a single jump with the profit function $q(t,x)$ given by the sum of the profit for first jump $q_1(t,x)$ and the value function $\overline{F}(t,x)$ for a problem with a single jump and some profit $\overline{q}(t,x)$.

All notation for the problem with profit $\overline{q}$ will be the same as in the previous sections of this chapter except for the bar above each symbol. We will assume that the problem with profit $\overline{q}$ satisfies Assumption 6.1, so that $\overline{\phi}(t,x) = \overline{F}(t,x)$, $\overline{\psi}_i(t,x) = \partial\overline{F}(t,x)/\partial x_i$, $q(t,x) = \overline{\phi}(t,x) + q_1(t,x)$ and $\partial q(t,x)/\partial x_i = \overline{\psi}_i(t,x) + \partial q_1(t,x)/\partial x_i$. Let $\overline{L}^j(t,x)$ corresponding to an optimal synthesis $\overline{\alpha}$ be the coefficient of $\overline{\alpha}^j$ in the Hamiltonian $\overline{\mathcal{H}}$. For the trajectory $y(t)$ corresponding to $\overline{\alpha}$ we have

$$\dot{q}(t,y(t)) = \dot{\overline{\phi}} + \dot{q}_1 = \sum_{k=1}^{m} \overline{\alpha}^k \left(p^k\overline{\phi} - p^k\overline{q}^k + \frac{\partial q_1}{\partial l^k}\right).$$

Subtracting from this equality the following equality

$$\sum_{k=1}^{m} \overline{\alpha}^k \left(\frac{\partial q}{\partial l^k} - \frac{\partial q}{\partial t}\right) = \sum_{k=1}^{m} \overline{\alpha}^k \left[\sum_{i=1}^{N} \overline{\psi}_i a_i^k + \frac{\partial q_1}{\partial l^k} - \frac{\partial q_1}{\partial t}\right],$$

and taking into account (6.19) and (6.20), we obtain

$$\frac{\partial q(t,y)}{\partial t} = -\sum_{k=1}^{m} \overline{\alpha}^k(t,y)\left[\overline{L}^k(t,y) - \frac{\partial q_1(t,y)}{\partial t}\right]. \tag{6.25}$$

Substituting formula (6.25) in (6.24) for the points $\Gamma^j x(t)$, $j = 1,\dots,m$, we obtain the formula connecting the coefficients $L^j$ and $\overline{L}^j$ for the two problems as

$$\begin{aligned}\sum_{j=1}^{m} \alpha^j \left[\dot{L}^j(t,x(t)) - \alpha p^*(x,t)L^j(t,x(t))\right] &= -\sum_{j=1}^{m} \alpha^j p^j(x(t)) \sum_{k=1}^{m} \overline{\alpha}^k(t,\Gamma^j x(t)) \\ &\quad\times\left[\overline{L}^k(t,\Gamma^j x(t)) - \frac{\partial q_1}{\partial t}(t,\Gamma^j x(t))\right].\end{aligned} \tag{6.26}$$

In studying the problem of maximization of the number of successes in the basic scheme with two devices and two hypotheses (i.e.



$m = N = 2$) and in the solution of the problem $B_k$, the formula for the derivative of the function $L := L^1 - L^2$ is interesting. For the case of the basic scheme it is not difficult to check that $r^{jk}$ and $p_i^{jk}$ in (6.23) equal 0 for any $m$ and $N$. Therefore, it follows from (6.23) that

$$\begin{aligned}\dot{L} - \alpha p^* L &= p^1\frac{\partial q^1}{\partial t} - p^2\frac{\partial q^2}{\partial t} + \left(\frac{\partial}{\partial l^2} - \frac{\partial}{\partial t}\right)p^1q^1 \\ &\quad - \left(\frac{\partial}{\partial l^1} - \frac{\partial}{\partial t}\right)p^2q^2 + p^1p^2(q^2 - q^1).\end{aligned} \tag{6.27}$$

