From this and from (2.74) the induction step is completed and the statement of the theorem follows. ■

For $\nu = \infty$, when Condition 3 of Theorem 2.4 makes no sense, the situation is more complex. We give an example showing that Conditions 1 and 2 are not sufficient for the statement of the theorem to hold. In this example, as before, all spaces $\mathcal{X}_n$, $\mathcal{A}_{n+1}$, $n = 0, 1, \ldots$, coincide with some $\mathcal{X}$ and $\mathcal{A}$, the strategy of nature depends only on the previous control and is deterministic and the cost function q depends only on the previous control. In Figure 7 the nodes correspond to the states, the directed arcs to controls and numbers on the arcs show the value of the function q for these controls.

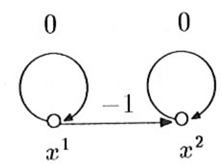

Figure 7
The transition digraph corresponding to Example 2.4

Example 2.4 *(See Figure 7.) The space $\mathcal{X}$ consists of two points x^1 and x^2. From the state x^1 it is possible to go to x^2 at cost -1 and stay in x^2 at cost 0 or otherwise to stay in x^1 at cost 0.*

It is obvious that $w_{s\infty}(x^1) = -1$, $w_{s\infty}(x^2) = 0$ for all s. The strategy which prescribes at each step staying at the previous point realizes the operator T_s for the function $w_{s\infty}$ for any s, but it is not optimal. The functions $f_s(x^1) = 0$, $f_s(x^2) = 0$ (for all s) satisfy equation (2.73), but they do not coincide with $w_{s\infty}$. ■

For $\nu = \infty$, two types of conditions may be formulated which are analogous to Conditions 1–3 of Theorem 2.4. In the first case, it is assumed that the value functions $w_{s\infty}(x)$ are known and conditions which allow the optimality of some strategies to be checked are given. An example of a theorem with conditions of this first type is given below in Theorem 2.5. In the second case, some functions $f_s(x)$ satisfying (2.73) are known, but the equalities $f_s(x) = w_{s\infty}(x)$ are not

known *a priori*, and the corresponding theorems state simultaneously these equalities and the optimality of certain strategies. Theorem 2.6 serves as an example.

Before giving statements of these theorems, we first formulate one more condition analogous to Condition (A2) from §2.3:

$$\textbf{(A3)} \qquad \lim_{n\to\infty} \inf \inf_{\pi\in\Pi_s} \sum_{r=n+1}^{\infty} E_x^\pi q_r(x_{r-1}a_r) \geq 0$$

for all $x \in \mathcal{X}_s$, $s \geq 0$.

From Condition (A3) it follows that for all $x \in \mathcal{X}_s$, $s \geq 0$,

$$\lim_{n\to\infty} \inf \inf_{\pi\in\Pi_s} E_x^\pi w_{n\infty}(x_n) \geq 0. \tag{2.75}$$

We give the proof of this fact only for the case of discrete transitions. As before, the proof also holds under the assumption of the existence of a uniformly ε-optimal strategy for any $\varepsilon > 0$ in each model Z_s, $s = 1, 2, \ldots$. From (2.34) we obtain that for any strategy $\pi \in \Pi_s$

$$E_x^\pi \sum_{r=n+1}^{\infty} q_r(x_{r-1}a_r) = E_x^\pi w_{n\infty}^{\pi'}(x_n), \tag{2.76}$$

where π' is a strategy in model Z_n, $n \geq s$ which is an extension of the strategy π and is defined with respect to π similarly to the definition of π'_{xa} in (2.63). By the assumption of discrete transitions, x_n takes not more than a countable number of values and therefore, as shown in the proof of Theorem 2.3, a $\pi_\varepsilon \in \Pi_n$ exists such that $w_{n\infty}^{\pi_\varepsilon}(x_n) \leq w_{n\infty}(x_n) + \varepsilon$ holds for all possible values x_n. From this

$$\begin{aligned} &\inf_{\pi\in\Pi_s} E_x^\pi w_{n\infty}^{\pi'}(x_n) \\ &\quad\leq \inf_{\pi\in\Pi_s} E_x^\pi w_{n\infty}^{\pi_\varepsilon}(x_n) \\ &\quad\leq \inf_{\pi\in\Pi_s} E_x^\pi w_{n\infty}(x_n) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, then from this, (2.76) and Condition (A3) the truth of (2.75) follows.

