6 APPLICATION OF PONTRYAGIN'S MAXIMUM PRINCIPLE TO CONTROL PROBLEMS WITH RANDOM JUMPS

6.1 The linear control problem with Poisson jumps

In Chapter 4 a problem of control of a random process $X(t)$ with an unknown random parameter θ was stated. An $I\!F$-predictable function $\beta(t)$ was called an action rule. A measure P_i^β on $\mathcal{F}_\infty^\theta$ was put in correspondence with each action rule β and each realization of $\theta = \theta_i$ $(i = 1, \ldots, N)$ and the measure $P_\xi^\beta := \sum_i \xi_i P_i^\beta$, for which the process $X(t) - \int_0^t p(\theta) \operatorname{diag} \beta(s) \, ds$ is an $I\!F^\theta$-martingale, was put in correspondence with the *a priori* distribution ξ. It was required to minimize the value of an additive functional depending on the process $X(t)$ and the unknown parameter θ for a fixed *a priori* distribution ξ.

Further, it was shown that this problem is equivalent to a control problem with complete data involving the process pair $X(t)$, $\xi(t)$, so that a measure on $\mathcal{F}_\infty$ (rather than on $\mathcal{F}_\infty^\theta$), for which the process $Y(t) := X(t) - \int_0^t p(\xi(s)) \operatorname{diag} \beta(s) \, ds)$ is an $I\!F$-martingale, was put in correspondence with the *a priori* distribution ξ and the action rule β and the process $\xi(t)$ satisfies a stochastic equation.

In the case of a Markov cost function which is independent of the values of the jumps of the process $X(t)$, it can be assumed that the measure is given simply on the trajectories of the process $\xi(t)$. An *action rule* $\beta := \beta(t)$, which is equivalent to a sequence of functions $\{\beta_n(j_{1,n-1}, t_{1,n-1}, t), \, n = 1, \ldots\}$ is conveniently replaced by a *strategy*, i.e. a sequence of functions $\{\alpha_n(t|t_{1,n-1}, \xi_{1,n-1}), \, n = 1, \ldots\}$. In the Markov case we can be restricted to *Markov* strategies, i.e. functions of the type $\alpha_n(t|t_{n-1}, \xi_{n-1})$.

In this chapter we will study the linear control problem with Poisson jumps, in which the behaviour of the controlled process $x(t)$ is similar to the behaviour of the process $\xi(t)$. Since the unknown

parameter is absent in the statement of this problem, the definition of action rule is not introduced here, and we consider only Markov strategies.

An interpretation and heuristic description of the linear control problem with Poisson jumps were presented in §1.10. According to this description such a problem is defined by the following elements:

(1) A system of differential equations describing the motion in the *state space* $\mathbb{R}^N$ of the point $x = (x_1, \ldots, x_N)$ in the intervals between jumps. We assume a *linear* equation with respect to the *controls* $\alpha := (\alpha^1, \ldots, \alpha^m)$ taking values in S^m, viz.

$$\dot{x} = \alpha A(x) + a^0(x) := f(\alpha, x), \qquad x \in \mathbb{R}^N, \qquad (6.1)$$

where $A(x) := \{a_i^j(x)\}$ is a matrix, $a^0(x)$ is a vector and $\dot{x}$ denotes the derivative with respect to s, $t_0 \le s \le \nu \le \infty$.

(2) A set of nonnegative functions $p(x) := \{p^1(x), \ldots, p^m(x)\}$, such that $p^j(x(s))\alpha^j(s)$ defines the local density of the probability of a jump of type j given that the system at time s is at the point $x(s)$ and the control $\alpha(s)$ is used.

(3) A set of single-valued maps $\Gamma(s) := (\Gamma^1(x), \ldots, \Gamma^m(x))$ of the space $\mathbb{R}^N$ into itself giving the change in location of the point in the event of a jump of jth type.

(4) A set of functions $q_n(t, x)$, $n = 1, 2, \ldots$, defining the cost incurred when the nth jump occurs at time t and after it the system arrives at state x.