We notice that, in contrast to Chapter 5, the derivatives along the trajectory are considered in direct time, therefore now $\partial/\partial l^j = (\partial/\partial t - \delta^j)(\partial/\partial y)$. If Assumption 6.1 holds for the problem obtained after the first jump, then using (6.25) for the points $y = \Gamma^1 x$ and $y = \Gamma^2 x$, taking into account that in the basic scheme $q_1(t,x) \equiv c$, we obtain

$$\begin{aligned}&\dot{L}(s) - \alpha(s)p^*(x(s))L(s) \\ &= -\sum_{j=1}^{2} p^j(x(s))\overline{\alpha}^j(s,\Gamma^j x(s))\overline{L}(s,\Gamma^j x(s)) \\ &\quad + \sum_{j=1}^{2}(-1)^j\Big[p^j(x(s))\overline{L}^{3-j}(s,\Gamma^j x(s)) \\ &\quad + \left(\frac{\partial}{\partial l^j} - \frac{\partial}{\partial t}\right)(p^{3-j}q^{3-j}) + p^1p^2q^j\Big].\end{aligned} \tag{6.28}$$

We show that the second sum converts to 0. For $\overline{L}^j(s,\Gamma^j x(s))$, we can write the following formula, similar to (6.19), using the facts that $\overline{\phi}(t) = q(t,x(t)) - c$ and $\overline{\psi}_i = \partial q(t,x(t))/\partial x_i$,

$$\overline{L}^j(s,y) = p^j(y)\overline{q}^j(s,y) - p^j(y)(q(s,y) - c) + \left(\frac{\partial}{\partial l^j} - \frac{\partial}{\partial t}\right)q(s,y).$$

For the case of the basic scheme we have the equalities

$$p^1(x)p^2(\Gamma^1 x) = p^2(x)p^1(\Gamma^2 x), \quad \Gamma^1\Gamma^2 x = \Gamma^2\Gamma^1 x,$$

$$-\frac{\partial p^i(y)}{\partial l^j} + p^1(y)p^2(y) - p^j(y)p^i(\Gamma^j y) = 0, \quad i \neq j, \tag{6.28a}$$

$$\frac{\partial q^j(s,x)}{\partial l^i} = \frac{\partial q(s,\Gamma^j x)}{\partial l^i}.$$

These equations are checked directly, and it is convenient to conduct the check in $\eta$ coordinates (see also formula (5.64)).

Substituting the expression for $\overline{L}^j$ at the corresponding points in the second sum in (6.28) and using the equalities given above, we obtain the following equality:

$$\dot{L}(s) - \alpha(s)p^*(x(s))L(s) = -\sum_{j=1}^{2} p^j(x(s))\overline{\alpha}^j(s,\Gamma^j x(s))\overline{L}(s,\Gamma^j x(s)). \qquad (6.29)$$

Recall that $x(t)$ and $\alpha^j := \alpha^j(t,x(t))$ correspond to an arbitrary synthesis in the problem $B(q)$ with $q := q(t,x) := q_1(t,x) + \overline{F}(t,x)$, where $\overline{F}(t,x)$ is the value function satisfying Assumption 6.1 in the problem $B(\overline{q})$ and $\overline{\alpha}(t,x)$ is the optimal synthesis for this problem.

**Remark 6.1** Formula (6.29) can be obtained without considering the Pontryagin problem, but simply using the optimality equation. Similar formulae were obtained in §5.3 (see (5.65)) in exactly this way. ■

## 6.5 Optimal syntheses for the case of symmetric hypotheses

In this section we will obtain the solution of problems $B_k$, $k = 1, 2, \ldots$, and as a consequence, the solution of the problem of maximization of the number of successes in a finite time interval. For the case of the basic scheme on the time interval $[0,\nu)$ we call the *problem* $B_k$ the problem of maximizing the probability that the $k^{\text{th}}$ jump occurs before time $\nu$, or, which is the same, that before time $\nu$ not less than $k$ jumps occur. Formally, the function $q_r$ has the form

$$q_i(s,x) :\equiv 0, \qquad i = 1,\ldots,k-1,$$

$$q_k(s,x) :\equiv \begin{cases} 1 & \text{if } s \le \nu, \\ 0 & \text{if } s > \nu. \end{cases}$$

As shown in §6.2, the solution of this problem reduces to the solution of a sequence of Pontryagin type problems and to a proof of the continuous differentiability of the value functions obtained here. The corresponding Pontryagin problems we will also call problems $B_k$.