Theorem 2.5 *Let*

(*1*) *the functions* $w_{s\infty}(x)$, $s = 0, 1, \ldots$, *satisfy the optimality equation (2.71),*

(*2*) *the distributions* $\overline{\pi}_s(\cdot|x_s)$ *realize the operator* T_s *for functions* $w_{s\infty}(x)$, $s = 1, 2, \ldots$,

(*3*) *the model* Z *satisfies (2.75).*

Then the randomized Markov strategy $\overline{\pi}^s := \{\overline{\pi}_{r+1}(\cdot \mid x_r),\ r = s, s+1, \ldots\}$ *is uniformly optimal for the model* Z_s.

Proof. For a randomized Markov strategy $\pi := \{\pi_{r+1}(\cdot|x_r),\ r = 0, 1, \ldots\}$ we will denote $\pi^s := \{\pi_{r+1}(\cdot|x_r),\ r = s, s+1, \ldots\} \in \Pi_s$ for any s, and for simplicity we will write $w^{\pi}_{s\infty}$ and T^{π}_s instead of $w^{\pi^s}_{s\infty}$ and $T^{\pi^s}_s$. From (2.34) it follows directly that for any randomized Markov strategy π and any function f defined on $\mathcal{X}_n$, for $0 \le s < n$ and $x \in \mathcal{X}_s$,

$$T^{\pi}_{s+1} T^{\pi}_{s+2} \ldots T^{\pi}_n f(x) = E^{\pi}_x \sum_{r=s+1}^{n} q_r(x_{r-1} a_r) + E^{\pi}_x f(x_n). \tag{2.77}$$

Replacing π and f by the strategy $\overline{\pi}$ and $w_{n\infty}(x)$ and using the fact that by Condition 1 of the theorem, $T^{\overline{\pi}}_r w_{r\infty}(x) = w_{r-1,\infty}(x)$, $r = 1, 2, \ldots$, we have

$$w_{s\infty}(x) = E^{\overline{\pi}}_x \sum_{r=s+1}^{n} q_r(x_{r-1} a_r) + E^{\overline{\pi}}_x w_{n\infty}(x_n).$$

As $n \to \infty$ the first term on the right-hand side of this equation by definition tends to $w^{\overline{\pi}}_{s\infty}(x)$ and the liminf of the second item is nonnegative by (2.75). From this it follows that $w^{\overline{\pi}}_{s\infty}(x) \le w_{s\infty}(x)$ and this means that, by the definition of $w_{s\infty}(x)$, we have that $w^{\overline{\pi}}_{s\infty}(x) = w_{s\infty}(x)$, which it was required to prove. ∎

We now formulate the second type of theorem.

Theorem 2.6 *Let the sequence of functions* $f_s(x)$ *and the randomized Markov strategy* $\overline{\pi} := \{\overline{\pi}_{r+1}(\cdot|x),\ r = 0, 1, \ldots\}$ *be such that for* $s = 1, 2, \ldots$, *Conditions 1 and 2 of Theorem 2.4 hold, and let the model* Z *for all* $x \in \mathcal{X}_s$, $s = 1, 2, \ldots$, *satisfy the conditions:*

(*3a*) $\limsup_{n\to\infty} E^{\overline{\pi}}_x f_n(x_n) \ge 0$

(*3b*) *for each* $\varepsilon > 0$ *and each* x *there exists an* ε*-optimal strategy* π^*_ε *such that* $\lim_{n\to\infty} E^{\pi^*_\varepsilon}_x f_n(x_n) \le \varepsilon$.

Then $f_s(x) = w_{s\infty}(x)$ *and the strategy* $\overline{\pi}^s := \{\overline{\pi}_r(\cdot \mid x),\ r = s+1, s+2, \ldots\}$ *is uniformly optimal for the model* Z_s, $s = 0, 1, \ldots$.

Proof. By (2.77) for $\pi := \overline{\pi}$ and $f := f_n$ we have

$$T^{\overline{\pi}}_{s+1} T^{\overline{\pi}}_{s+2} \ldots T^{\overline{\pi}}_n f_n(x) = E^{\overline{\pi}}_x \sum_{r=s+1}^{n} q_r(x_{r-1} a_r) + E^{\overline{\pi}}_x f_n(x_n). \tag{2.78}$$

By Conditions 1 and 2 of Theorem 2.4, for f_s and $\overline{\pi}$ the left-hand side of the last equality coincides with $f_s(x)$, the first term on the right-hand side tends to $w^{\overline{\pi}}_{s\infty}(x)$ as $n \to \infty$, and Condition 3a holds for the second term. From this, $f_s(x) \ge w^{\overline{\pi}}_{s\infty}(x) \ge w_{s\infty}(x)$. On the other hand, assuming for simplicity that the strategy π^*_ε is Markovian, and by Conditions 1 and 2 and the definition of the operator T_{s+1}, we have that

$$f_s(x) = T^{\overline{\pi}}_{s+1} f_{s+1}(x) = T_{s+1} f_{s+1}(x) \le T^{\pi^*_\varepsilon}_{s+1} f_{s+1}(x).$$

