \cdot 3 = 6$.

8. A math class has between 15 and 40 students. Exactly 25% of the class knows how to play poker. On a certain Wednesday, 3 students were absent (because they were participating in a math contest). On that day, exactly 20% of the students attending the class knew how to play poker. How many students attending the class on that day knew how to play poker?

   (A) 3 (B) 4 (C) 5 (D) 6 (E) 7

   Answer: C.
   Solution. Suppose there are usually s students in the math class. Since exactly $1/4$ know how to play poker, s must be a multiple of 4. Similarly, $s - 3$ must be a multiple of 5. The only multiple of 4 between 15 and 40 that is 3 more than a multiple of 5 is $s = 28$. On Wednesday, there were $28 - 3 = 25$ students, and 5 knew how to play poker.

9. Pansies have five petals and lilacs have four petals. A bouquet has twenty flowers with a total of 92 petals. How many pansies does the bouquet have?

   (A) $3 \le P \le 7$ (B) $8 \le P \le 10$ (C) $11 \le P \le 14$ (D) $15 \le P \le 17$ (E) $P \ge 18$

   Answer: C.
   Solution. If all the 20 flowers were lilacs, then there would be a total of $20 \cdot 4 = 80$ petals. Hence some lilacs should be replaced with pansies. How many? Each such replacement adds one more petal. Therefore, $92 - 80 = 12$ lilacs should be replaced with pansies.

10. A dictionary contains pages numbered 1 through 852. How many times does the number 8 appear as a digit in this numbering?

    (A) 85 (B) 146 (C) 165 (D) 217 (E) 218

    Answer: E.
    Solution. The number 8 appears in the ones place 85 times $(8, 18, 28, \ldots, 848)$, in the tens place 80 times $(80. \ldots, 89;\ 180, \ldots, 189;\ 280. \ldots, 289; \ldots; 780, \ldots, 789)$, and in the hundreds place 53 times $(800, 801, \ldots, 852)$. Hence it appears a total of $85 + 80 + 53 = 218$ times.

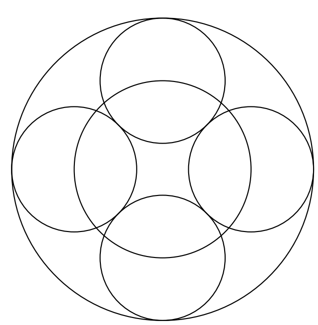
11. Two married couples have purchased theater tickets and are seated in a row consisting of just four seats. If they take their seats in a completely random order, what is the probability that Jim and Paula (husband and wife) sit in the two seats on the far left?

    (A) $\dfrac{1}{12}$ (B) $\dfrac{1}{24}$ (C) $\dfrac{1}{4}$ (D) $\dfrac{1}{2}$ (E) $\dfrac{1}{6}$

    Answer: E.
    Solution. The number of ways four people can sit in a row is $4! = 24$. The number of favorable outcomes is four: (two possibilities for Jim and Paula) $\times$ (two ways to seat the second couple). Thus, the probability sought for equals $4/24 = 1/6$.

12. Inside a large circle of radius 10 are four small circles and one medium size circle as in the diagram below. The four small circles all have the same radius and each is tangent to the large circle and to two other small circles. The center of the medium circle is the same as the center of the large circle and the medium circle passes through the center of each small circle. What is the radius of the medium circle?

    (A) $10\left(2 - \sqrt{2}\right)$
    (B) $5\sqrt{2}$
    (C) $3\sqrt{5}$
    (D) $10\left(\sqrt{3} - 1\right)$
    (E) $2\left(2 + \sqrt{2}\right)$



Answer: A.
Solution. Let s be the radius of each small circle and let m be the radius of the medium size circle. We have $m + s = 10$. The centers of the four small circles are the vertices of a square with side length $2s$ and diagonal $2m$. So we have $s = m/\sqrt{2}$. Thus

$$m = \frac{10}{1 + 1/\sqrt{2}} = \frac{20}{2 + \sqrt{2}} = \frac{20(2 - \sqrt{2})}{4 - 2} = 10(2 - \sqrt{2}).$$

13. Suppose N is the size of a certain caterpillar population in the UNC Charlotte botanical gardens, M is the maximum possible population size (the carrying capacity), and $x = N/M$ is the relative size of the population. Scientists have observed that if the population one spring has relative size x, then the following spring the population has relative size $4x(1 - x)$. There is at most one value x_0 ($0 < x_0 < 1$) with the property that if the relative size one spring is x_0, then the relative size for all following springs will be x_0 as well. Which of the following statements is true?

    (A) $0 < x_0 \le 1/2$ (B) $1/2 < x_0 \le 3/5$ (C) $3/5 < x_0 \le 7/8$ (D) $7/8 < x_0 < 1$ (E) no such x_0 exists

    Answer: C.
    Solution. Set $x = 4x(1 - x)$, and solve to find $x = 3/4$.