For this problem $\lambda_1^1 = \lambda_2^2 = \lambda^1$, $\lambda_1^2 = \lambda_2^1 = \lambda^2$, $\delta^2 = -\delta^1 = \lambda^2 - \lambda^1 > 0$. The differential equations for the state variables are of the following type (see (4.28),(5.14))

$$\dot{\xi} = -\xi(1-\xi)[\delta^1\alpha + \delta^2(1-\alpha)],$$

or, in logarithmic coordinates $\eta := \tilde{\eta}(\xi) := \ln[\xi/(1-\xi)]$,

$$\dot{\eta} = -[\delta^1\alpha + \delta^2(1-\alpha)] = \delta^1(1-2\alpha). \qquad (6.30)$$

The function $z(s)$ defining the probability of no jump satisfies the equation

$$\dot{z} = -z[\alpha\tilde{p}^1(\eta) + (1-\alpha)\tilde{p}^2(\eta)], \qquad (6.31)$$

where

$$\tilde{p}^2(-\eta) = \tilde{p}^1(\eta) := p^1(\tilde{\xi}(\eta)) = \lambda^1\tilde{\xi}(\eta) + \lambda^2(1-\tilde{\xi}(\eta)) = (\lambda^1 e^\eta + \lambda^2)/(e^\eta + 1) \qquad (6.32)$$

Moreover $\Gamma^1\eta = \eta + \gamma^1$, $\Gamma^2\eta = \eta + \gamma^2$, where $\gamma^2 = -\gamma^1 = \ln \lambda^2/\lambda^1 > 0$. The criterion functional in the Pontryagin problem $B_k$ has the form

$$F_k^\alpha(\tau,\eta) = \int_t^\nu z(s)\sum_{j=1}^{2}\alpha^j(s)\tilde{p}^j(\eta(s))F_{k-1}(s,\eta(s)+\gamma^j)ds, \qquad (6.33)$$

where $F_0 :\equiv 1$, $F_{k-1}(s,\eta) := \sup_{\alpha\in i} F_{k-1}^\alpha(s,\eta)$, and controls $\alpha = \alpha(\cdot)$ are considered to be measurable functions defined on $[t,\nu]$ such that $0 \le \alpha(\cdot) \le 1$. We denote by $\alpha^*(s,\eta)$ the following synthesis

$$\alpha^*(s,\eta) := \alpha^*(\eta) := \begin{cases} 1 & \text{if } \eta < 0 \\ 1/2 & \text{if } \eta = 0 \\ 0 & \text{if } \eta > 0 \end{cases} \qquad (6.34)$$

and describe the regions $G_k$ and $\overline{G}_k$ $(k = 1, 2, \ldots)$ introduced in the statement of the theorem given below, viz.

$$\overline{G}_k := \{(t,\eta) : t \ge 0, |\eta| \le \eta_k(t)\}$$
$$G_k := \{(t,\eta) : t > 0, (t,\eta) \in \overline{G}_k\} \qquad (6.35)$$

where $\eta_k(t) := \delta^2 t + (k-1)\gamma^1$. Notice that similar regions were considered in §5.3 for the case of an arbitrary $2 \times 2$ matrix.

**Theorem 6.1** *The following statements hold for the problem $B_k$:*

*(a) The synthesis $\alpha^*(\eta)$ defines an optimal control (which is unique in the region $(\nu - t, \eta) \in G_k$).*

*(b) For $(\nu - t, \eta) \in \overline{G}_k$, any control is optimal up to the time of exit from this region, and the optimal value function for $(\nu - t, \eta) \in \overline{G}_k$ is given by*

$$F_k(t,\eta) = 1 - F_k(t)e^{\eta/2}/(1+e^{\eta}), \tag{6.36}$$

*where $F_0(s) :\equiv 0$, and for $k \geq 1$*

$$\begin{aligned} F_k(t) := {} & \sqrt{\lambda^1\lambda^2}\int_t^{\nu+(k-1)\gamma^1} \exp\{-(\lambda^1+\lambda^2)(s-t)/2\}F_{k-1}(s)\,ds \\ & + 2\exp\{-(\lambda^1+\lambda^2)(\nu+(k-1)\gamma^1-t)/2\} \\ & \times [1 - F_{k-1}(\nu+(k-1)\gamma^1, 0)]. \end{aligned} \tag{6.37}$$

*(c) The value function $F_k(t,\eta)$ is a continuously differentiable function and $F_k(t,\eta) = F_k(t,-\eta)$.* ■

**Corollary 6.1** *The control given by the synthesis $\alpha^*(\eta)$ is the unique optimal control in the symmetric problem of maximization of the number of successes for horizon $\nu < \infty$.*