Similarly, expanding $f_{s+1}(x)$ in terms of T_{s+2} and so on, we obtain

$$f_s(x) \le T^{\pi^*_\varepsilon}_{s+1} T^{\pi^*_\varepsilon}_{s+2} \ldots T^{\pi^*_\varepsilon}_n f_n(x)$$
$$= E^{\pi^*_\varepsilon}_x \sum_{r=s+1}^{n} q_r(x_{r-1} a_r) + E^{\pi^*_\varepsilon}_x f_n(x_n).$$

The first term of the right-hand side of this inequality tends to $w^{\pi^*_\varepsilon}_{s\infty}(x) \le w_{s\infty}(x) + \varepsilon$ and Condition 3b holds for the second. From this, $f_s(x) \le w_{s\infty}(x) + 2\varepsilon$. Since ε is arbitrary, $f_s(x) = w_{s\infty}(x)$. Again, using equation (2.78) we have that $w_{s\infty}(x) = \lim_{n\to\infty} E^{\overline{\pi}}_x \sum_{r=s+1}^{n} q_r(x_{r-1} a_r) = w^{\overline{\pi}}_{s\infty}$ and therefore the strategy $\overline{\pi}^s$ is optimal for the model Z_s. ∎

Remark 2.9 At first glance the application of Theorem 2.6 to the search for optimal strategies is complicated because of the necessity of checking Conditon 3b. In practice, in concrete problems this may be effected for a whole class of strategies which will certainly contain the optimal strategy and in some cases this condition will hold for all strategies. ■

Remark 2.10 Theorems similar to those given above may be formulated for finding ε-optimal strategies. ■

* * *

An important subclass of general Markov models is that of *homogeneous models*, i.e. models in which all spaces of states and controls coincide respectively with some $\mathcal{X}$ and $\mathcal{A}$ and the sets F_n, transition probabilities $p_n(\cdot|x_{n-1}a_n)$ and loss functions $q_n(x_{n-1}a_n)$ do *not* depend on n. For homogeneous models, the operators T_n and T_n^a do *not* depend on n and we will denote them simply as T and T^a.

For $\nu \leq \infty$, as before (see Remark 2.2), it suffices to consider strategies for the initial model on the interval $[0,\nu)$ and for the s^{th} remaining model on the interval $[s,\nu)$. Let $\pi := \{\pi_{r+1}(\cdot \mid x),\ r = 0,1,\ldots,\nu-1\}$ be some Markov strategy on the time interval $[0,\nu)$ and $\tilde{\pi} := \{\tilde{\pi}_{r+1}(\cdot|x),\ r = s,\ldots,s+\nu-1\}$ be a strategy in the model on the interval $[s,s+\nu)$ such that $\tilde{\pi}_{s+r}(\cdot|x) = \pi_r(\cdot|x)$. Then it is obvious that in the homogeneous case $w_{0\nu}^{\pi}(x) = w_{s,s+\nu}^{\tilde{\pi}}$. From this it follows that $w_{0\nu}(x) = w_{s,s+\nu}(x)$. We therefore introduce *new notation* for the homogeneous case, defining

$$w_s(x) := w_{0s}(x), \tag{2.79}$$

so that the optimality equation has the form

$$w_{s+1}(x) = Tw_s(x). \tag{2.80}$$

Now the value function $w_s(x)$ corresponds to any intermediate model in which the time remaining for control equals s. Correspondingly, it is convenient to change the notation for Markov strategies in homogeneous models, and we assume that in the model on the interval $[0,s)$ a strategy has the form

$$\pi := \{\pi_s(\cdot|x),\ \pi_{s-1}(\cdot|x),\ldots,\pi_1(\cdot|x)\}. \tag{2.81}$$

For the sake of brevity, we will call the notation introduced for the homogeneous case notation in *time remaining*.

The homogeneous case applies, for example, to the problem of minimization of loss in the basic scheme and the notation $w_\nu(\xi)$ for minimal loss in this scheme agrees with the notation just introduced above. For the homogeneous case, Theorems 2.4–2.6 have a simpler form. For example, Theorem 2.6 may be reduced to the following statement.

Corollary 2.1 *If the Markov model is homogeneous, the function f satisfies the equation $f(x) = Tf(x)$, the transition function $\pi(\cdot|x)$ realizes the operator T for the function f and Conditions 3a and 3b of Theorem 2.6 hold for $f_n(x) :\equiv f(x)$, then the stationary strategy $\pi = \{\pi(\cdot|x), \pi(\cdot|x),\ldots\}$ is uniformly optimal and $f(x) = w_\infty(x)$.* ■

* * *

In §2.3, the existence of an optimal action rule for the basic scheme for any fixed ξ was proven. However, the question of existence of a uniformly optimal strategy was not touched upon. In the general situation, uniformly optimal and uniformly ε-optimal strategies might not exist. However, in the case where all control spaces are finite and for $\nu = \infty$ condition (2.75) holds, there exist uniformly optimal Markov strategies.

