ELIMINATION AND INSERTION OPERATIONS FOR FINITE MARKOV CHAINS

Abstract. A Markov chain (MC) observed only when it is outside of a subset D is again a MC with a well-known transition matrix P_D. This matrix can be obtained also in a few iterations, each requiring $O(n^2)$ operations, when the states from D are "eliminated" one at a time. We modify these iterations to allow for a state previously eliminated to be "reinserted" into the state space in one iteration. This modification sheds a new light on the relationship between an initial and censored MC, and introduces a new operation - "insertion" into the theory of MCs.

Key words. Markov chain, censored Markov chain, State Elimination Algorithm.

1. Introduction. Let X be a finite state space, $|X| = n, P = \{p(x,y)\}$ a stochastic matrix indexed by elements of X, and (Z_n) a Markov chain (MC) defined by X, P with some initial distribution. Let $D \subset X$, $S = X \setminus D$. It is well known that a MC observed only at visits to the subset S is again a MC (sometimes called a censored or embedded MC) with a new transition matrix P_D, see formula (2.4) below. This formula can be found e.g. in the classical text [3]. We say that set D is "eliminated." Such a transition matrix has an especially simple form when D consists of only one point, $D = \{z\}$. In this case, if we denote $P_{\{z\}} = P'$, it is easy to see that

$$p'(x,y) = p(x,y) + p(x,z)n(z)p(z,y), x \in X, y \in S, \tag{1.1}$$

where $n(z) = \sum_{n=0}^{\infty} p^n(z,z) = 1/(1-p(z,z))$. According to formula (1.1), each row-vector of the new stochastic matrix P' is a linear combination of two rows of P (with the z-column deleted). This transformation corresponds formally to one step of the Gaussian elimination and requires $O(n^2)$ operations. We call such a transformation of a matrix an *iteration*. If $|D| = k$ then the transition matrix P_D can be obtained in k iterations.

Censored MCs have had numerous applications in Probability Theory and Linear Algebra, see e.g. [11] and [4]. A relatively recent method for recursively calculating many important characteristics of MCs based on formula (1.1), is the so called *state reduction (SR) approach*. This was initiated by the papers [1] and [6], where the so-called GTH/S algorithm to calculate the invariant distribution for an ergodic Markov chain was introduced. See also [5], where similar ideas were analyzed mainly from the algebraic point of view.

Briefly this approach can be described as follows. If an initial Markov model $M = (X,P)$ is finite, $|X| = n$, and only one point is eliminated each time, then a sequence of stochastic matrices $(P_k), k = 1,...,n-1$ can be calculated recursively on the basis of formula (1.1) or (2.4) below. Such a sequence of stochastic matrices provides an opportunity to calculate many characteristics of the initial Markov model M recursively starting from some reduced model $M_s, 1 < s \leq n$.

Another application of formula (1.1) in the area of the Optimal Stopping (OS) of MCs was started in [8], where the so called the State Elimination Algorithm was introduced. According to this algorithm all points that do not belong to an optimal

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stopping set are eliminated one by one, or more generally in the countable case at some steps a subset may be eliminated. The order in which states are eliminated is defined by some auxiliary procedure. Another algorithm based on the State Elimination algorithm was used to calculate the Generalized Gittins index in [9].