The functions $A(x)$, $p(x)$, $q_n(t, x)$ and the maps $\Gamma^j(x)$ are assumed to be smooth. For reasons of simplicity, we assume that $a^0(x) :\equiv 0$.

The class of *admissible strategies* is the set of functions $\pi := \{\alpha_n(s|t, x), \ n = 1, 2, \ldots\}$, where for fixed n, t and x the function $\alpha_n(\cdot)$ is an element of the set $\mathcal{A}$ (see §4.4) depending in a measurable way on t, x. The function $\alpha_{n+1}(s|t, x)$ is called the *control* between the nth and $(n+1)^{\text{st}}$ jumps given that the nth jump occurred at time t to the point x.

The measure $P_{x_0}^\pi$ on the trajectories of the right continuous random process $x(s)$ corresponding to the *fixed* initial point x_0 and the strategy π is defined recursively as follows. If the nth jump occurs at time t

and the process takes the value $x = x(t)$ after the jump, after which no jump occurs up to moment $s > t$, then on the interval $[t, s)$ the process trajectory is deterministic and satisfies equation (6.1), where α is replaced by $\alpha_{n+1}(s|t, x)$ and the initial condition is $x(t) = x$. On the interval $(s, s + \Delta)$ the $(n+1)^{\text{st}}$ jump occurs with probability $\alpha_{n+1}^j(s|t, x)p^j(x(s))(\Delta + o(\Delta))$ and the process takes the value $\Gamma^j x(s)$, $j = 1, \ldots, m$, or with probability $1 - \sum_{j=1}^m \alpha_{n+1}^j(s|t, x)p^j(x(s))(\Delta + o(\Delta))$ the process continues its motion according to equation (6.1).

Since we consider the Markov case, the description of the measures given above corresponds to defining the conditional probabilities at jump moments of the process $x(t)$. Obviously these conditional measures can be defined under the assumption that the kth jump occurs not later than time t, the $(k+1)^{\text{st}}$ after time t and the process at time t took the value x. Such conditional measures do not depend on the strategies employed up to time t and they can be considered to be appropriate measures in the remaining model (see §2.4). In this chapter we will consider problems of the *maximization* of functionals of the type

$$F_{kn}^\pi(t, x) := E_x^\pi[\sum_{r=k+1}^n q_r(\tau_r, x(\tau_r)) \mid \tau_k \le t < \tau_{k+1}, x(t) = x], \quad (6.2)$$

where τ_r $(r = 1, 2, \ldots)$ is the time of the rth jump of the process $x(s)$, $\tau_0 = t_0$, $x(\tau_0) = x_0$ and $n \le \infty$. We consider given a *horizon* ν $(t_0 < \nu \le \infty)$, such that $q_r(t, x) := 0$ for $t > \nu$, $r = 1, 2, \ldots$, but we do not show the dependency on ν of the *cost function* $q_r(t, x)$ and the corresponding functional.

This problem can be reformulated in the terminology of Chapter 4 in the following way. On the probability space $(\Omega, \mathcal{F})$ we are given an m-variate counting process $X(t)$ generating the σ-algebras $\mathcal{F}_t$ with $\mathcal{F} := \mathcal{F}_\infty$. For each value of the parameter $x_0 \in \mathbb{R}^n$ and strategy $\pi = \{\alpha_n(s|t, x), \ n = 1, 2, \ldots\}$ there corresponds a measure $P_{x_0}^\pi$ on $\mathcal{F}_\infty$, an $\mathbb{F}$-predictable function $\beta(s)$ and a right continuous process $x(t)$, $t_0 \le t < \infty$, such that $X(t) - \int_{t_0}^t p(s(s)) \operatorname{diag} \beta(s) \, ds$ is $\mathbb{F}$-martingale, the process $x(t)$ satisfies equation

$$x(t) = x_0 + \int_{t_0}^t f(\beta(s), x(s)) \, ds + \int_0^t [\Gamma x(s_-) - x(s_-) d \, X^*(s), \quad (6.3)$$

and for the function $\beta(s)$ for any $n \geq 0$,

$$I[\tau_n < s \leq \tau_{n+1}]\beta(s) = \alpha_{n+1}(s|\tau_n, x(\tau_n))I[\tau_n < s \leq \tau_{n+1}]. \quad (6.4)$$

The problem of maximization of the number of successes up to time ν for the basic scheme is obtained if we put $k = 0$, $n = \infty$, $t_0 = 0$, $q_r(t, x) \equiv 1$ for $0 < t < \nu$ and take the appropriate p, f and Γ. The only difference from the basic scheme consists in the fact that for convenient presentation in (6.2) $q_r(\tau_r, x(\tau_r^-))$ of Chapter 4 is replaced by $q_r(\tau_r, x(\tau_r))$.

