1. How many points (x, y) in the plane satisfy both $x^2 + y^2 = 25$ and $x^2 - 10x + y^2 - 24y = -105$?

   (A) none (B) 1 (C) 2 (D) 3 (E) more than 3

   Answer: B.
   Solution. The second equation, upon completing the squares, becomes $(x - 5)^2 + (y - 12)^2 = 64$. Therefore, both equations represent circles. Their centers are 13 units apart and the sum of their radii is also 13.

2. Let $f(x) = (2x+3)^3$ and $g(x) = x^3+x^2-x-1$. Denote the sum of the coefficients of the polynomial $h(x) = f(g(x))$ by s. Which of the following statements is true?

   (A) $s \le 0$ (B) $1 \le s \le 6$ (C) $7 \le s \le 20$ (D) $21 \le s \le 36$ (E) $s > 36$

   Answer: D.
   Solution. The sum of the coefficients of any polynomial $p(x)$ equals $p(1)$. Therefore, $s = h(1) = f(g(1))$. We have $g(1) = 0$ and hence $h(1) = f(0) = 27$.

3. The graph of the function $f(x) = ||2x| - 10|$ on the interval $[-10, 10]$ looks like

   (A) M (B) W (C) V (D) Λ (E) none of these

   Answer: B.
   Solution. The graph of the function $g(x) = |2x|$ on $[-10, 10]$ is a letter "V" with vertices at the points $(-10, 20)$, $(0, 0)$ and $(10, 20)$. The graph of $h(x) = g(x) - 10$ is the graph of $g(x)$ shifted downward by 10 units – the half of its height. The graph of $f(x) = |h(x)|$ is obtained from the graph of $g(x)$ by reflecting its part lying below the x-axis (a smaller "V") across the x-axis, thus creating a letter "W".

4. There is a unique positive number r such that the two equations $y+2x = 0$ and $(x-3)^2+(y-6)^2 = r^2$ have exactly one simultaneous solution. Which of the following statements is true?

   (A) $0 < r < 1$ (B) $1 \le r < 3$ (C) $3 \le r < 5$ (D) $5 \le r < 6$ (E) $r \ge 6$

   Answer: D.
   Solution. The first equation $y + 2x = 0$ describes a line, while the second equation $(x - 3)^2 + (y - 6)^2 = r^2$ describes a circle of radius r centered at the point $(3, 6)$. The uniqueness of the simultaneous solution of the two equations means that the circle is tangent to the line, that is, r is the shortest distance from the circle's center to the line. To calculate r, consider the triangle with vertices $P(3, 6)$, $Q(-3, 6)$ and $O(0, 0)$. Considering the sides PQ or OQ as its bases, we obtain two expressions for the triangle's area which are, therefore, equal: $(1/2)6 \cdot 6 = (1/2)\sqrt{6^2 + 3^2} \cdot r$. Consequently, $r = (6\cdot 6)/\sqrt{45} = 12/\sqrt{5}$. It follows that $5 < r < 6$ since $r^2 = \frac{144}{5} = 28\frac{4}{5}$ is between 5^2 and 6^2.

5. The vertices of a triangle are the centers of the circles $C_1 = \{(x, y) \mid x^2 + y^2 = 1\}$, $C_2 = \{(x, y) \mid (x-4)^2 + y^2 = 1\}$ and $C_3 = \{(x, y) \mid x^2 - 14x + y^2 - 16y = 0\}$. Let S be the area of the triangle. Which of the following statements is true?

   (A) $S \le 6$ (B) $6 < S \le 9$ (C) $9 < S \le 12$ (D) $12 < S \le 15$ (E) $S > 15$

   Answer: E.
   Solution. The centers are $(0, 0)$ $(4, 0)$ and $(7, 8)$, so the area of the triangle is $\frac{1}{2}(4)(8) = 16$

6. How many real solutions does the following system have?

$$\begin{cases} x + y & = 2, \\ xy - z^2 & = 1. \end{cases}$$

   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

   Answer: B.
   Solution. The second equation implies that $xy = z^2 + 1 \ge 1 > 0$, so that x and y are of the same sign. It follows then from the first equation that they are positive. Hence the Arithmetic Mean – Geometric Mean Inequality applies, which says that $(x + y)/2 \ge \sqrt{xy}$. In our case, $(x + y)/2 = 1$ while $\sqrt{xy} \ge 1$ so the inequality turns into equality; in this case it implies that $x = y$. Therefore, $x = y = 1$ and $z = 0$.

7. Let $a > 1$. How many positive solutions has the equation

$$\sqrt{a - \sqrt{a + x}} = x\ ?$$

   (A) 1 (B) 2 (C) 0 (D) 3 (E) 4

   Answer: A.
   Solution. The numbers $x \ge 0$ for which the function $f(x) = \sqrt{a - \sqrt{a + x}} - x$ is defined form an interval $I = [0, b]$, where $b = a^2 - a$. In this interval the function $f(x)$ is continuous and decreasing from $\sqrt{a - \sqrt{a}} > 0$ to $-b < 0$. Therefore, there is exactly one value x_0 in this interval such that $f(x_0) = 0$. Since $f(0) > 0$ and $f(b) < 0$, we have $0 < x_0 < b$, so that x_0 is positive as required.