**Proof.** Indeed, $\alpha^*(\eta)$ defines the unique optimal control for each problem $B_k$ in the whole strip $\{(t,\eta) : 0 \leq t \leq \nu\}$. Therefore, from the equality $E\xi = \sum_{k=1}^{\infty} P\{\tau_k \leq \nu\}$, where $\xi$ is the number of jumps up to moment $\nu$, and $\tau_k$ is the time of the $k^{\text{th}}$ jump, it follows that $\alpha^*$ defines the unique optimal control in the problem of the maximization of the number of jumps also. ■

**Proof** (of Theorem 6.1). The proof is conducted by induction in which the maximum principle and formula (6.29) derived in the previous section have important rôles. Let $\tilde{L}_k(s,\eta,z\phi,z\psi) := zL_k(s,\eta,\phi,\psi)$ be the coefficient of $\alpha$ in the Hamiltonian $\tilde{\mathcal{H}}$ for the problem $B_k$; the function $\tilde{L}_k(s) = z(s)L_k(s) = z(s)L_k(s,\eta,\phi(s),\psi(s))$ corresponds to some fixed control $\alpha(s)$ and some initial point and $L_k(s,\eta) :=$



$L_k(s,\eta,\phi(s,\eta),\psi(s,\eta))$, where $\phi(s,\eta)$, $\psi(s,\eta)$ correspond to the synthesis $\alpha^*(\eta)$.

From the maximum principle it follows immediately that if $\tilde{L}_k(s) > 0$ then for optimal control $\alpha(s) = 1$, and if $L_k(s) < 0$, then $\alpha(s) = 0$.

We denote by $\tilde{L}_k(s|t,\eta)$ the value function $\tilde{L}_k(s)$ corresponding to the control $\alpha^*(\eta(s))$ and to the initial point $(t,\eta)$.

It will be shown by induction that in addition to statements (a), (b) and (c) of the theorem, the following statements also hold:

(d) Assumption 6.1 holds for problem $B_k$.

(e) $L_k(s,\eta) = -L_k(s,-\eta)$

$$= \begin{cases} 0 & \text{if } (\nu - s, \eta) \in \overline{G}_k \cup \{\eta = 0\} \\ < 0 & \text{if } (\nu - s,\eta) \in G_k, \quad \eta > 0, \quad s < \nu \end{cases}$$

$$L_1(\nu,\eta) < 0 \ \text{ for } \eta > 0, \ \ L_k(\nu,\eta) = 0 \ \text{ for } \eta > 0, \ \ k > 1.$$

(f) $\tilde{L}_k(s|t,\eta)$ (and in particular, $L_k(t,\eta) := \tilde{L}_k(t|t,\eta)$) is decreasing with increasing $\eta$ for $(\nu - s, \eta) \in G_k$, where $s \leq \nu$ for $k = 1$ and $s < \nu$ for $k > 1$.

Assume that statements (a)–(f) hold for every $k < r$ and consider the problem $B_r$, $r \geq 1$. From statement (d) for $k = r - 1$ it follows that formula (6.29) can be applied and has the form

$$\begin{aligned} \dot{\tilde{L}}_r(s)/z(s) &= \dot{L}_r(s) + \alpha\tilde{p}^*(\eta(s))L_r(s) \\ &= -\tilde{p}^1(\eta(s))\alpha^*(\eta(s)+\gamma^1)L_{r-1}(s,\eta(s)+\gamma^1) \\ &\quad - \tilde{p}^2(\eta(s))(1-\alpha^*(\eta(s)+\gamma^2))L_{r-1}(s,\eta(s)+\gamma^2) \\ &:= X_{r-1}(s,\eta(s)). \end{aligned} \tag{6.38}$$

First we prove that

$$\dot{\tilde{L}}_1(s) \equiv 0, \tag{6.39}$$

and for $r > 1$

$$\dot{\tilde{L}}_r(s) = \begin{cases} 0 & \text{if } (\nu - s, \eta(s)) \in \overline{G}_r \cup \{\eta(s) = 0\} \\ < 0 & \text{if } (\nu - s, \eta(s)) \in G_r, \quad \eta(s) < 0 \\ > 0 & \text{if } (\nu - s, \eta(s)) \in G_r, \quad \eta(s) > 0. \end{cases} \tag{6.40}$$