Recently in [10] we presented an algorithm for finding an optimal strategy and the value function for a Markov Decision Process (MDP) model where at each moment of discrete time a decision maker (DM) can apply one of three possible actions - *continue* when MC evolves according to the transition matrix P, *quit* when the evolution of a MC is stopped, and *restart* when the MC is moved to one of a finite number m of fixed "restarting" points. A decision at state x brings a corresponding reward, positive or negative, $c(x), q(x)$ or $r_i(x), i = 1, ..., m$. The goal of a DM is to maximize the total expected discounted reward. Such a model is a generalization of a model of Katehakis and Veinott in [2], where a restart to a unique point was allowed without any fee and the quit action was absent. Both models are related to the well-known Gittins index. For the case $m = 1$, a recursive algorithm to solve this model by performing $O(n^3)$ operations was proposed. An important part of this algorithm is a sequence of recursive steps when a transition matrix is transformed. It turns out that at some steps the points are eliminated but on other steps they need to be included (inserted) back. Assume, for example, that three points, z_1, z_2 and z_3, are subsequently eliminated. Reversing formula (1.1), it is easy to restore point z_3. This requires one iteration step. What if we wish to restore point z_1, i.e. to have only points z_2 and z_3 be eliminated? At first glance it seem that we either have to keep the matrix P_1 in memory and eliminate points z_2 and z_3 or restore (in three iteration steps) points z_3, z_2 and z_1 and then eliminate points z_2 and z_3. It turns out that we can insert point z_1 in one iteration if we will keep in memory a *nonstochastic* matrix W_D similar to matrix P_D. This matrix is also obtained by iterations. This result was given by Theorem 3 in [10]. The proof was somewhat tedious, part of it was given in Appendix and we wrote that "we fail to find a simpler proof though one likely exists". The main goals of this note are: first, to describe this new operation of *insertion* and to present corresponding formulas, and second, to give simpler, shorter and more transparent proof of this theorem, based on two new lemmas, keeping the formulation of the theorem the same. We also note that these "new elimination" steps allow us as a byproduct to obtain a recursive algorithm to calculate a fundamental matrix $N = N_D = (I - Q)^{-1}$ corresponding to any transient MC with substochastic matrix Q. In Section 3 we prove this theorem and we discuss the transformation of transition matrices of MCs under elimination and insertion. At the end we give two small numerical examples.

2. Censored MC and Elimination. An important and traditional tool for the study of Markov chains (MCs) is the notion of a Censored (Embedded) MC. Two operations on stochastic and related matrices are introduced in this section. They serve as building blocks for the algorithm in [10].

A pair $M = (X,\ P)$, where X is a state space and P is a stochastic matrix is called Markov model. Let us assume that a Markov model $M = (X,\ P)$ is given and $D \subset X$, $S = X \setminus D$. Then the matrix $P = \{p(x, y)\}$ can be decomposed as follows

$$P = \begin{bmatrix} Q & T \\ R & P_0 \end{bmatrix}, \tag{2.1}$$

where the substochastic matrix Q describes the transitions inside of D, P_0 describes the transitions inside of S and so on. Let us introduce the sequence of Markov times $\tau_0, \tau_1, ..., \tau_n, ...,$ where $\tau_0 = 0$, and $\tau_n, n \geq 1$ are the times of first, and so on, *return* of the MC (Z_n) to the set S, i.e., $\tau_{n+1} = \min\{k > \tau_n, Z_k \in S\}$. Let us consider the random sequence $Y_n = Z_{\tau_n}, n = 0, 1, 2, ...$ and assume that $Z_0 = x \in S$. The strong Markov property and standard probabilistic reasoning imply the following basic lemma of the SR approach which should probably be credited to Kolmogorov and Doeblin.

LEMMA 2.1 (Elimination Lemma). *(a) The random sequence* (Y_n) *is a Markov chain in a model* $M'_D = (S, P'_D)$, *where*

(b) the transition matrix $P'_D = \{p'(x, y), x, y \in S\}$ *is given by the formula*

$$P'_D = P_0 + RU = P_0 + RNT. \tag{2.2}$$

Here U is the matrix of the distribution of the MC at the time of the first visit to S starting from $x \in D$ and $N = N(D)$ is the fundamental matrix for the substochastic matrix Q, i.e. $N = \sum_{n=0}^{\infty} Q^n = (I - Q)^{-1}$, where I is the $|D| \times |D|$ identity matrix. This representation is given, for example, in the classical text [3].