Theorem 2.7 *If all control spaces are finite, then:*

(a) For $\nu < \infty$ there exists a uniformly optimal Markov strategy.

(b) If the value function $w_{s\infty}(x)$ which satisfies the optimality equation is measurable and satisfies condition (2.75), then a uniformly optimal Markov strategy also exists for $\nu = \infty$ and in the homogeneous model it may be chosen to be stationary.

Proof. We index controls by the numbers $1, 2, \ldots, m$ and replace T_s^a by T_s^j. Let $\nu < \infty$. Consider a sequence of functions $f_s(x)$, $s = 1, \ldots, \nu$, with $f_\nu(x) := 0$ and $f_{s-1}(x) := T_s f_s(x)$. These functions are measurable by the measurability of the functions $T_s^j f_s(x)$ and by the measurability of the minimum of a finite number of measurable functions. Let $\mathcal{X}_s^1 := \{x : T_s f_s(x) = T_s^1 f_s(x)\}, \mathcal{X}_s^2 := \{x : x \notin \mathcal{X}_s^1, T_s f_s(x) = T_s^2 f_s(x)\}$, and so on. These sets are also measurable and disjoint and $\mathcal{X}_s := \bigcup_{j=1}^m \mathcal{X}_s^j$. It is obvious that the distribution $\pi_s(\cdot|x)$ which is concentrated on the control j for $x \in \mathcal{X}_s^j$ realizes the operator T_s for $f_s(x)$. The functions $f_s(x)$ constructed and the Markov strategy $\pi := \{\pi_{s+1}(\cdot|x),\ s = 0, 1, \ldots\}$ satisfy the conditions of Theorem 2.4, from which the uniform optimality of the strategy π follows. To prove the theorem for $\nu = \infty$ it is sufficient, similarly to the case for $\nu < \infty$, to construct the sets $\mathcal{X}_s^j$ utilizing the functions $w_{s\infty}(x)$, take the corresponding $\pi_s(\cdot|x)$ which realizes the operator T_s for $w_{s\infty}(x)$ and refer to Theorem 2.5. ■

We now give an example showing that without the condition (2.75) the statement of Theorem 2.7 is not true for $\nu = \infty$.

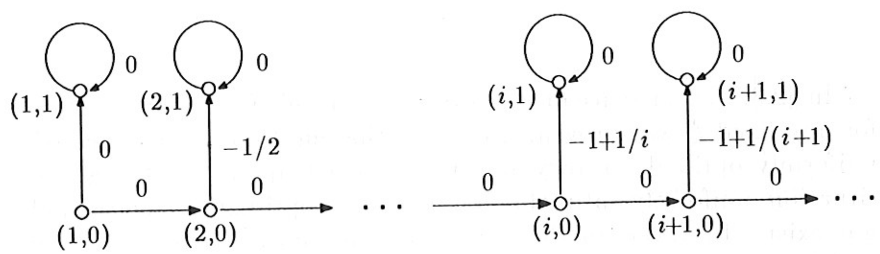

Figure 8

The transition digraph corresponding to Example 2.5

Example 2.5 *(See Figure 8.) Consider the homogeneous model represented in Figure 8. The space $\mathcal{X}$ consists of the points (i, j), $i = 1, 2, \ldots$, $j = 0, 1$. From the state $(i, 0)$ it is possible to go to $(i, 1)$ at a cost $-1 + 1/i$ or go to $(i+1, 0)$ at cost 0. The state $(i, 1)$ is absorbing with cost 0.*

It is obvious that $w_{s\infty}(i, 0) = -1$, $w_{s\infty}(i, 1) = 0$ and the functions

$w_{s\infty}$ satisfy the optimality equation, but for the points $(i, 0)$ optimal strategies do not exist. ■

We turn now to the basic scheme with the assumption that the loss functions f_n do not depend on previous history, i.e. $f_n := f_n(\Delta X(n-1), a(n), \Delta X(n))$. At the end of §2.1 (see (2.17) and (2.18)), the process $\xi(n)$ corresponding to the *a posteriori* probabilities was defined and it was shown that action rules $\beta := \{\beta(s),\ s \geq 1\}$ may be considered as functions of the previous controls and the previous values of the process $\xi(n)$ and that the cost functions q_n have the form $q_n(\xi(n-1),\ a(n))$.

