UNIVERSITY OF NORTH CAROLINA CHARLOTTE
1996 HIGH SCHOOL MATHEMATICS CONTEST
March 4, 1996

1. If

$$f(x,y) = (\max(x,y))^{min(x,y)}$$

and

$$g(x,y) = \max(x,y) - \min(x,y),$$

then

$$f(g(-1,-\frac{3}{2}), g(-4,-1.75)) =$$

(A) -0.5 **(B)** 0 **(C)** 0.5 **(D)** 1 **(E)** 1.5

(E) First, evaluate the two occurences of g, and then take f of the results.

$$\begin{aligned} g\left(-1,-\frac{3}{2}\right) &= \max\left(-1,-\frac{3}{2}\right) - \min\left(-1,-\frac{3}{2}\right) \\ &= -1-\left(-\frac{3}{2}\right) \\ &= \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} g(-4,-1.75) &= \max(-4,-1.75) - \min(-4,-1.75) \\ &= -1.75-(-4) \\ &= 2.25 \\ &= \frac{9}{4}. \end{aligned}$$

Then,

$$\begin{aligned} f\left(\frac{1}{2},\frac{9}{4}\right) &= \max\left(\frac{1}{2},\frac{9}{4}\right)^{\min(\frac{1}{2},\frac{9}{4})} \\ &= \left(\frac{9}{4}\right)^{\frac{1}{2}} \\ &= \frac{3}{2}. \end{aligned}$$

Notice that $g(x,y) = |x-y|$ is just the distance between the numbers x and y.

2. Which of the following statements below could be used to disprove "If p is a prime number, then p is three less than a multiple of four."

(A) Some even numbers are not prime.

(B) Not all odd numbers are prime.

(C) Seven is prime.

(D) Nine is not prime.

(E) Five is prime.

(C) We want a number which satisfies the hypothesis but fails the conclusion. Seven is such a number.

---

3. If $\log x + \log 5 = \log x^2 - \log 14$, then $x =$

(A) 2^{70} **(B)** 0 **(C)** 70 **(D)** either 0 or 70 **(E)** 70^2

(C) Note that $5x = x^2/14$, so $x = 70$. Even though 0 satisfies $5x = x^2/14$, it does not satisfy $\log x + \log 5 = \log x^2 - \log 14$ because $\log 0$ is undefined.

---

4. A line l_1 has a slope of -2 and passes through the point $(r, -3)$. A second line, l_2, is perpendicular to l_1, intersects l_1 at (a, b), and passes through the point $(6, r)$. The value of a is

(A) r **(B)** $\dfrac{2}{5}r$ **(C)** 1 **(D)** $2r - 3$ **(E)** $\dfrac{5}{2}r$

(B) An equation for l_1 is $y + 3 = -2(x - r)$, so $y = -2x + (2r - 3)$. An equation for l_2 is $y - r = \frac{1}{2}(x - 6)$, so $y = \frac{1}{2}x + (r - 3)$. Hence we have $-2x + 2r - 3 = \frac{1}{2}x + r - 3$, so $x = \frac{2}{5}r$.

---

5. What is the probability of obtaining an ace on both the first and second draws from a deck of cards when the first card is not replaced before the second is drawn?

(A) 1/13 **(B)** 1/17 **(C)** 1/221 **(D)** 30/221 **(E)** 4/221

(C) The probability of obtaining an ace on the first draw is $4/52 = 1/13$. If the first card drawn is an ace there are 3 aces remaining in the deck, which now consists of 51 cards. Thus, the probability of getting an ace on the second draw is $3/51 = 1/17$. The required probability is the product of the two, which is c) 1/221. The problem can be solved by dividing: $\frac{\binom{4}{2}}{\binom{52}{2}} = \frac{4!/2!2!}{52!/50!2!} = 1/221$.

---

6. If a, b, c, and d are nonzero real numbers, $\frac{a}{b} = \frac{c}{d}$, and $\frac{a}{d} = \frac{b}{c}$, then which one of the following must be true?

(A) $a = \pm b$ **(B)** $a = \pm c$ **(C)** $a = \pm d$ **(D)** $b = \pm c$

(E) none of **A**, **B**, **C** or **D**

(A) From the second equation, we have $\frac{a}{b} = \frac{d}{c}$. But the first equation asserts that $\frac{a}{b} = \frac{c}{d}$. Thus $\frac{c}{d} = \frac{d}{c}$, and so $\frac{a}{b} = \pm 1$.To see that none of the other choices can be correct, let $a = 1, \quad b = -1, \quad c = 2$, and $d = -2$.