Similarly to §4.4, after transformation to a Markov model with discrete time, the question regarding the *existence* of an optimal strategy can be considered. However, in this chapter we are interested only in *necessary* conditions for optimality. Therefore, from the beginning we assume that there exists a uniformly optimal strategy and that the optimal value function is a measurable function.

6.2 Reduction of the initial formulation to a Pontryagin type problem and description of results

First we study the problem with a single jump, i.e. the problem of maximizing the functional $F_{01}^\pi(t, x)$. In such a problem the specification of a strategy reduces to the specification of a single function $\alpha_1(s|t, x)$.

Let $a := (t, x, \alpha(\cdot))$, where $\alpha(\cdot) \in \mathcal{A}$, and denote by $x(s|a)$ the solution of equation (6.1) with initial condition $x(t|a) = x$ and control $\alpha(\cdot)$. Let $z(s|a)$ be the *probability* that in the time interval $[t, s)$ *no jump occurs* given that at time t the process was in state x and the control $\alpha(\cdot)$ was applied. When it is obvious about which initial conditions and control we are speaking we will sometimes write simply $x(s)$, $z(s)$ or $z(s|t, x)$, $x(s|t, x)$. The jump probabilities are defined so that the function $z(s|a)$ satisfies the equation

$$\frac{dz(s|a)}{ds} = -z(s|a)\alpha(s)p^*(x(s|a)), \quad (6.5)$$

and the *probability density* that a jump of the jth type *occurs* in the small time interval $(s, s + ds)$ is given as $z(s|a)\alpha^j(s)p^j(x(s|a))$. Using

the total probability formula we find that the problem of maximization of the functional $F_{01}^\pi(t, x)$ reduces to the following. For fixed t and x, the control $\alpha(\cdot) \in \mathcal{A}$ must be chosen to maximize the functional

$$F^\alpha(t, x) := \int_t^\nu z(s|a) \sum_{j=1}^m \alpha^j(s)p^j(x(s|a))q_1(s, \Gamma^j(x(s|a)))\, ds, \quad (6.6)$$

and then it should be checked that optimal control can be given by some synthesis.

Thus, for fixed t and x we obtain a problem of optimal control with state variables z, x satisfying equations (6.1), (6.5), the integral functional (6.6) and fixed left and free right end points. Let us call this problem of optimal control with $q_1 = q$ the *problem B(q)*. For the solution of this type of problem the *Pontryagin maximum principle* (see Pontryagin *et al.* 1961; Boltyanski 1969), giving *necessary* conditions for optimality, is widely applied. Its formulation for $B(q)$ problems, differential equations for the conjugate variables and some statements concerning the optimal value function are given in §6.3.

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Now consider the problem with a finite number of jumps, i.e. the problem of maximization of the functional $F_{kn}^\pi(t, x)$, $0 \leq k < n < \infty$. The case $n - k = 1$ does not differ from the case $k = 0$, $n = 1$ discussed above, therefore assume that $n - k > 1$. Let

$$F_{kn}(t, x) := \sup_\pi F_{kn}^\pi(t, x).$$

Since we are considering only Markov strategies, then from the assumptions of the existence of a uniformly optimal strategy and the measurability of the optimal value function, we obtain directly the following optimality equation, viz.

$$F_{k-1,n}(t, x) = \sup_{\alpha(\cdot) \in \mathcal{A}} \int_t^\nu z(s|a) \sum_{j=1}^m \alpha^j(s)p^j(x(s|a))$$
$$\times \left[q_k(s, \Gamma^j x(s|a)) + F_{k,n}(s, \Gamma^j x(s|a)) \right]\, ds. \quad (6.7)$$

If the function $F_{k,n}(t, x)$ were continuously differentiable with respect to t and x (i.e. a *smooth* function), then we again would obtain a problem of the type $B(q)$, where $q = q_k(t, x) + F_{k,n}(t, x)$.