The matrix $N = N(D)$ has the following well-known probabilistic interpretation, $N = \{n(x, y), x, y \in D\}$, where $n(x, y) = E_x \sum_{n=0}^{\tau} I_y(Z_n)$, and τ is the time of the *first visit* to S, i.e. $\tau = \min(n \geq 0 : Z_n \in S)$. Thus $n(x, y)$ is the expected number of visits to y starting from x until τ. The matrix N also satisfies the equalities

$$N = I + QN = I + NQ. \tag{2.3}$$

MC (Y_n) is called an *embedded* (*censored*) MC.

An important case is when the set D consists of one nonabsorbing point z. In this case formula (2.2) takes the form (1.1), where $n(z) = 1/(1-p(z, z))$, is a "fundamental matrix".

In Lemma 2.1, the Markov model M'_D has a reduced state space S. Sometimes, it is more convenient to have all stochastic matrices of equal full size. Then we consider a MC (Y_n) with initial points x not only in S but also in D, though after the first step the MC (Y_n) is always in S. Lemma 2.1 remains true but now we obtain a Markov model $M_D = (X, P_D)$. In addition to (2.2) for $x, y \in S$, we have the equality $T + QNT = (I + QN)T = NT$ for $x \in D, y \in S$. The last equality is true by (2.3). Thus instead of (2.2) we have the following full size transition matrix

$$P_D = \begin{bmatrix} 0 & NT \\ 0 & P_0 + RNT \end{bmatrix}. \tag{2.4}$$

Note that the rows of matrix P_D give the distribution of MC (Z_n) at the time τ_1 of the *first return* to set S, i.e. $\tau_1 = \min\{n > 0 : Z_n \in S\}$ and $P(Z_{\tau_1} = y) = P_D(Y_1 = y)$ when $x \in X, y \in S$. For $x \in D$, the moment of the first return coincides with the moment of the first visit and this distribution is given by submatrix NT. For the points from $x \in S$ the corresponding distribution is given by submatrix $P_0 + RNT$. If $D = \{z\}$, i. e. when state z is eliminated, formula (2.4) is replaced by the one-state elimination formula, written here for columns, $(P_{\{z\}} = P')$,

$$p'(\cdot, z) = 0, \quad p'(\cdot, y) = p(\cdot, y) + p(\cdot, z)\frac{p(z, y)}{1 - p(z, z)}, y \neq z. \tag{2.5}$$

We say that matrix P' is obtained from P in one *iteration.* Thus matrix P_D can be calculated directly by (2.4) or recursively using formula (2.5) in $|D|$ iterations.

3. Elimination vs Insertion. Suppose that the set $D = \{z_1, z_2, ...z_k\}$ is eliminated in an initial model M, and P_D is the corresponding matrix obtained recursively by formula (2.5). Let $z \in D$, say $z = z_1$. How can one obtain the transition matrix $P_{D\setminus z}$, i.e. when only the points $z_2, ...z_k$ are eliminated? Of course we can obtain this matrix starting from an initial matrix P and eliminating these points, performing $k-1$ iterations. Is there a way to obtain this matrix in just one iteration?

The answer for this rhetorical question is Yes. In this case we say that point z is inserted (restored). To do this, initially, instead of the stochastic matrices $P_1, P_2, ..., P_k = P_D$, we must calculate recursively similar but different nonstochastic (!) matrices $W_1, W_2, ..., W_k = W_D$. We do this by applying the second part of formula (2.5) to *all columns*, including previously eliminated states, i.e. when state z is eliminated using the formula

$$w'(\cdot, y) = w(\cdot, y) + w(\cdot, z)\frac{w(z, y)}{1 - w(z, z)}, y \in X. \tag{3.1}$$

Let us show immediately that this transformation of matrix W into W' can be reversed, i.e. the following statement holds

LEMMA 3.1. *If matrix W' is obtained from matrix W by elimination formula (3.1) then matrix W can be obtained from matrix W' by insertion formula*

$$w(\cdot, y) = w'(\cdot, y) - w'(\cdot, z)\frac{w'(z, y)}{1 + w'(z, z)}, y \in X. \tag{3.2}$$