---

7. If $\sqrt{2 + \sqrt{x}} = 3$, then $x =$

(A) 1 **(B)** 7 **(C)** 11 **(D)** 49 **(E)** 121

(D)

$$\sqrt{2 + \sqrt{x}} = 3 \Rightarrow 2 + \sqrt{x} = 9 \Rightarrow \sqrt{x} = 7 \Rightarrow x = 49.$$

---

8. A cyclist rides his bicycle over a route which is $\frac{1}{3}$ uphill, $\frac{1}{3}$ level, and $\frac{1}{3}$ downhill. If he covers the uphill part of the route at the rate of 16 miles per hour and the level part at the rate of 24 miles per hour, what rate in miles per hour would he have to travel the downhill part of the route in order to average 24 miles per hour for the entire route?

(A) 32 **(B)** 36 **(C)** 40 **(D)** 44 **(E)** 48

(E) Let $3d$ be the number of miles in the entire route. Then the time t, in hours, for the cyclist to cover the entire route is $t = \frac{d}{16} + \frac{d}{24} + \frac{d}{r}$ where r is the rate in miles per hour over the downhill portion of the route. Hence, the average rate in miles per hour over the entire route is $\frac{3d}{t} = 3\frac{(48r)}{5r+48}$. Equating this result to 24 gives $r = 48$.

---

9. The square of $2^{\sqrt{2}}$ equals

(A) 2^2 **(B)** $4^{\sqrt{2}}$ **(C)** 4^2 **(D)** $4^{2\sqrt{2}}$ **(E)** $4^{\sqrt{2}^2}$

(B)

$$(2^{\sqrt{2}})^2 = 2^{2\sqrt{2}} = (2^2)^{\sqrt{2}} = 4^{\sqrt{2}}$$

---

10. Let A be the ratio of the volume of a sphere to the volume of a cube each of whose faces is tangent to the sphere, and let B be the ratio of the surface area of this sphere to the surface area of the cube. Then

(A) $\dfrac{A}{B} > 1$ **(B)** $\dfrac{A}{B} = 1$ **(C)** $1 > \dfrac{A}{B} > \dfrac{1}{2}$ **(D)** $\dfrac{A}{B} = \dfrac{1}{2}$
(E) $\dfrac{A}{B} < \dfrac{1}{2}$

(B) Let r be the radius of the sphere. Then $2r$ is the edge length of the cube, $A = \frac{\frac{4}{3}\pi r^3}{(2r)^3} = \frac{\pi}{6}$, and $B = \frac{4\pi r^2}{6(2r)^2} = \frac{\pi}{6}$. Thus $\frac{A}{B} = 1$.

---

11. The sum of the odd positive integers from 1 to n is 9,409. What is n?

(A) 93 **(B)** 97 **(C)** 103 **(D)** 167 **(E)** 193

(E) The sum of the first k odd integers, $1 + 3 + 5 + \cdots + (2k - 1)$ is k^2. Therefore, n is the $\sqrt{9409}$th $=$ 97th positive odd integer, which is $2 \cdot 97 - 1 = 193$.

---

12. What is the area of the region of the plane determined by the inequality $3 \le |x| + |y| \le 4$?

(A) 7 **(B)** 9 **(C)** 14 **(D)** 16 **(E)** 32

(C) The region is an 'annular square' whose outside bounding square has area 32 and whose inside bounding square has area 18. Hence, the area of the region is $32 - 18 = 14$.

---

13. For how many positive integers n do there exist n consecutive integers that sum to -1? (The sum of 1 consecutive integer is just the number itself.)

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

(C) Suppose there is a sequence of n consecutive integers beginning with u having a sum of -1. Regrouping and factoring we obtain,

$$\begin{aligned} u + (u+1) + (u+2) + \cdots + (u+n-1) &= (n) \cdot u + 1 + 2 \cdots + n - 1 \\ &= (n)\left(u + \frac{n-1}{2}\right) \\ &= -1 \end{aligned}$$

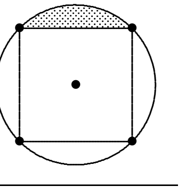
Since $n \ge 1$, there are only two possible ways to get -1 as such a product: $n = 1$, in which case we get $u = -1$ and $n = 2$, in which case we get $u = -1$. Thus, there are exactly two ways to write -1 as the sum of one of more consecutive integers: $-1 = -1$ and $-1 + 0 = -1$.