The optimality principle would then make possible the solution of the problem with a finite number of jumps by the sequential solution of problems with a single jump. In some problems (as, for example, in the symmetric case of the general scheme for $m = N = 2$ considered in §6.5), the smoothness of the function $F_{k,n}(t, x)$ can be proved based on a constructed synthesis which is obviously optimal.

It is well known (see, for example, Boltyanski 1969), that in general the optimal value function of control problems need not be smooth. However, the probabilistic special features of the problem $B(q)$ lead one to suspect the smoothness of the value function (see Assumption 6.1 and its discussion).

Derivation of the optimal control by the Pontryagin maximum principle is connected with the study of the *Hamiltonian* $\widetilde{\mathcal{H}}$. In the problems considered here the control enters the Hamiltonian $\widetilde{\mathcal{H}}$ linearly. We denote by zL_{kn}^j the coefficient of α^j in the Hamiltonian corresponding to the functional $F_{kn}^\pi(t, x)$. An important property in linear control problems with Poisson jumps consists in the fact that a definite relation exists between the functions L_{kn}^j and $L_{k-1,n}^j$ which can be used as a necessary condition of optimality. This relation was already presented in the solution of the basic scheme in Chapter 5. In §6.4, the corresponding statements are proved for the general case. This is one of the basic results of the present chapter.

In §6.5, as an example of the application of the maximum principle and the relations of §6.4, the solution of the problem of maximization of the probability that not less than k jumps occur in a fixed time (the problems B_k, $k = 1, 2, \ldots$) is given in the framework of the basic scheme for the case $m = N = 2$ and symmetric hypothesis. As a consequence, for this case the solution of the problem of maximization of the number of successes (minimization of loss) for a fixed time interval is once more obtained.

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We consider also the problem with an infinite number of jumps ($n = \infty$) and assume that $\nu < \infty$ and the functions $q_k(t, x)$ have the form $q_k(t, x) := \lambda^k q(t, x)$. The problem of maximization of the expected number of jumps ($\lambda := 1$, $q_k(t, x) :\equiv 1$) is a special case of this type of problem. If $F(t, x)$ is the optimal value of the functional in such a problem, then for $F(t, x)$ the following equality is obtained

from (6.7) by replacing $F_{kn}(\cdot)$ and $F_{k-1,n}(\cdot)$ by $F(\cdot)$:

$$
\begin{aligned}
F(t, x) = \sup_{\alpha(\cdot) \in \mathcal{A}} \int_t^\nu & z(s|a) \sum_{j=1}^m \alpha^j(s) p^j(x(s|a)) \left[\lambda q(s, \Gamma^j x(s|a)) \right. \\
& \left. + F(s, \Gamma^j x(s|a)) \right] ds.
\end{aligned}
\tag{6.8}
$$

If it is known that the function $F(t, x)$ is smooth, then the problem obtained can again be considered as a problem of optimal deterministic control in continuous time and the maximum principle can be used to write necessary conditions of optimality. It is possible to be in the situation in which the function $F(t, x)$ is unknown and we know only that it satisfies relation (6.8), but nevertheless a synthesis satisfying the necessary conditions of optimality can be constructed. It is also likely in this situation that if π is the strategy corresponding to this synthesis and the function

$$
\tilde{F}(t, x) := E_{x_0}^\pi \left[\sum_{k=1}^\infty \lambda^k q(\tau_k, x(\tau_k)) | \tau_1 > t, x(t) = x \right]
\tag{6.9}
$$

is smooth, then $\tilde{F}(t, x) = F(t, x)$. The discussion of this hypothesis and its connection with related statements is contained in §6.6.