8. The top of a rectangular box has area 40 square inches, the front has area 48 square inches, and the side has area 30 square inches. How high is the box?

   (A) 3 (B) 4 (C) 5 (D) 6 (E) 8

   Answer: D.
   Solution. Let x, y, z be the width, depth, and height of the box, respectively. Then $xy = 40$, $xz = 48$ and $yz = 30$. Hence

$$x^2y^2z^2 = (xy)(xz)(yz) = 40 \cdot 48 \cdot 30 = 57600.$$

   Therefore, $xyz = 240$, and $z = (xyz)/(xy) = 240/40 = 6$.

9. The lower two vertices of a square lie on the x-axis, while the upper two vertices of the square lie on the parabola $y = 15 - x^2$. What is the area of the square?

   (A) 9 (B) $10\sqrt{2}$ (C) 16 (D) 25 (E) 36

   Answer: E.
   Solution. The parabola, and hence the whole picture, is symmetric with respect to the y-axis. Therefore, the midpoint of the bottom side of the square is at the origin, so that the upper right vertex (x, y) of the square satisfies two equations $y = 2x$ and $y = 15 - x^2$. They imply that $2x = 15 - x^2$. This quadratic equation has one nonnegative root $x = 3$. Therefore, the sides of the square have length $2 \cdot 3 = 6$.

10. Pansies have 5 petals while lilacs have 4 petals. A bouquet has 20 flowers with a total of 92 petals. Let P be the number of pansies in the bouquet. Which of the following statements does P satisfy?

    (A) $3 \le P \le 7$ (B) $8 \le P \le 10$ (C) $11 \le P \le 14$ (D) $15 \le P \le 17$ (E) $P \ge 18$

    Answer: C.
    Solution. If all the 20 flowers were lilacs, then there would be a total of $20 \cdot 4 = 80$ petals. Hence some lilacs should be replaced with pansies. How many? Each such replacement adds one more petal. Therefore, $92 - 80 = 12$ lilacs should be replaced with pansies.

11. A three-digit number abc is *palindromic* if $a = c$. What is the number of distinct three-digit palindromic numbers?

    (A) 72 (B) 84 (C) 88 (D) 90 (E) 100

    Answer: D.
    Solution. The pair of equal digits $a = c$ can be selected in 9 ways $(1, 2, \ldots, 9)$ while, for any such selection, b can be chosen in 10 different ways $(0, 1, 2, \ldots, 9)$. Hence the number sought for is $9 \cdot 10 = 90$.

12. The double of a positive number is the triple of its cube. The number is:

    (A) $\sqrt{2/3}$ (B) 1 (C) $\sqrt{3/2}$ (D) $\sqrt[3]{2}/\sqrt{3}$ (E) $\sqrt[3]{3}/\sqrt{2}$

    Answer: A.
    Solution. Let x be the number sought for. Then $2x = 3x^3$, or, equivalently, $x(2 - 3x^2) = 0$. Since $x > 0$, it follows that $x^2 = 2/3$ so that $x = \sqrt{2/3}$.

13. Suppose a, b and c are positive integers with $a < b < c$ such that $1/a + 1/b + 1/c = 1$. What is $a + b + c$?

    (A) 6 (B) 8 (C) 9 (D) 11 (E) no such integers exist

    Answer: D.
    Solution. First note that $a = 1$ is impossible since this would imply that $1/a + 1/b + 1/c > 1$. Similarly, $a \ge 3$ would imply that $1/a+1/b+1/c \le 1/3+1/4+1/5 < 1$, which is not possible either. Therefore, $a = 2$. Now we need to find integers b and c such that $3 \le b < c$ and $1/b + 1/c = 1/2$. Again, $b \ge 4$ is ruled out since in this case $1/b + 1/c \le 1/4 + 1/5 < 1/2$. Therefore, $b = 3$ and hence $c = 6$, so that $a + b + c = 2 + 3 + 6 = 11$.

14. A quadratic equation $x^2 - 9x + a = 0$ has two distinct roots, one of them being twice the other. Which of the following statements is true?

(A) $a \le 5$ (B) $5 < a \le 10$ (C) $10 < a \le 15$ (D) $15 < a \le 20$ (E) $a > 20$

Answer: D.
Solution. Let the roots of the quadratic equation be u and $2u$. Then the equation can be rewritten in the form $(x-u)(x-2u) = 0$, or, equivalently, $x^2 - 3ux + 2u^2 = 0$. It follows that $3u = 9$, so that $u = 3$ and hence $a = 2 \cdot 3^2 = 18$.