In studying the question of finiteness of loss on an infinite time interval, a significant rôle is played by the asymptotic behaviour of the process $\xi(n)$ which depends on the action rule β and the parameter value θ. The corresponding analysis will be considered in §2.5.

In studying other problems it is more convenient to use a representation of the basic scheme, derived from the general results of this section, in the form of a Markov model in which the parameter θ is absent.

Consider that the controls have m possible values $1, \ldots, m$. Transition probabilities for $j = 1, \ldots, m$ are given by

$$P\{\xi(n) = \Gamma^{1j}\xi | \xi(n-1) = \xi,\ a(n) = j\} = p^j(\xi)$$
$$P\{\xi(n) = \Gamma^{0j}\xi | \xi(n-1) = \xi,\ a(n) = j\} = 1 - p^j(\xi), \qquad (2.82)$$

where $p^j(\xi)$, $\Gamma^{1j}\xi$, Γ^{0j} are defined by formulae (2.16) and (2.17)).

The operator T_s^j corresponding to the control $a(s) = j$ has the form

$$T_s^j f(\xi) = q_s(\xi, j) + p^j(\xi) f(\Gamma^{1j}\xi) + (1 - p^j(\xi)) f(\Gamma^{0j}\xi), \qquad (2.83)$$

and in the problem of loss minimization $q_s(\xi, j)$ has the form

$$q_s(\xi, j) := q^j(\xi) = \sum_{i=1}^N \xi_i(\lambda_i - \lambda_i^j). \qquad (2.84)$$

The optimality equation in the problem of loss minimization becomes

$$W_s(\xi) = \min_{1 \leq j \leq N} T^j W_{s-1}(\xi), \qquad (2.85)$$

where, in light of the discussion of homogeneous problems, s is the time remaining.

From the results of this section, it follows that an optimal strategy for the problems of the basic scheme may be found in the form of a sequence of functions $\{\pi_{n+1}(\xi),\ n = 0, 1, \ldots\}$ taking values in S^N (randomized Markov strategies), and moreover, actually taking values in $\widehat{S}^N$ (Markov strategies). Here, for fixed $\xi(0) := \xi$ each strategy defines an action rule by the simple formula

$$\beta(n, \omega) := \pi_n(\xi(n)), \quad n = 1, 2, \ldots . \tag{2.86}$$

From theorems proved in this section applied to the case of the basic scheme, the following result, which we will use in the sequel, may be derived.

Corollary 2.2 *For problems of the basic scheme:*

(I) For $\nu < \infty$ always, and for $\nu = \infty$ when condition (2.75) holds, there exists a uniformly optimal Markov strategy.

(II) If in the homogeneous case for $\nu = \infty$ there exists ξ_0 such that $F(\xi_0) < \infty$, then there exists a uniformly optimal stationary strategy.

Proof. Since the control space in the case of the basic scheme is finite, then for $\nu < \infty$ Theorem 2.7 may be applied directly, and for $\nu = \infty$ it suffices to prove that the optimal value functional $W(\xi)$ exists (is finite), is measurable, and satisfies the optimality equation. Finiteness of $W(\xi) := F(\xi)$ follows from (a) of Theorem 2.2, and its measurability follows from convexity, which was stated in the same theorem. It was proved in Theorem 2.3 that $F(\xi)$ satisfies the optimality equation. ■

2.5 Evolution of *a posteriori* probabilities

At the end of §2.1 we gave the recurrence relation (2.18) for the *a posteriori* probability $\xi(n)$ and showed that the problem of loss minimization may be considered to be a problem of the optimal control of the process $\xi(n)$. As will be seen below (see §3.2), in studying loss

an important rôle is played by the asymptotic behaviour of $\xi(n)$ as $n \to \infty$. In this section Theorem 2.8 is proved, from which it follows (see Corollary 2.3) that if the elements of each column of the matrix $\Lambda := \{\lambda_i^j\}$ are different from each other and different from 0 and 1, then for *any* action rule β *discrimination of hypotheses* holds with probability 1, i.e. the relation

$$P_\xi^\beta\{\lim_{n\to\infty} \xi(n) = \theta\} = 1 \tag{2.87}$$

holds.

If some elements of the matrix coincide, or any are equal to 0 or 1, then relation (2.87) does not always hold, as is easy to see from the example of the Bellman matrix with $0 \le \lambda_1^1 < \lambda_1^2 = \lambda_2^2 < \lambda_2^1 < 1$. In this case, using the action rule $\beta^1(n) \equiv 0$, $\beta^2(n) \equiv 1$ we have $\xi(n) \equiv \xi(0)$ and (2.87) does not hold. However, analogues of relation (2.87) and estimates of rates of convergence for $\xi(n)$ hold with less rigorous assumptions on the matrix Λ as in Theorem 2.8. This result is basic for the study of loss on an infinite time interval in §3.2.