6.3   The problem $B(q)$

According to the general theory of (deterministic) optimal control for the solution of the problem $B(q)$, with state variables z, x satisfying equations (6.1), (6.5) and functional (6.6), it is necessary to consider the *Hamiltonian*

$$
\begin{aligned}
\widetilde{\mathcal{H}}(\alpha, s, x, z, \phi, \tilde{\psi}) &:= z\alpha Q^*(s, x) - z\alpha p^*(x)\phi + \alpha A(x)\tilde{\psi}^* \\
&= \sum_{j=1}^m \alpha^j \left[zp^j(x)(q^j(s, x) - \phi) + \sum_{i=1}^N \tilde{\psi}_i a_i^j(x) \right] \\
&:\equiv \sum_{j=1}^m \alpha^j \tilde{L}^j(s, x, z, \phi, \tilde{\psi}) \\
&:= z \sum_{j=1}^m \alpha^j L^j(s, x, \phi, \tilde{\psi}/z),
\end{aligned}
\tag{6.10}
$$

where $\alpha := (\alpha^1, \ldots, \alpha^m)$, $Q(s,x) := (p^1(x)q^1(s,x), \ldots, p^m(x)q^m(s,x))$, $q^j(s,x) := q(s, \Gamma^j x)$ and ϕ and $\tilde{\psi} = (\tilde{\psi}_1, \ldots, \tilde{\psi}_m)$ are *conjugate variables*.

The conjugate variables ϕ and $\tilde{\psi}_i$ satisfy the following equations:

$$\dot{\phi} = -\frac{\partial \widetilde{\mathcal{H}}}{\partial z}, \qquad \dot{\tilde{\psi}}_i = -\frac{\partial \widetilde{\mathcal{H}}}{\partial x_i}.$$

Make the change of variables $\tilde{\psi}_i$ to $\psi_i := \tilde{\psi}_i / z$. Such a change is always possible because $z(s) > 0$ for all s. The equations for ϕ, ψ_i then have the form

$$\dot{\phi} = -\alpha Q^* + \phi \alpha p^*$$
$$\dot{\psi}_i = -\alpha \frac{\partial Q}{\partial x_i} + \phi \alpha \frac{\partial p^*}{\partial x_i} - \alpha \frac{\partial A}{\partial x_i} \psi^* + \alpha p^* \psi_i \quad i = 1, \ldots, N. \quad (6.11)$$

Since the boundary condition for the problem $B(q)$ has the form $F^\alpha(\nu, x) = 0$, then the *transversality condition* reduces to

$$\phi(\nu) = \psi_i(\nu) = 0. \quad (6.12)$$

If we define $\mathcal{H} := \widetilde{\mathcal{H}}/z$, then $\mathcal{H} := \mathcal{H}(\alpha, s, x, \phi, \psi)$ will not depend on z and in all subsequent formulations the variable z need not be mentioned.

If $a := (t, x, \alpha(\cdot))$, where $\alpha(s)$ is some control and (t, x) is the initial point, then the trajectory $x(t) := x(t|a)$ can be constructed (as the solution of equation (6.1)). Then we can substitute $\alpha(s)$ and $x(s|a)$ into (6.11) and solve equations (6.11), (6.12). As a result we obtain $\phi(s) := \phi(s|a)$ and $\psi(s) := \psi(s|a)$ corresponding to the control $\alpha(\cdot)$ and the initial point (t, x).

The maximum principle states that if $\alpha(s)$ is the optimal control for the problem $B(q)$ and $x(s)$ is the corresponding optimal trajectory, then there exists a nontrivial solution $\phi(s)$, $\psi(s)$ of equations (6.11), (6.12) such that at any time s, $t \le s \le \nu$, the *maximality condition*

$$\max_\alpha \mathcal{H}(\alpha, s, x(s), \phi(s), \psi(s)) = \mathcal{H}(\alpha(s), s, x(s), \phi(s), \psi(s)) \quad (6.13)$$

holds.

We formulate the set of conditions (Assumption 6.1) which will be used in the following section to derive the formula establishing the connection between coefficients of the Hamiltonians in two sequential problems (these conditions apply to the problem after the first jump).