Proof. Using formula (3.1) for $x = z$ we obtain the equality $w'(z, y) = w(z, y)/(1 - w(z, z))$ and hence formula (3.1) can be written as

$$w(\cdot, y) = w'(\cdot, y) - w(\cdot, z)w'(z, y), y \in X. \tag{3.3}$$

Applying this formula for $y = z$ we obtain $w(\cdot, z) = w'(\cdot, z)/(1 + w'(z, z))$ and hence formula (3.3) can be written as (3.2). $\square$

Note that by the definition of matrices W_D and P_D their columns for $y \notin D$ coincide but the matrix P_D has zero columns for $y \in D$. The interpretation of the nonzero columns in W_D is given in Theorem 3.4.

Let us introduce the matrix $N^+ = N^+(D) = \{n^+(x, y|D)\}, x \in X, y \in D$, where $n^+(x, y|D)$ is the expected number of visits of a MC (Z_n) to state y *after the initial moment* until the moment of first *return* (visit) to set S. As with matrix $N = N(D)$ we usually will skip D. Note the differences between matrices N^+ and $N = \{n(x, y)\}, x, y \in D$: first, N^+ is the $|X| \times |D|$ matrix, N is the $|D| \times |D|$ matrix; second, $n^+(x, y)$ counts the number of visits to y after the initial moment, where as $n(x, y)$ including the initial moment. We also have obvious equalitiess: if $x, y \in D$ and $x \neq y$ then $n^+(x, y) = n(x, y)$, and $n^+(x, x) = n(x, x) - 1$. If $y \in D.x \notin D$ then $n^+(x, y) = n(x, y) = \sum_{z\in D} p(x, z)n(z, y)$.

To describe the structure of matrix N^+ it is convenient to introduce also an auxiliary matrix, $P(D)$. It consists of the first D columns of matrix P, see (2.1), i.e. has dimension $|X| \times |D|$ and contains blocks Q and R, LEMMA 3.2. *(One point Lemma). Let $D \subset X$, and $N, N^+, P(D)$ are matrices defined above. Then*

a) matrix $N^+ = P(D)N$, i.e.

$$N^+ = \begin{bmatrix} QN \\ RN \end{bmatrix} = \begin{bmatrix} N - I \\ RN \end{bmatrix}, \tag{3.4}$$

b) the columns of N^+ can also be obtained by formula

$$n^+(\cdot, y|D) = \frac{p_{D\setminus y}(\cdot, y)}{1 - p_{D\setminus y}(y, y)} = p_{D\setminus y}(\cdot, y)n(y, y|D), y \in D. \tag{3.5}$$

First, we show that it is sufficient to consider only the case when D contains only one point, $D = \{y\}$. The explains the name for this lemma. The reason is that all points in D except y one can be eliminated without changing $n(y, y|D)$ or $n^+(\cdot, y|D)$. More precisely

PROPOSITION 3.3. *Let $D \subseteq X, y \in D$. Then*

$$n(\cdot, y|D) = n_{D\setminus y}(\cdot, y), n^+(\cdot, y|D) = n^+_{D\setminus y}(\cdot, y).$$

Proof. Intuitively this statement is almost obvious: the expected number of visits to state y starting in y before exit to S remains the same if model M is transformed to model $M_{D\setminus y}$. The strict proof of this and more general statement about trajectories in an initial and reduced model was given in [8].