To study the asymptotic behaviour of the process $\xi(n)$ for an arbitrary matrix Λ it is convenient to change to coordinates corresponding to the logarithm of the maximum likelihood. The transformed process satisfies *linear* equations (see (2.93)), while equation (2.18) for the process $\xi(n)$ is *nonlinear*.

First, we define the function $\ln(x/y)$ in the case where x or y may be equal to 0 as

$$\ln(x/y) := \begin{cases} 0 & \text{if } x = y = 0 \\ +\infty & \text{if } x > 0,\ y = 0 \\ -\infty & \text{if } x = 0,\ y > 0\,, \end{cases} \tag{2.88}$$

and set

$$0 \cdot \infty := 0 \cdot (-\infty) := 0 \quad \text{and} \quad -\infty + a := -\infty \text{ if } a < \infty. \tag{2.89}$$

Define the matrix $B(\xi) := \{B_{ik}(\xi)\}$, where

$$B_{ik}(\xi) := \ln(\xi_k/\xi_i) \qquad \xi \in S^N, \tag{2.90}$$

and consider the process

$$\eta(n) := \theta B(\xi(n)). \qquad (2.91)$$

It is obvious that all coordinates of the process $\eta(n)$ take values less than $+\infty$ with probability 1, since on the set $\{\theta_i = 1\}$ it follows from $\xi_i(0) > 0$ that $\xi_i(n) > 0$ for any n. Similarly to $\xi(n)$, the process $\eta(n)$ depends on the initial point ξ, but we assume that ξ is *fixed* and do not indicate this dependence.

According to (2.10), in studying loss on an infinite time interval for some fixed ξ and a corresponding action rule $\overline{\beta}$, it suffices to consider expectations with respect to P_i^{β} for those i for which $\xi_i > 0$.

Theorem 2.8 *If the elements of each column of the matrix Λ are different from each other and differ from 0 and 1, then for any action rule β, any constant $a > 0$ and any i such that $\xi_i > 0$,*

$$\sum_{n=1}^{\infty} P_i^{\beta}\{\eta_k(n) > -a\} < \infty, \quad k = 1, 2, \ldots, N, \quad k \neq i. \qquad (2.92)$$

Proof. The proof is similar for all i, therefore we consider only the case $i = N$. Accordingly, all events are assumed conditional on the event $\{\theta_N = 1\}$ and, correspondingly, the multiplier depending on θ_N is omitted in equalities between random variables. In addition, we will write simply P in place of P_N^{β}. Since on the set $\{\theta_N = 1\}$ it is true that $\eta_N(s) = 0$, then we will consider the $(N-1)$-dimensional process $(\eta_1(s), \ldots, \eta_{N-1}(s))$, retaining the notation $\eta(s)$ for it.

In (2.18) multiply the k^{th} coordinate for $1 \le k \le N-1$ and the N^{th} coordinate by $a^j(n)\Delta X^j(n)$. Divide the first of the equalities so obtained by the second and take the logarithm of the coefficients on both sides of the result. Then repeat the operation, multiplying on both sides of the result by $a^j(n)(1 - \Delta X^j(n))$, and sum the expressions obtained this time by $a^j(n)(1-\Delta X^j(n))$, and sum the expressions obtained with respect to j from 1 to m. As a result we obtain the following recurrence relations for $\eta(n)$ holding on the set $\{\theta_N = 1\}$, viz.

$$\eta(n) = \eta(n-1) + \sum_{j=1}^{m} a^j(n)[\gamma^{1j}\Delta X^j(n) + \gamma^{0j}(1 - \Delta X^j(n))], \qquad (2.93)$$

where

$$\gamma^{rj} := (\gamma_1^{rj}, \ldots, \gamma_{N-1}^{rj}), \qquad j = 1, \ldots, m, \quad r = 0, 1,$$

$$\gamma_k^{1j} := \ln \frac{\lambda_k^j}{\lambda_N^j}, \quad \gamma_k^{0j} := \ln \frac{1-\lambda_k^j}{1-\lambda_N^j}, \qquad k = 1, \ldots, N-1, \quad (2.94)$$
$$j = 1, \ldots, m.$$