Assumption 6.1

(a) There exists an optimal synthesis.

(b) The optimal value function $F(t, x)$ is smooth with respect to t and x.

(c) The equalities

$$F(t, x(t)) = \phi(t), \qquad \frac{\partial F}{\partial x_i}(t, x(t)) = \psi_i(t) \quad (6.14)$$

hold, where $x(t)$ is the optimal trajectory and $\phi(t)$ and $\psi_i(t)$ are the corresponding conjugate variables. ∎

In specific problems (see §6.5) it can sometimes be proved that Assumption 6.1 holds, which gives the possibility of constructing an optimal synthesis. There are examples of problems $B(q)$ for which (a) holds but (b) does not (the appropriate counter-example was shown by Zelikina 1985). At the same time we show in Lemma 6.1 that the first of the equalities of (6.14) holds for an arbitrary (not necessarily optimal) synthesis and for the corresponding $F^\alpha(t, x)$.

The question of conditions implying assumptions (a), (b) and (c) for the general case is open. It seems that (c) always follows from (b). We show also (Lemma 6.2) that from regularity of the optimal synthesis and continuous dependency of the trajectories on the initial point, continuous differentiability with respect to t of the function $F(t, x)$ follows.

Lemma 6.1 *For an arbitrary synthesis $\alpha(t, x)$, the value of the functional $F^\alpha(t, x)$ coincides with the value of $\phi(t)$ on the corresponding trajectories, i.e. $F^\alpha(t, x(t)) = \phi(t)$.*

Proof. We show that $F^\alpha(t, x)$ satisfies the same differential equation as ϕ on the corresponding trajectories. The statement of the lemma follows from this, since for $t = \nu$ we have $\phi(\nu) = F^\alpha(\nu, x(\nu)) = 0$. We

denote by $\alpha(\cdot|t,x)$ the control corresponding to the synthesis $\alpha(\cdot,\cdot)$ and the initial point (t,x), and by $R_{t,x}$ the operation of differentiation along a trajectory at the point (t,x). Notice that for the given synthesis $x(s|t,x) = x(s|t_1, x(t_1|t,x))$ and $\alpha(s|t,x) = \alpha(s|t_1, x(t_1|t,x))$ for $t < t_1 < s$. Therefore

$$R_{t,x}x(s|t,x) = 0, \qquad R_{t,x}\alpha(s|t,x) = 0. \qquad (6.15)$$

Differentiating the equation

$$F^\alpha(t,x) = \int_t^\nu z(s|t,x)\alpha(s|t,x)Q^*(s, x(s|t,x))\,ds$$

along the corresponding trajectory, taking account of (6.15), we obtain

$$R_{t,x}F^\alpha(t,x) = -\alpha(t,x)Q^*(t,x)$$
$$+ \int_t^\nu [R_{t,x}z(s|t,x)]\,\alpha(s|t,x)Q^*(s, x(s|t,x))\,ds. \quad (6.16)$$

Since, according to (6.5),

$$z(s|t,x) = \exp\left\{-\int_t^s \alpha(u|t,x)p^*(x(u|t,x))\,du\right\},$$

then by (6.15) we have

$$R_{t,x}z(s|t,x) = -\alpha(t,x)p^*(x)z(s|t,x).$$

From this and from (6.16) we obtain that

$$R_{t,x}F^\alpha(t,x) = -\alpha(t,x)Q^*(t,x) + \alpha(t,x)p^*(x)F^\alpha(t,x). \qquad (6.17)$$

So $F^\alpha(t,x)$ satisfies the same equation as $\phi(t)$. Therefore $\phi(t) \equiv F^\alpha(t, x(t))$. ∎

To prove the continuous differentiability of the function $F(t,x)$ with respect to t, we write the functional (6.6) as

$$F^\alpha(t,x) = \int_t^\nu \int_{\mathbb{R}^n} \mu^\alpha(ds, dy|t,x)q(s,y),$$

where $\mu^\alpha(ds, dy|t,x) := \mu^\alpha(s, dy|t,x)ds$ is a joint distribution for a jump moment and the position after the jump. By μ denote the corresponding distribution for the optimal synthesis.