Now we can prove lemma 3.2. Let $D = \{y\}$. Then $p_{D\setminus y}(\cdot, y) = p(\cdot, y)$ and $n(y, y)$ is the expectation of a geometric random variable with the P(success)= P(exit from D) $= 1 - p(y, y)$ and hence $n(y, y) = 1/(1 - p(y, y)$. Correspondingly $n^+(y, y) = n(y, y) - 1 = p(y, y)/(1 - p(y, y)$ and $n^+(x, y) = p(x, y)n(y, y)$. □

Before formulating our main theorem, recall that both matrices W_D and P_D are defined recursively, starting from the same stochastic matrix P and their columns for $y \notin D$ are the same, i.e. P_D is a part of W_D. The rows of P_D have a simple probabilistic interpretation as the distribution of MC (Z_n) at the moment of the first return to set $S = X \setminus D$. Thus there is no question whether the matrix P_D for $D = \{z_1, ...z_k\}$ depends on the order of states in D, it does not. For the y-columns of matrix W_D for $y \in D$ this is initially an open question, but formula (3.6) (or (3.8)) shows that as with P_D the order is irrelevant. Let us denote by P^0_D a matrix which consists only of non zero columns of matrix P_D, see (2.4), and thus has dimension $|X| \times |S|$ and contains blocks NT and $P_0 + RNT$, and denote by $p_{D\setminus y}(\cdot, y)$ the columns of matrix $P_{D\setminus y}$.

THEOREM 3.4 (Insertion Theorem). *Let $D \subset X, S = X \setminus D$, the transition matrix P is decomposed as in (2.1) and N, N^+, P^0_D are matrices defined above. Then*

a) the matrix $W_D = [N^+|P^0_D]$, i.e. the first $|D|$ columns of W_D for $y \in D$ coincide with columns of matrix $N^+ = N^+(D)$ and the remaining columns, for $y \in S$ coincide with columns of matrix P^0_D, i.e.

$$W_D = \begin{bmatrix} QN & NT \\ RN & P_0 + RNT \end{bmatrix}, \tag{3.6}$$

b) given matrix W_D, any point $z \in D$ can be inserted in one iteration and matrix $W_{D\setminus z}$ can be obtained by the formula (3.2) with $W = W_{D\setminus z}, W' = W_D$, i.e.

$$w_{D\setminus z}(\cdot, y) = w_D(\cdot, y) - w_D(\cdot, z)\frac{w_D(z,y)}{1 + w_D(z,z)}, y \in X. \tag{3.7}$$

Corollary 3.5. *The equality (3.7) (the y-th column of matrix W_D for $y \in D$), can be described also by the formula*

$$w_D(\cdot, y) = n^+(\cdot, y|D) = \frac{p_{D\setminus y}(\cdot, y)}{1 - p_{D\setminus y}(y,y)}, y \in D, \tag{3.8}$$

this corollary follows immediately from point a) of Theorem 3.4 and formula (3.5).

Before to prove the theorem note that by formula (2.3) the submatrix QN in formula (3.6) coincides with $N - I$. Since matrix W_D is calculated recursively, it means that our theorem implies immediately

Corollary 3.6. *For any set $D \subset X$ and corresponding substochastic matrix Q the fundamental matrix $N = N(D)$ can be also calculated recursively as a part of matrix W_D. It means also that the insertion of one point into fundamental matrix can be also obtained in one iteration.*

To prove the theorem we need the following key lemma.

Lemma 3.7. *(Two point Lemma). Let $D \subset X, y \in D, z \notin D, G = D \cup \{z\}$, then*

$$n^+(\cdot, y|G) = n^+(\cdot, y|D) + n^+(\cdot, z|G)n^+(z, y|D). \tag{3.9}$$

Proof. Similarly to the proof of (3.2), using Proposition 3.3, it is sufficient to consider only the case when $D = \{y\}, G = \{y, z\}$, thus the name of this lemma. In this case $p_{D\setminus y}(\cdot,\cdot) = p(\cdot,\cdot)$. The proofs of (3.9) for all three possible cases $x = z, x = y, x \notin \{y, z\}$ are very similar so we consider only the more difficult case $x = z$. In this case $n^+(z, y) = n(z, y)$ and formula (3.5) applied separately for the sets D and G, and correspondingly for point $x = z$ and columns y and z, implies the equalities

$$n^+(z, y|D) = p(z, y)n(y, y)|D) = \frac{p(z,y)}{1 - p(y,y)},\ n^+(z, z|G) = \frac{p_D(z,z)}{1 - p_D(z,z)}.$$