14. Suppose a, b, and c are positive integers with $a < b < c$ such that $1/a + 1/b + 1/c = 1$. What is $a + b + c$?

    (A) 7 (B) 8 (C) 9 (D) 11 (E) no such integers exist

    Answer: D.
    Solution. First note that $a = 1$ is impossible since this would imply that $1/a + 1/b + 1/c > 1$. Similarly, $a \ge 3$ would imply that $1/a+1/b+1/c \le 1/3+1/4+1/5 < 1$, which is not possible either. Therefore, $a = 2$. Now we need to find integers b and c such that $3 \le b < c$ and $1/b + 1/c = 1/2$. Again, $b \ge 4$ is ruled out since in this case $1/b + 1/c \le 1/4 + 1/5 < 1/2$. Therefore, $b = 3$ and hence $c = 6$, so that $a + b + c = 2 + 3 + 6 = 11$.

15. One of the roots of the quadratic equation $x^2 - 9x + a = 0$ is twice the other root. Which of the following statements is true?

    (A) $a \le 5$ (B) $15 < a \le 10$ (C) $10 < a \le 15$ (D) $15 < a \le 20$ (E) $a > 20$

    Answer: D.
    Solution. Let the roots of the quadratic equation be u and $2u$. Then the equation can be rewritten in the form $(x - u)(x - 2u) = 0$, or, equivalently, $x^2 - 3ux + 2u^2 = 0$. It follows that $3u = 9$, so that $u = 3$ and hence $a = 2 \cdot 3^2 = 18$.

16. In the quadratic equation $x^2 - 7x + a = 0$ the sum of the squares of the roots equals 39. Find a.

    (A) 8 (B) 7 (C) 6 (D) 5 (E) 4

    Answer: D.
    Solution. Let x_1 and x_2 be the roots of the equation. The latter can be rewritten as $(x-x_1)(x-x_2) = 0$ so that $x_1 + x_2 = 7$ and $x_1x_2 = a$. Then $x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = 7^2 - 2a$. It follows that $49 - 2a = 39$ and hence $a = (49 - 39)/2 = 5$.

17. The sides of a right triangle form an arithmetic sequence, while their sum equals 48. Find the area of the triangle.

    (A) 24 (B) 96 (C) 48 (D) 54 (E) 84

    Answer: B.
    Solution. Let $a < b < c$ be the lengths of the legs and the hypotenuse of the triangle. Then $c - b = b - a = h$ for some $h > 0$. By the Pythagorean theorem, $(b - h)^2 + b^2 = (b + h)^2$ or $(b+h)^2-(b-h)^2 = b^2$. Then $4bh = b^2$. It follows that $h = b/4$. The equality $(b-h)+b+(b+h) = 48$ implies that $b = 16$ so that $h = 4$ and the lengths of the legs are 12 and 16. Therefore, the area of the triangle equals $(1/2)12 \cdot 16 = 96$.

18. The ratio of the legs in a right triangle equals $3/2$, while the length of the hypotenuse is $\sqrt{52}$. Find the area of the triangle.

    (A) 12 (B) 13 (C) 26 (D) 30 (E) 169

    Answer: A.
    Solution. The lengths of the legs of the triangle are a and $\left(\frac{3}{2}\right)a$ for some $a > 0$. By the Pythagorean theorem, $a^2 + \left(\left(\frac{3}{2}\right)a\right)^2 = 52$, so that $\left(\frac{13}{4}\right)a^2 = 52$. It follows that the legs have lengths $a = 4$ and $\left(\frac{3}{2}\right)4 = 6$. Therefore, the area of the triangle is $\left(\frac{1}{2}\right)6 \cdot 4 = 12$.

19. Fresh cucumbers are 90% water. It is known that in a week after they are picked the amount of water reduces to 80%. How much will 20 pounds of such cucumbers weigh after a week?

    (A) 8 (B) 10 (C) 12 (D) 15 (E) 18

    Answer: B.
    Solution. The amount of non-watery part in 20 pounds of fresh cucumbers is $20 \cdot 0.1 = 2$ pounds. The same amount after a week makes 20% or $1/5$ of the total weight. Hence, the weight is $2 \cdot 5 = 10$.

20. For how many pairs of digits (a, b) does $\sqrt{0.aaaaa\ldots} = 0.bbbbb\ldots$?

    (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

    Answer: D.
    Solution. We rewrite the equation as $\sqrt{a/9} = b/9$ or $9a = b^2$. This equation is satisfied by 4 pairs of digits: $(0, 0)$, $(1, 3)$, $(4, 6)$, $(9, 9)$.