Lemma 6.2 *If the optimal synthesis $\alpha^*(s,x)$ is regular and such that to close initial points correspond close trajectories (in the uniform norm), then the derivative of the value function $F(t,x)$ with respect to t exists, is continuous and is given by*

$$\frac{\partial F(t,x)}{\partial t} = \int_t^\nu \int_{\mathbb{R}^N} \mu(s, dy|t,x)\frac{\partial q(s,y)}{\partial s}\,ds$$
$$- \int_{\mathbb{R}^N} \mu(\nu, dy|t,x)q(\nu,y)dy. \qquad (6.18)$$

Proof. We denote by $\alpha^+ := \alpha_\Delta^+$ and $\alpha^- := \alpha_\Delta^-$ respectively the strategies corresponding to the following synthesis defined on the strips $\{s,x : t \le s \le \nu\}$ and $\{s,x : t + \Delta \le s \le \nu\}$, $\Delta > 0$,

$$\alpha_\Delta^+(s,x) := \begin{cases} \alpha^*(s + \Delta, x) & \text{if } t \le s < \nu - \Delta \\ \alpha^*(s,x) & \text{if } \nu - \Delta \le s \le \nu, \end{cases}$$
$$\alpha_\Delta^-(s,x) := \alpha^*(s - \Delta, x) \qquad \text{if } t + \Delta \le s \le \nu.$$

By the definition of the value function we have that

$$F(t,x) \ge F^{\alpha^+}(t,x), \qquad F(t + \Delta, x) \ge F^{\alpha^-}(t + \Delta, x).$$

From this we can assess the increment in the value function as

$$I_1 := F^{\alpha^-}(t + \Delta, x) - F(t,x)$$
$$\le F(t + \Delta, x) - F(t,x)$$
$$\le F(t + \Delta, x) - F^{\alpha^+}(t,x) := I_2.$$

Consider first I_1. We have

$$I_1 := \int_{t+\Delta} \int_{\mathbb{R}^N} \mu^{\alpha^-}(s, dy|t + \Delta, x)q(s,y)\,ds$$
$$- \int_t^\nu \int_{\mathbb{R}^N} \mu(s, dy|t,x)q(s,y)\,ds.$$

By the (temporal) autonomy of the system (6.1) and the definition of the strategy α_Δ^-, we have $\mu^{\alpha^-}(s + \Delta, dy|t + \Delta, x) = \mu(s, dy|t,x)$ and

therefore

$$I_1 := \int_t^{\nu-\Delta} \int_{\mathbb{R}^N} \mu(s, dy|t, x)(q(s+\Delta, y) - q(s, y))\, ds$$

$$- \int_{u-\Delta}^\nu \int_{\mathbb{R}^N} \mu(s, dy|t, x) q(s, y)\, ds.$$

From continuous differentiability of the cost function $q(s, y)$ it follows that after dividing by Δ the first integral in I_1 will converge as $\Delta \to 0$ to the first integral in (6.18). From the regularity of synthesis it follows that along the trajectory the function $\alpha(t)$ is continuous from the left and this means that $\mu(s, dy|t, x)$ is continuous from the left at s. Thus, after dividing by Δ the second integral in I_2 converges to the second integral in (6.18). Thus, $\lim I_1/\Delta$ coincides with the right-hand side of formula (6.18). Similar statements are obtained by considering the expression I_2. So, formula (6.18) is proved. The continuity of the right-hand side of (6.18) follows from the assumption made with respect to the optimal synthesis. ∎

