Optimal Stopping of Seasonal Observations and Calculation of Related Fundamental Matrices

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Reality vs Theory ! Resistance levels. Do they exist ?

...despite the upturn in investor sentiment, the Dow met with staunch *resistance* near the 11,700 level yesterday...

...though *support* could materialize just below 1,090 at the SPX's 50-day moving average...

Support for the Dow has materialized in the 11,550 region, with additional *support* from its 10-week trendline hovering just below the average. *Resistance*, meanwhile, should materialize near 10,750 if the bulls remain asleep...

Resistance levels 2

"Model": Suppose that *all believe* that

1) support level d and resistance level D do exist;

2) The stock market is a MC with three states:

a, bearish = random walk with a negative drift,
n, neutral = random walk,
b, bullish = random walk with a positive drift.

What is your optimal strategy if stock market is in a, n, b ?

Obvious: buy in b, sell in a, wait in n, but only inside of (d, D).

Then, we will have self-fulfilling prophecy - markets will follow the model !

Resistance levels 3

Suppose that *you do not believe but other believe*.

What is your optimal strategy if stock market is in a, n, b ?

Your strategy will be the same !

Then, again - markets will follow the model !

Outline

- Motivation - done
- Seasonal Observations
- Optimal Stopping (OS) of Markov Chains (MCs)
- State Elimination Algorithm (SEA)
- Projections of MCs
- Key transformation
- Open Problems

Seasonal observations

The simplest problem of optimal stopping is tossing a die.

The following problem of OS of MC was dubbed previously by an author as OS of "Seasonal" Observations.
A DM (decision maker, player) observes a MC $(Z_n), Z_n = (U_n, Y_n)$ with two components, where the first one is an "*underlying*" finite MC (U_n) with m states and given transition matrix U, and the other component is one of m independent sequences of i.i.d. r.v. $(Y_n(k))$ with known distr-s $F = \{f(\cdot|k)\}$.
There are m dice and which of them is tossed at a given moment is specified by a position of a MC (U_n).
The goal of a player is to maximize the discounted expected reward over all possible stopping times. The crucial point is that $P = U \times F$. *Hidden MC* if a DM observes only (Y_n).

Theorem (1, Presman, Sonin 2010, Theory of Prob. and Appl.)

(a) There are threshold values $d_*(1), ..., d_*(m)$, *such that the optimal stopping set* $S_* = \cup_j S_*(j)$, $S_*(j) = \{x : g(i,j) \geq d_*(j)\}$, $j \in B = \{1, ..., m\}$,
(b) $d_* = (d_*(s), s \in B)$ *satisfies the equation*

$$d_*(s) = \sum_{k \in B} l_*(s,k) \sum_{j \in D_*(k)} g(k,j)\, f(j|k),$$

where the matrix $L_* = \{l_*(s,k),\ s,k \in B\}$ *is defined by the equality*

$$L_* = [I - UF_d(D_*)]^{-1} U,$$

c) only a finite number of steps k_* *is necessary to obtain an optimal stopping set...*

$N = [I - Q]^{-1}$ is a fundamental matrix for a subst-c matrix Q.

Optimal Stopping (OS) of Markov Chain (MC)

T. Ferguson: *"Most problems of optimal stopping without some form of Markovian structure are essentially untractable...".*

OS Model $M = (X, P, c, g, \beta)$:

- X finite (countable) state space,
- $P = \{p(x, y)\}$, stochastic (transition) matrix
- $c(x)$ one step cost function,
- $g(x)$ terminal reward function,
- β discount factor,$0 \leq \beta \leq 1$
- (Z_n) MC from a family of MCs defined by a Markov Model $M = (X, P)$
- $v(x) = \sup_{\tau \geq 0} E_x[\sum_{i=0}^{\tau-1} \beta^i c(Z_i) + \beta^\tau g(Z_\tau)]$ value function

Description of OS Continues

- **Remark !** absorbing state e, $p(e,e)=1$,
  $p(x,y) \longrightarrow \beta p(x,y)$, $p(x,e)=1-\beta$,
  $\beta \longrightarrow \beta(x)=P_x(Z_1 \neq e)$ probability of "survival".
- $S_* = \{x : g(x) = v(x)\}$ optimal stopping set.
- $Pf = Pf(x) = \sum_y p(x,y) f(y)$.