---

14. A circle with radius $\sqrt{2}$ is centered at $(0,0)$. The area of the smaller region cut from the circle by the chord from $(-1,1)$ to $(1,1)$ is

(A) π **(B)** $\sqrt{2} - 1$ **(C)** $\frac{\pi}{2} - 1$ **(D)** $\sqrt{2}(1 - \frac{\pi}{4})$ **(E)** $\frac{\sqrt{2}}{\pi}$

(C) The area of the circle is 2π and that of the inscribed square 4, so the shaded area is $\frac{2\pi-4}{4} = \frac{\pi}{2} - 1$.



---

15. In a math class of size 50, the average score on the final exam is 68. The best ten exams are all 100. The average of the other 40 is

(A) 50 **(B)** 55 **(C)** 60 **(D)** 65 **(E)** 70

(C) Since the average of all 50 scores is 68, the sum of the scores is $50 \times 68 = 3400$, and the sum of the other 40 scores is $3400 - 1000 = 2400$. Therefore the average of these 40 scores is $2400/40 = 60$.

---

16. What is the base x for which $\log_x 729 = 4$?

(A) $3\sqrt{3}$ **(B)** $7/\sqrt{2}$ **(C)** 5 **(D)** $2\sqrt{5}$ **(E)** $\pi\sqrt{3}$

(A) $\log_x 729 = \log_x 3^6 = 4 \Rightarrow x^4 = 3^6 = (3\sqrt{3})^4 \Rightarrow x = 3\sqrt{3}$.

---

17. For what positive value of c does the line $y = -x + c$ intersect the circle $x^2 + y^2 = 1$ in *exactly* one point?

(A) $\ln 4$ **(B)** $4^{1/3}$ **(C)** $\frac{3}{2}$ **(D)** $\sqrt{2}$ **(E)** $\sin^{-1}(1)$

(D) Since $x^2 + (-x+c)^2 = 1$, we must have $x^2 + x^2 - 2xc + c^2 = 1$, so $2x^2 - 2xc + c^2 - 1 = 0$ for exactly one value of c. Thus the discriminant of the quadratic must be zero. Hence, $4c^2 - 4(2(c^2 - 1)) = 0$ and $c = \pm\sqrt{2}$.

---

18. If a, b and c satisfy $a^2 + b^2 = 208$, $b^2 + c^2 = 164$, and $c^2 + a^2 = 244$, then $a^2 + b^2 - c^2 =$

(A) -36 **(B)** 20 **(C)** 108 **(D)** 120 **(E)** 180

(C) Subtract the second equation from the first to get $a^2 - c^2 = 44$. Conbine this with the third equation to get $a^2 = 144$; then it follows that $c^2 = 244 - 144 = 100$, and $b^2 = 164 - 100 = 64$.

---

19. A grocer has c pounds of coffee that are divided equally among k sacks. She finds n empty sacks and decides to redistribute the coffee equally among the $k+n$ sacks. When this is done, how many fewer pounds of coffee does each of the original sacks hold?

(A) $\dfrac{c}{k+n}$ **(B)** $\dfrac{c}{k+cn}$ **(C)** $\dfrac{c}{k^2+kn}$ **(D)** $\dfrac{cn}{k+n}$ **(E)** $\dfrac{cn}{k^2+kn}$

(E) Originally each sack holds $\frac{c}{k}$ pounds of coffee. With $k+n$ sacks, each sack holds $\frac{c}{k+n}$ pounds, so the difference is

$$\frac{c}{k}-\frac{c}{k+n}=\frac{c(k+n)-ck}{k(k+n)}=\frac{cn}{k^2+kn}.$$

---

20. The quantities x, y, and z are positive and $xy=\frac{z}{4}$. If x is increased by 50% and y is decreased by 25%, how must z be changed so that the relation $xy=\frac{z}{4}$ remains true?

(A) z must be decreased by 12.5%

(B) z must be increased by 12.5%

(C) z must be decreased by 25%

(D) z must be increased by 25%

(E) z must be increased by 50%

(B) The quantity xy becomes $\frac{3}{2}x\cdot\frac{3}{4}y$ which represents a $\frac{1}{8}=12.5\%$ increase in $\frac{z}{4}$, hence also for z.

---

21. If N is the cube of a certain positive integer, which of the following is the square of the next positive integer?

(A) $\sqrt{(N+1)}$ **(B)** $\sqrt[3]{(N+1)}$ **(C)** N^2+1 **(D)** $N^{2/3}+2N^{1/3}+1$
(E) $N^{2/3}-2N^{1/3}+1$

(D) Let x denote the integer in question. Then $N^{1/3}=x$, so $(x+1)^2=(N^{1/3}+1)^2=N^{2/3}+2\cdot N^{1/3}+1$.