15. In the quadratic equation $x^2 - 7x + a = 0$ the sum of the squares of the roots equals 39. Find a.

(A) 8 (B) 7 (C) 6 (D) 5 (E) 4

Answer: D.
Solution. Let x_1 and x_2 be the roots of the equation. The latter can be rewritten as $(x-x_1)(x-x_2) = 0$ so that $x_1 + x_2 = 7$ and $x_1 x_2 = a$. Then $x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = 7^2 - 2a$. It follows that $49 - 2a = 39$ and hence $a = (49 - 39)/2 = 5$.

16. Evaluate $S = \cot 1^\circ \cot 2^\circ \cot 3^\circ \ldots \cot 89^\circ$.

(A) $\frac{\pi}{2}$ (B) $\frac{2}{\pi}$ (C) 1 (D) $\frac{\sqrt{2}}{2}$ (E) 2

Answer: C.
Solution. We have
$S = \cot 1^\circ \ \cot 2^\circ \ \cot 3^\circ \ \ldots \cot 89^\circ$ and
$S = \cot 89^\circ \cot 88^\circ \cot 87^\circ \ldots \cot 1^\circ$.
Note that $\cot(90^\circ - x^\circ) = \tan x^\circ$ and hence $\cot(90^\circ - x^\circ) \cdot \cot x^\circ = 1$. It follows that
$S^2 = (\cot 1^\circ \cot 89^\circ)(\cot 2^\circ \cot 88^\circ) \ldots (\cot 89^\circ \cot 1^\circ) = 1$. Since $S > 0$, we also have $S = 1$.

17. The sides of a right triangle form an arithmetic sequence, while their sum equals 48. Find the area of the triangle.

(A) 24 (B) 96 (C) 48 (D) 54 (E) 84

Answer: B.
Solution. Let $a < b < c$ be the lengths of the legs and the hypotenuse of the triangle. Then $c - b = b - a = h$ for some $h > 0$. By the Pythagorean theorem, $(b-h)^2 + b^2 = (b+h)^2$ or $(b+h)^2 - (b-h)^2 = b^2$. Then $4bh = b^2$. It follows that $h = b/4$. The equality $(b-h)+b+(b+h) = 48$ implies that $b = 16$ so that $h = 4$ and the lengths of the legs are 12 and 16. Therefore, the area of the triangle equals $(1/2)12 \cdot 16 = 96$.

18. The ratio of the legs in a right triangle equals $3/2$, while the length of the hypotenuse is $\sqrt{52}$. Find the area of the triangle.

    (A) 12 (B) 13 (C) 26 (D) 30 (E) 169

    Answer: A.
    Solution. The lengths of the legs of the triangle are a and $\left(\frac{3}{2}\right)a$ for some $a > 0$.

    By the Pythagorean theorem, $a^2 + \left(\left(\frac{3}{2}\right)a\right)^2 = 52$, so that $\left(\frac{13}{4}\right)a^2 = 52$. It follows that the legs have lengths $a = 4$ and $\left(\frac{3}{2}\right)a = 6$. Therefore, the area of the triangle is $\left(\frac{1}{2}\right)6 \cdot 4 = 12$.

19. The Chebyshev polynomial of the first kind of order n is defined by $T_n(\cos\alpha) = \cos n\alpha$, so that $T_0(\cos\alpha) = 1$ and hence $T_0(x) = 1$; $T_1(\cos\alpha) = \cos\alpha$, hence $T_1(x) = x$; $T_2(\cos\alpha) = \cos 2\alpha = 2\cos^2\alpha - 1$ so that $T_2(x) = 2x^2 - 1$, etc. What is the value of $T_{10}(\sin\alpha)$?

    (A) $\cos 10\alpha$ (B) $\sin 10\alpha$ (C) $-\sin 10\alpha$ (D) $-\cos 10\alpha$ (E) $\frac{\sin 10\alpha}{\cos\alpha}$

    Answer: D.
    Solution. We have $\sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right)$ so that

    $$T_{10}(\sin\alpha) = T_{10}\left(\cos\left(\frac{\pi}{2} - \alpha\right)\right) = \cos\left(10\left(\frac{\pi}{2} - \alpha\right)\right)$$

    $$= \cos(5\pi - 10\alpha) = -\cos(-10\alpha) = -\cos 10\alpha.$$

20. Find the maximum value of the expression $f(x, y) = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$ over the square Q : $-1 \le x \le 1,\ -1 \le y \le 1$.

    (A) 1 (B) $\sqrt{\frac{3}{2}}$ (C) $\frac{3}{2}$ (D) $\frac{2}{\sqrt{3}}$ (E) $\frac{4}{3}$

    Answer: A.
    Solution. Let $u = \sqrt{1 - x^2}$ and $v = \sqrt{1 - y^2}$. Using the Arithmetic Mean – Geometric Mean Inequality, we obtain:

    $$f(x, y) = xv + yu \le \left(\frac{1}{2}\right)(x^2 + v^2) + \left(\frac{1}{2}\right)(y^2 + u^2) = \left(\frac{1}{2}\right)(x^2 + u^2 + y^2 + v^2) = \left(\frac{1}{2}\right)(1 + 1) = 1.$$

    Therefore, $f(x, y) \le 1$ for all x, y in the square Q. On the other hand, $f(1, 0) = 1$.