Theorem (Shiryayev 1969)

(a) The value function $v(x)$ is the minimal solution of Bellman equation ...

$$v = \max(g, c + Pv),$$

(b) if state space X is finite then set S_* is not empty and $\tau_0 = \min\{n \geq 0 : Z_n \in S_*\}$ is an optimal stopping time. ...

Basic methods of solving OS of MC, $c \equiv 0$

- The direct solution of the Bellman equation
- The value iteration method : one considers the sequence of functions $v_n(x) = \sup_{0 \le \tau \le n} E_x \ldots, v_{n+1}(x) = max(g(x), Pv_n(x))$, $v_0(x) = g(x)$. Then $v_0(x) \le v_1(x) \le \ldots v_n(x)$ converges to $v(x)$.
- The linear programming approach $(\ |X| < \infty)$, $min \sum_{y \in X} v(y), \ v(x) \ge \sum_y p(x,y)v(y), v(x) \ge g(x), x \in X.$
- Davis and Karatzas (1994), interesting interpretation of the Doob-Meyer decomposition of the Snell's envelope

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- The *State Elimination Algorithm* (SEA) Sonin (1995, 1999, 2005, 2008, 2010); A. Irle (1980, 2006); E. Presman (2009), continuos time

State Elimination Algorithm for OS of MC

OS = Bellman equation $v(x) = max(g(x), c(x) + Pv(x))$;
$M_1 = (X_1, P = P_1, c = c_1, g), S_* = S_{1*}$. Three simple facts:

1. It may be *difficult* to find the states where it is optimal *to stop*, $g(x) \geq c(x) + P_1 v(x)$, but it is *easy* to find a state (states) where it is optimal *not to stop: do not stop if* $g(z)c(z) + P_1 g(z) \leq c(z) + P_1 v(z)$.
2. After identifying these states, set G, we can "eliminate" the subset $D \subset G$, and recalculate $P_1 \longrightarrow P_2$ and $c_1 \longrightarrow c_2,\ g$. *Elimination theorem:* $S_{1*} = S_{2*}, v_1 = v_2$. Repeat these steps until $g(x) \geq c_k(x) + P_k g(x)$ for all remaining $x \epsilon X_k$. Then
3. **Proposition 1.** *Let* $M = (X, P, c, g)$ *be an optimal stopping problem, and* $g(x) \geq c(x) + Pg(x)$ *for* all $x \in X$. *Then* X *is the optimal stopping set in the problem* M, *and* $v(x) = g(x)$ *for all* $x \in X$.

Eliminate state(s) z, (set D) and recalculate probabilities

Embedded Markov chain (Kolmogorov, Doeblin) $M_1 = (X_1, P_1)$, $D \subset X_1$, $X_2 = X_1 \setminus D = S$; MC (Z_n); $\tau_0, \tau_1, ..., \tau_n, ...$, the moments of zero, first, and so on, visits of (Z_n) to the set X_2. Let $Y_n = Z_{\tau_n}, n = 0, 1, 2, ..$

Lemma (KD)

(a) The random sequence (Y_n) is a Markov chain in a model $M_2 = (X_2, P_2)$, where $P_2 = \{p_2(x, y)\}$ is given by formula

$$P_1 = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} D & S \\ \left[\begin{array}{c} Q_1 \\ R_1 \end{array}\right. & \left.\begin{array}{c} T_1 \\ P_{01} \end{array}\right] \end{array}, \quad P_2 = P_S = P_{01} + R_1 U_1 = P_{01} + R_1 N_1 T_1,$$

$N_1 = N_D$ is a (transient) fundamental matrix, i.e. $N_1 = (I - Q_1)^{-1}$.

$$N = I + Q + Q^2 + ... = (I - Q)^{-1}, N = \{n(x, y)\}, U = NT.$$

State Elimination Algorithm, $c \equiv 0$

If $D = \{z\}$ then

$$p_2(x, y) = p_1(x, y) + p_1(x, z)n_1(z)p_1(z, y),$$

where $n_1(z) = 1/(1 - p_1(z, z))$. GTH/S algorithm (1985), inv. distr.