6.4   Formula for the derivative of the Hamiltonian along trajectories

First we derive a formula for the derivatives of the functions $L^j(s, x, \phi, \psi)$ along trajectories of the system (6.1) (see formulae (6.10)–(6.12)) for some fixed control. In this section we assume that all controls (optimal and non-optimal) are given by synthesis. Let $\alpha(\cdot|t, x)$ be the control for the initial point (t, x) corresponding to a given synthesis. We define the functions $\phi(s, x)$ and $\psi(s, x)$ by the formulae $\phi(t, x) := \phi(t|a)$, $\psi(t, x) := \psi(t|a)$, where $a := (t, x, \alpha(\cdot|t, x))$. Then for any trajectory corresponding to this synthesis $\phi(t) = \phi(t, x(t))$, $\psi(t) = \psi(t, x(t))$, and by Lemma 6.1, $\phi(t, x) = F^\alpha(t, x)$. Correspondingly, set $L^j(s, x) := L^j(s, x, \phi(s, x), \psi(s, x))$. Then for any trajectory $L^j(t) = L^j(t, x(t), \phi(t), \psi(t)) = L^j(t, x(t))$ and we have

$$L^j(s) = p^j(x(s))q^j(s, x(s)) - p^j(x(s))\phi(s) + \sum_{i=1}^N \psi_i(s)a_i^j(x(s)). \quad (6.19)$$

For a function $f(s, x)$ we denote by $\dot{f}$ the derivative along a trajectory of system (6.1) and $\partial f/\partial l^k := \partial f/\partial t + \sum_{i=1}^N (\partial f/\partial x_i)a_i^k$. Then for a

smooth function f we have

$$\dot{f} = \sum_{k=1}^m \alpha^k \frac{\partial f}{\partial l^k}. \quad (6.20)$$

Differentiating (6.19) we obtain

$$\dot{L}^j = \sum_{k=1}^m \alpha^k \left[\frac{\partial p^j q^j}{\partial l^k} - \phi \frac{\partial p^j}{\partial l^k} + \sum_{i=1}^N \psi_i \frac{\partial a_i^j}{\partial l^k} \right] - p^j \dot{\phi} + \sum_{i=1}^N \dot{\psi}_i a_i^j. \quad (6.21)$$

From (6.11) it follows that

$$\sum_{i=1}^N \dot{\psi}_i a_i^j = -\sum_{k=1}^m \alpha^k \left[\frac{\partial p^k q^k}{\partial l^j} - \frac{\partial p^k q^k}{\partial t} - \phi \frac{\partial p^k}{\partial l^j} + \sum_{i=1}^N \psi_i \left(\frac{\partial a_i^k}{\partial l^j} - p^k a_i^j \right) \right]. \quad (6.22)$$

Substituting (6.22) and the expression for $\dot{\phi}$ in (6.11) in (6.21) we obtain

$$\dot{L}^j - \alpha p^* L^j = \sum_{k=1}^m \alpha^k \left[p^k \frac{\partial q^k}{\partial t} + \phi \left(-\frac{\partial p^j}{\partial l^k} + \frac{\partial p^k}{\partial l^j} \right) + \sum_{i=1}^N \psi_i \left(\frac{\partial a_i^j}{\partial l^k} - \frac{\partial a_i^k}{\partial l^j} \right) \right.$$

$$\left. + \left(\frac{\partial p^j q^j}{\partial l^k} - \frac{\partial p^k q^k}{\partial l^j} + p^j p^k [q^k - q^j] \right) \right]. \quad (6.23)$$

We denote the items in round brackets in formula (6.23) by r^{jk}, p_i^{jk}, s^{jk} respectively. It is obvious that $r^{jk} = -r^{kj}$, $p_i^{jk} = -p_i^{kj}$, $s^{jk} = -s^{kj}$, and therefore $\sum_{j=1}^m \sum_{k=1}^m \alpha^k \alpha^j r^{jk} = 0$, and so on. From this, premultiplying (6.23) by α^j and summing on j, we obtain

$$\sum_{j=1}^m \alpha^j (\dot{L}^j - \alpha p^* L^j) = \sum_{k=1}^m \alpha^k p^k \frac{\partial q^k}{\partial t}. \quad (6.24)$$

If all the $L^j(s, x)$ are smooth functions, then (6.24) can be written in the form

$$\sum_{j=1}^m \alpha^j \sum_{k=1}^m \alpha^k \left(\frac{\partial L^j}{\partial l^k} - p^k L^j - p^j \frac{\partial q^j}{\partial t} \right) = 0.$$

Consider now the case where the number of jumps equals two or more. Then (see (6.7)) it can be assumed that the problem considered has