We show that under the conditions of Theorem 2.8 the process $\eta(n)$ may be represented as a sum

$$\eta(n) := \eta^1(n) + \eta^2(n), \quad \eta^1(0) := \eta(0), \quad \eta^2(0) := \eta(0), \qquad (2.95)$$

where $(\eta^1(n), \mathcal{F}_n)$ is a *martingale* with bounded increments and $\eta^2(n)$ is a *strictly decreasing process* with respect to all coordinates. Set

$$\Delta\eta^1(n) := \sum_{j=1}^{m} a^j(n)(\gamma^{1j} - \gamma^{0j})(\Delta X^j(n) - \lambda_N^j a^j(n)), \qquad (2.96)$$

$$\Delta\eta^2(n) := \sum_{j=1}^{m} a^j(n)((1-\lambda_N^j)\gamma^{0j} + \lambda_N^j \gamma^{1j}), \qquad (2.97)$$

$$\eta^1(0) := 0, \quad \eta^2(0) := \eta(0), \quad \eta^r(n) := \eta^r(0) + \sum_{s=1}^{n} \Delta\eta^r(s),$$
$$r = 1, 2.$$

Then, according to (2.93), $\eta(n) = \eta^1(n) + \eta^2(n)$.

From (2.4) it follows that the process $\{\eta^1(n), \mathcal{F}_n\}$ is a martingale, and further that the random values $\Delta\eta_k^1(n)$ are bounded in modulus by $c_1 := \max_{j,k}(|\gamma_k^{1j} - \gamma_k^{0j}| \max\{\lambda_N^j, 1 - \lambda_N^j\})$.

We show next that the process $\eta^2(n)$ is strictly decreasing with respect to each coordinate. First we prove a useful lemma.

Lemma 2.4 *For any x, y such that $0 < x, y < 1$,*

$$(1-x)\ln\frac{1-y}{1-x} + x \ln\frac{y}{x} \le 0, \qquad (2.98)$$

and equality holds only for $x = y$.

Proof. Denoting the left-hand side of (2.98) by $f(x,y)$ and taking the derivative with respect to y, we have that $\partial f/\partial y > 0$ for $y < x$, and $\partial f/\partial y < 0$ for $y > x$. Since $f(x,x) = 0$, the statement of Lemma 2.4 follows. ■

From Lemma 2.4 with $x = \lambda_N^j$, $y = \lambda_k^j$ it follows that under the condition of Theorem 2.8 the coefficient of $a^j(n)$ in (2.97) is negative for every j. Since $\sum_{j=1}^m a^j(n) = 1$, then this means that the process $\eta_k^2(n)$ is strictly decreasing. The jumps of this process are not less in modulus than a positive number c_2 $(= \min_j |(1-\lambda_N^j)\gamma_k^{0j} + \lambda_N^j \gamma_k^{1j}|)$, and this means that for fixed $\eta_k(0)$ there exists an n_0 such that for $n > n_0$

$$P\{\eta_k^2(n) > -c_2 n/2\} = 0. \qquad (2.99)$$

To study the behaviour of the process $\eta^1(n)$ we prove the following lemma, which will also be used in §3.2.

Lemma 2.5 *Let* $(S_n, \mathcal{F}_n, n \geq 0)$ *be a martingale such that* $S_0 := 0$, $|S_n - S_{n-1}| < c$, $n \geq 1$, *where* c *is a constant. Then for any* $\alpha > 0$

$$P\{S_n > \alpha n\} \leq \exp(-n\alpha^2/4c^2). \qquad (2.100)$$

Proof. We need the inequality

$$e^x - x \leq e^{x^2}, \qquad (2.101)$$

which for $x \geq 1$ is obvious, and for $x < 1$ can be proved, for example, by expansion in series or by the elementary inequalities $e^{x^2} - 1 > x^2 > e^x - x - 1$.

Using the equality $E[(S_n - S_{n-1})|\mathcal{F}_{n-1}] = 0$, which holds by the definition of a martingale, the inequality (2.101) and the boundedness of $|S_n - S_{n-1}|$, we have

$$\begin{aligned} E[e^{(S_n - S_{n-1})t}|\mathcal{F}_{n-1}] &= E[(e^{(S_n - S_{n-1})t} - t(S_n - S_{n-1}))|\mathcal{F}_{n-1}] \\ &\leq E[e^{(S_n - S_{n-1})^2 t^2}|\mathcal{F}_{n-1}] \leq e^{(ct)^2}. \end{aligned}$$

Using this inequality sequentially, we have

$$Ee^{S_n t} = E[e^{S_{n-1}t} E[e^{(S_n - S_{n-1})t}|\mathcal{F}_{n-1}]] \leq e^{(ct)^2} E e^{S_{n-1}t} \leq e^{n(ct)^2}.$$

Substituting this estimate in Chebychev's inequality

$$P\{S_n > \alpha n\} = P\{e^{S_n t} > e^{\alpha n t}\} \leq e^{-\alpha n t} E e^{S_n t},$$

which holds for $\alpha t > 0$, and setting $t := \alpha/2c^2$, we obtain the statement of the lemma. ■