State Elimination Algorithm

$$g(x) - (Pg(x) + c(x)) = g - Pg$$

$\swarrow \qquad \searrow$

$g(x) - P_1g(x) \geq 0$ for *all* x $\qquad$ there is $z : g(z) - P_1g(z) < 0$

$\Downarrow \qquad \Downarrow$

$X_1 = S$ $\qquad$ $M_1 \longrightarrow M_2 : g(x) - P_2g(x)$

$\swarrow \qquad \searrow$

... and so on

Forget about OS, just MCs.

Let $M_i = (X_i,\ P_i)$ be two Markov models. Let $i = 1, 2$ and let $h : X_1 \longrightarrow X_2$ be a mapping, $H(t) = h^{-1}(t),\ t \in X_2$.

Let (Z_n) be a MC in M_1 and (Y_n) is defined by $Y_n = h(Z_n)$. Generally (Y_n) is not a MC. When it is ?

A necessary and sufficient condition for a MC to be "lumpable" (Kemeny, Snell, 1960), "mergeable" (Howard, 1971):

X_1 is partitioned into $H(t), t \in X_2$,

$$\sum_{y \in H(k)} p_1(x, y) = \sum_{y \in H(k)} p_1(x', y)$$

for any $x, x' \in H(s)$ and any $s, k \in X_2$.

Projections of Markov Models (MCs)

Model M_2 is an *S-projection* of a basic model M_1 (under h) (Seas.) if there is a stochastic matrix P_2) and m dice, i.e.

$$p_1(x, y) = p_2(h(x), h(y))f(y|h(y))$$

for all $x, y \in X_1$, where $f(y|z)$ is a probability distribution on a set $H(z) = h^{-1}(z) = \{y \in X_1 : h(y) = z\}$, defined for each $z \in X_2$.

Model M_2 is a *B-projection* of a basic model M_1 (under h) if $h(x) \neq h(y)$, then transitions occur as above, and if $h(x) = h(y)$, then

$$p_1(x, y) = p_2(h(x), h(x))q_1(x, y|h(x)),$$

where stoch. matrix $Q_1(k) = \{q_1(x|, y|k)\}$ is defined for each k.
Model M_2 is an *A-projection* of...if $f(y|h(y))$ is replaced by $f(y|h(x), h(y))$, i.e. there are not m but m^2 dice, and maybe stochastic matrices $Q_1(k), k \in X_2$.

Back to OS of Seasonal Observations

The key element in applying SEA is calculation of P_S using N_D, where...

The key element in the proof of Theorem 1 - the transformation of the equality $P = U \times F$ into equality $P_S = U_S \times F_S$.

Let $D \subset X_1, S = X_1 \setminus D$ and we consider MC $(Z_{n,D})$ stopped at $S = X \diagdown D$.

By SEA to find matrix P_S, we have to find $N_{1,D} = n_{1,D}(x, y), x, y \in X_1$.

Let MC $(U_{n,D})$ in model M_2, be the "projection" of MC $(Z_{n,D})$, defined by the equality $U_{n,D} = h_D(Z_{n,D})$, where function $h_D(x) = h(x)$ if $x \in D$ and $h_D(X) = e$ if $x \in S$.

Let $N_{2,D}$ be fundamental matrix for this MC.

Our goal is: $P_S = U_S \times F_S$. Not. $F_d(A)$ diagonal matrix with elements $F(A(k)) = \sum_{j \in A(k)} f(j|k)$, $A(k) = A \cap H(k)$, and $U_A^F = UF_d(A)$.

Theorem (2)

The fundamental matrices in the original and the projected models, $N_{1,D}$ and $N_{2,D}$ are related by the equalities valid for all $x, y \in X_1$, $n_{1,D}(x,y) = n_{2,D}(s,k) f(y|k)/F(D(k)), s = h(x), k = h(y)$.

Using this theorem we can obtain the key lemma in PS 2010:

Lemma

$$F_S = \{f_S(y|k) = f(y|k)/F(S(k))\},$$

and

$$U_S = U_S^F + U_D^F(I - U_D^F)^{-1}U_S^F = (I - U_D^F)^{-1}U_S^F.$$

Open Problems.

- Transformation of Fundamental matrices for all projections
- OS of Hidden MC
- explanation of world financial crisis
- Thank you for your attention ! Danke schon ! Merci beaucoup !

  Spasibo !

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