Then the right side of formula (3.9) with $x = z$ can be written as equality

$$p(z, y)n(y, y)|D)(1 + n^+(z, z|G)) = \frac{p(z,y)}{(1 - p(y,y))(1 - p_D(z,z))}. \tag{3.10}$$

Thus to prove lemma we need to prove that the left side of (3.9), i.e. $n^+(z, y|G) \equiv n(z, y)$ coinsides with (3.10).

Using the second equality in (2.3) the left side of (3.9) can be represented (skipping G) as $n(z, y) = n(z, y)p(y, y) + n(z, z)p(z, y)$ and hence $n(z, y) = n(z, zz)p(z, y)/(1 - p(y, y))$. By point b) of (3.2) we have $n(z, z) \equiv n(z, z|G) = 1/(1 - p_D(z, z))$. Thus $n(z, y)$ coincides with the right side of (3.10). Lemma is proved.

Now we can prove theorem 3.4. To prove point a) is equivalent to prove (3.8). For the case when $D = \{y\}$ by formula (3.1) with $W' = P_D, W = P$ and $z = y$ we have

$$w_D(\cdot, y) = p(\cdot, y) + p(\cdot, y)\frac{p(y,y)}{1 - p(y,y)} = \frac{p(\cdot,y)}{1 - p(y,y)}, y \in D,$$

i.e. formula (3.8) is valid. Suppose that (3.8) holds for a set $D \subset X$. Let us prove that then this equality holds for any set $G = D \cup z, z \notin D$. To simplify notation let us denote matrices and columns

$$W_D = W_1, W_G = W_2, p_{D\backslash y}(\cdot, y) = s_1(\cdot, y), p_{G\setminus y}(\cdot, y) = s_2(\cdot, y).$$

Thus our goal is to prove the first equality in the following formula

$$(3.11) \qquad w_2(\cdot, y) = n^+(\cdot, y|G) = \frac{s_2(\cdot, y)}{1 - s_2(y, y)}, y \in G,$$

where the second equality holds by (3.5).

If formula(3.8) holds for a set $D \subseteq X$, then two equivalent statements are true

$$(3.12) \qquad w_1(\cdot, y) = n^+(, y|D) = \frac{s_1(\cdot, y)}{1 - s_1(y, y)}, s_1(\cdot, y) = \frac{w_1(\cdot, y).}{1 + w_1(y, y)}, y \in D.$$

The second formula follows from the first one if we apply the first formula for $x = y$ to obtain the equality $1 - s_1(y, y) = 1/(1 + w_1(y, y))$.

For $y = z$ formula (3.11) can be checked directly: formula (3.1) with $W' = W_2, W = W_1$, state z eliminated, and with $y = z$ implies $w_2(\cdot, z) = w_1(\cdot, z)/(1 - w_1(z, z))$; since $z \notin D$ by definition of s_2 we have $s_2(\cdot, z) = p_{G\setminus z}(\cdot, z) = p_D(\cdot, z) = s_1(\cdot, z)$, and by definition of s_1 we have $s_1(\cdot, z) = w_D(\cdot, z)$.

For $y \in D$ by definition of W_2, i.e. by formula (3.1) with $W' = W_2, W = W_1$, state z eliminated, using the first formula in ((3.12)), the equalities $w_D(\cdot, z) = p_D(\cdot, z), w_D(z, z) = p_D(z, z)$ (since $z \notin D$), and the first formula in (3.12) for $x = z$, we have

$$(3.13) \qquad w_2(\cdot, y) = n^+(\cdot, y|D) + \frac{p_D(\cdot, z)}{1 - p_D(z, z)} n^+(z, y|D), y \in D, z \notin D.$$

By formula (3.5) in Lemma 3.2 applied to the case when set D is replaced set G and y is replaced by z, we have

$$\frac{p_D(\cdot, z)}{1 - p_D(z, z)} = n^+(\cdot, z|G).$$

Therefore applying lemma 3.7, i.e. formula (3.9) the right side of (3.13) coincides with $n^+(\cdot, z|G)$, i.e. formula (3.11) is proved.