---

22. In a certain class, two-thirds of the female students and half of the male students speak Spanish. If there are three-fourths as many girls as boys in the class, what fraction of the entire class speaks Spanish?

(A) $\dfrac{5}{6}$ **(B)** $\dfrac{4}{7}$ **(C)** $\dfrac{4}{5}$ **(D)** $\dfrac{1}{3}$ **(E)** none of **A**, **B**, **C** or **D**

(B) Suppose there are m male students in the class. Then there are $3m/4$ females. The number of Spanish speaking students is therefore $\frac{2}{3} \cdot \frac{3m}{4} + \frac{1}{2}m$. Thus, the fraction of the class who can speak Spanish is $(\frac{1}{2}m + \frac{1}{2}m)/(m + \frac{3}{4}m) = \frac{4}{7}$. OR
(B) Suppose there are 42 students in the class. Then 18 are female, and 12 of these speak Spanish. There are 24 males and 12 of these speak Spanish, so the fraction of the class who can speak Spanish is $24/42 = 4/7$.

---

23. If $x \neq y$ and $\frac{x^3-y^3}{x-y} = 8$, then $x^2 + xy + y^2 =$

(A) 2 **(B)** 5 **(C)** 8 **(D)** 64

(E) It cannot be determined from the information given.

(C) The expression $\frac{x^3-y^3}{x-y}$ is the same as $x^2 + xy + y^2$ in case $x \neq y$, so $x^2 + xy + y^2 = 8$.

---

24. The number of integers from 1 to 10000 (inclusive) which are divisible neither by 13 nor by 51 is

(A) 9030 **(B)** 9050 **(C)** 9070 **(D)** 9090 **(E)** 9110

(B) By the principle of inclusion-exclusion, the number is

$$10000 - \left\lfloor \frac{10000}{13} \right\rfloor - \left\lfloor \frac{10000}{51} \right\rfloor + \left\lfloor \frac{10000}{13 \times 51} \right\rfloor =$$

$$10000 - 769 - 196 + 15 = 9050.$$

---

25. In order that 9986860883748524N5070273447265625 equal 1995^{10}, the letter N should be replaced by the digit

(A) 1 **(B)** 2 **(C)** 4 **(D)** 7 **(E)** 8

(B) Because 1995 is divisible by 3, all its higher powers are divisible by 9. The principle of *casting out nines* says that any positive integer differs from the sum of its digits by a multiple of 9. Hence the sum of the digits of 1995^{10} must also be divisible by 9. This requires that N be replaced by 2.

---

26. If $x = a+bi$ is a complex number such that $x^2 = 3+4i$ and $x^3 = 2+11i$ where $i = \sqrt{-1}$, then $a + b =$

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

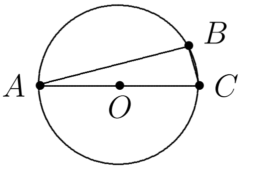
(B)

$$\begin{aligned} x &= \frac{x^3}{x^2} \\ &= \frac{2+11i}{3+4i} \times \frac{3-4i}{3-4i} \\ &= \frac{6+(33-8)i-44i^2}{9-16i^2} \\ &= \frac{(6+44)+25i}{9+16} \\ &= \frac{50+25i}{25} \\ &= 2+i. \end{aligned}$$

---

27. Three chords in a circle have lengths a, b, and c, where $c = a + b$. If the chord of length a subtends an arc of $30°$ and the chord of length b subtends an arc of $90°$, then the number of degrees in the smaller arc subtended by the chord of length c is

(A) 120 **(B)** 130 **(C)** 140 **(D)** 150 **(E)** 160



(D) Let O be the center of the circle as shown with diameter AC of length d and chord AB of length c. Let $\overparen{AB} = \theta$. Then $\angle ACB = \frac{\theta}{2}$. Since $\triangle ABC$ is inscribed in a semicircle, $\angle ABC$ is a right angle and $c = d \sin \frac{\theta}{2}$. Then for chords of lengths a and b in the same circle, with a subtending an arc of $30°$ and b subtending an arc of $90°$, $a = d \sin 15°$ and $b = d \sin 45°$. Since $c = a + b$, $\sin \frac{\theta}{2} = \sin 15° + \sin 45°$. For any angle u, $\sin(\frac{\pi}{3} + u) - \sin(\frac{\pi}{3} - u) = \sin u$. Letting $u = 15°$ and noticing that $\frac{\pi}{3} = 60°$, we get $\sin 75° = \sin 15° + \sin 45°$. Hence $\frac{\theta}{2} = 75°$ and the number of degrees in the smaller arc subtended by a chord of length c is 150.