Applying Lemma 2.5 to the sequence $\eta_k^1(n)$, we find that for any $\alpha > 0$

$$P\{\eta_k^1(n) > \alpha n\} \leq \exp\{-n\alpha^2/4c_1^2\}. \qquad (2.102)$$

From this, by (2.99), we have for $n > n_0$

$$P\{\eta_k(n) > -c_2 n/4\} \leq P\{\eta_k^1(n) > c_2 n/4\} < \exp\{-n\left(\frac{c_2}{4c_1}\right)^2\},$$

and this means that for any $a > 0$

$$\sum_{n=1}^{\infty} P\{\eta_k(n) > -a\} < \infty. \qquad (2.103)$$

Theorem 2.8 is proven. ■

Corollary 2.3 *Under the conditions of Theorem 2.8, (2.87) holds.*

Proof. From (2.103) we obtain by the Borel–Cantelli lemma that for $k \neq i$ such that $\xi_i > 0$

$$P\{\lim_{n\to\infty} \eta_k(n) = -\infty\} = 1. \qquad (2.104)$$

From this, taking into account the equality $\sum_{i=1}^N \xi_i(n) = 1$, (2.87) follows directly. ■

In the general case, where λ_i^j may be equal to 0 or 1 or may coincide for several values of i for fixed j, relation (2.93) also holds, but in this case one needs to take into account that the coefficient of $a^j(n)$ for some j may be $-\infty$. Therefore we must separate the sum with respect to these j into a third process $\eta^3(n)$. For the remaining

j we preserve the decomposition (2.95)–(2.97), omitting those terms with coefficients 0. Accordingly, for $1 \le k \le N-1$ define

$$\begin{aligned} R'_k &:= \{j : \lambda^j_k = 1,\ 0 \le \lambda^j_N < 1 \text{ or } \lambda^j_k = 0,\ 0 < \lambda^j_N \le 1\}, \\ R''_k &:= \{j : j \notin R'_k,\ \lambda^j_k \ne \lambda^j_N\} \end{aligned} \tag{2.105}$$

and represent $\eta(n)$ as the sum of three processes

$$\eta(n) := \eta^1(n) + \eta^2(n) + \eta^3(n), \tag{2.106}$$

where

$$\eta^1(0) := \eta^3(0) := 0,\quad \eta^2(0) := \eta(0) \tag{2.107}$$

$$\Delta\eta^1_k(n) := \sum_{j\in R''_k} a^j(n)(\gamma^{1j}_k - \gamma^{0j}_k)(\Delta X^j(n) - \lambda^j_N a^j(n)) \tag{2.108}$$

$$\Delta\eta^2_k(n) = \sum_{j\in R''_k} a^j(n)((1-\lambda^j_N)\gamma^{0j}_k + \lambda^j_N\gamma^{1j}_k) \tag{2.109}$$

$$\Delta\eta^3_k(n) = \sum_{j\in R'_k} a^j(n)(\gamma^{1j}_k \Delta X^j(n) + \gamma^{0j}_k(1-\Delta X^j(n))). \tag{2.110}$$

Note that these indices j, for which $\lambda^j_N = 1$, $\lambda^j_k \ne 1$, $\lambda^j_k \ne 0$ or $\lambda^j_N = 0$, $\lambda^j_k \ne 0$, $\lambda^j_k \ne 1$, are included in R''_k, but not in R'_k, for the following reason. Despite the fact that γ^{0j}_k or γ^{1j}_k take value $-\infty$ for these j, by convention (2.89), the coefficient of $a^j(n)$ for these j in (2.108) equals 0 with probability 1, and the corresponding coefficient in (2.109) has a negative value.

The process $\{\eta^1_k(n), \mathcal{F}_n\}$ is again a martingale with bounded jumps, and the inequality (2.102) is applied to it. The process $\eta^3_k(n)$ may have an increment equal to 0, but under the condition that the increment does not equal 0 it takes a positive value (bounded by a fixed constant) with positive probability not equal to 1 and with the complementary probability taking the value $-\infty$.

The truth of the conclusion of Theorem 2.8 is connected with the fact that under its conditions the process $\eta^2_k(n)$ is strictly decreasing. According to Lemma 2.4 the process $\eta^2_k(n)$ is nonincreasing since all

coefficients of $a^j(n)$ in the sum defining $\Delta\eta^2(n)$ are negative. However, in the general case the presence of jumps for $\eta^2(n)$ depends on the strategy, since not all indices j are included in the sums defining $\Delta\eta^2(n)$.

In §3.2 the behaviour of the processes $\eta^1_k(n)$, $\eta^2_k(n)$ and $\eta^3_k(n)$ will be considered for some special strategies.