Remark 3.8. *In the MDP problems with a current reward function $c(x)$ it is possible to introduce a transformation of the cost function $c(x)$ (or any function $f(x)$) defined on X into the cost function $c'_D(x)$ under transition from model M to model M'_D or correspondingly into function $c_D(x)$ under transition to model M_D. This formula was used first in [7] in the context of MDP. A corresponding transformation can be obtained for the insertion situation as well.*

4. Example. Example shows elimination and insertion for the Markov chain with 5 states. Initially, the first 3 states are eliminated one by one, then states #2 and #3 are inserted.

All calculations are performed with double floating-point precision (16 decimal significant digits), the values in tables are rounded to 4 digits after decimal point.

Table 4.1

The elimination of states #1, #2, #3.

0.4000	**0.3000**	**0.2000**	**0.1000**	**0.0000**
0.3000	**0.5000**	**0.1000**	**0.1000**	**0.0000**
0.2000	**0.3000**	**0.4000**	**0.1000**	**0.0000**
0.1000	**0.1000**	**0.1000**	**0.5000**	**0.2000**
0.1000	**0.1000**	**0.2000**	**0.2000**	**0.4000**

0.6667	**0.5000**	**0.3333**	**0.1667**	**0.0000**
0.5000	**0.6500**	**0.2000**	**0.1500**	**0.0000**
0.3333	**0.4000**	**0.4667**	**0.1333**	**0.0000**
0.1667	**0.1500**	**0.1333**	**0.5167**	**0.2000**
0.1667	**0.1500**	**0.2333**	**0.2167**	**0.4000**

1.3810	1.4286	**0.6190**	**0.3810**	**0.0000**
1.4286	1.8571	**0.5714**	**0.4286**	**0.0000**
0.9048	1.1429	**0.6952**	**0.3048**	**0.0000**
0.3810	0.4286	**0.2190**	**0.5810**	**0.2000**
0.3810	0.4286	**0.3190**	**0.2810**	**0.4000**

3.2188	3.7500	2.0313	**1.0000**	**0.0000**
3.1250	4.0000	1.8750	**1.0000**	**0.0000**
2.9688	3.7500	2.2813	**1.0000**	**0.0000**
1.0313	1.2500	0.7188	**0.8000**	**0.2000**
1.3281	1.6250	1.0469	**0.6000**	**0.4000**

Table 4.2

The insertion of states #2 and #3.

0.8750	**0.7500**	0.6250	**0.2500**	**0.0000**
0.6250	**0.8000**	0.3750	**0.2000**	**0.0000**
0.6250	**0.7500**	0.8750	**0.2500**	**0.0000**
0.2500	**0.2500**	0.2500	**0.5500**	**0.2000**
0.3125	**0.3250**	0.4375	**0.2750**	**0.4000**

0.6667	**0.5000**	**0.3333**	**0.1667**	**0.0000**
0.5000	**0.6500**	**0.2000**	**0.1500**	**0.0000**
0.3333	**0.4000**	**0.4667**	**0.1333**	**0.0000**
0.1667	**0.1500**	**0.1333**	**0.5167**	**0.2000**
0.1667	**0.1500**	**0.2333**	**0.2167**	**0.4000**

The original transition matrix and the result of consecutive elimination of states #1, #2, #3 is shown in Table 4.1. The Table 4.2 shows result of consecutive insertion of states #2 and #3. Bold columns correspond to non-zero columns in P_D, these columns form a stochastic matrix. Note, that the last matrix in Table 4.2 coincides with second matrix in Table 4.1 (with eliminated state #1), which is expected.

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