Indeed, the relation (6.39) is obtained from (6.27), (6.28a) and the fact that in this case $q^2 := q^1 := 1$. Moreover, from $(\nu - s, \eta) \in \overline{G}_r$ it follows that $(\nu - s, \eta + \gamma^j) \in \overline{G}_{r-1}$, and by (6.38) and statement (e) for $k = r - 1$ we obtain that $d\tilde{L}_r(s)/ds \equiv 0$ for $(\nu - s, \eta(s)) \in \overline{G}_k$. Finally, from (e) and (f) for $k = r - 1$ and the monotonicity of $\tilde{p}^1(\eta)$ and $\tilde{p}^2(\eta)$, it follows that for $(\nu - s, \eta) \in G_k$, the function $X_{r-1}(s, \eta)$ (see (6.38)) is strictly increasing with respect to $\eta$ for fixed $s$ and $X_{r-1}(s, 0) = 0$. Since $L_{r-1}(s, \eta) = -L_{r-1}(s, -\eta)$, the last equality holds for all $s$, not only in $G_k$. From this (6.40) follows.

Consider statement (a). The existence of an optimal control was proved in §4.4 and follows also from the general results of the theory of optimal control.

We show that $\alpha^*$ defines the unique control in region $G_k$. For $r = 1$ with any control, the trajectory passing through the point $(\nu - s, \eta(s)) \in G_1$, $\eta(s) > 0$, arrives at $s = \nu$ at some point $\eta(\nu) > 0$. Since by (6.19) and (6.12), $\tilde{L}_1(\nu) = z(\nu)[\tilde{p}^1(\eta(\nu)) - \tilde{p}^2(\eta(\nu))]$, then $\tilde{L}_1(\nu) < 0$ for $\eta(\nu) > 0$, and this means, using (6.39), that $\tilde{L}_1(s) = \tilde{L}_1(\nu) < 0$ for the trajectory considered. Therefore, by the maximum principle, the optimal control is given by $\alpha(s) \equiv 0$ for $(\nu - s, \eta(s)) \in G_1$, $\eta(s) > 0$. Similarly, $\alpha(s) \equiv 1$ for $(\nu - s, \eta(s)) \in G_1$, $\eta(s) < 0$.

We consider next the case $r > 1$. Let a trajectory corresponding to an optimal control $\tilde{\alpha}(\cdot)$ pass through the point $(\nu - s, \eta(s)) \in G_r$, $\eta(s) > 0$, and let $\tilde{L}_r(s) \geq 0$. Denote by $\tau$ the exit time of this trajectory from the region $G'_r := \{(s, \eta) : (\nu - s, \eta) \in G_r, \eta > 0\}$. From (6.40) it follows that $\tilde{L}_r(v) > 0$ holds for all $v$ such that $s < v < \tau$, and therefore it follows from the maximum principle that $\tilde{\alpha}(v) = 1$ for this $v$. From this, by equation (6.30), $\eta(s)$ is monotonically increasing and $\eta(\tau) > 0$, and this means that $\tau = \nu$. But $L_r(\nu, \eta(\nu)) = 0$, because $F_{r-1}(\nu, \eta) = 0$, $\phi(\nu) = 0$, $\psi(\nu) = 0$. This is a contradiction and thus $L_r(s, \eta(s)) < 0$ and $\tilde{a}(\eta(s)) = 0 = \alpha^*(\eta(s))$ for $(\eta - s, \eta(s)) \in G'_r$.

Similarly, we obtain that in the region $G''_r := \{(s, \eta) : (\nu - s, \eta) \in G_r, \eta < 0\}$ the optimal control also coincides with $\alpha^*$. From this it follows that on the points of a trajectory belonging to $G'_r$ (correspondingly $G''_r$), $\eta(s)$ is monotonically decreasing (increasing). From this fact it follows immediately that the optimal trajectory passing through the point $(\nu - s, 0)$, $\nu - s < (r - 1)|\gamma^1|$ cannot reach the regions $G'_r$, $G''_r$, consequently $\eta(v) = 0$ for all $v > s$ and the corresponding optimal control $\tilde{\alpha}(v) = 1/2 = \alpha^*(\eta(v))$. Thus, it is proved that the synthesis $\alpha^*(\eta)$ defines the unique optimal control in the region $G_r$.



Consider the proof of statement (b) for $k = r \geq 1$. From this the optimality of $\alpha^*(\eta)$ in the region $\overline{G}_r$ will be proved, which completes the proof of statement (a) for $k = r$.

Consider an initial point $(t, \eta)$ such that $(\nu - t, \eta) \in \overline{G}_r$ and a control which is arbitrary up to the time of exit from the region $\overline{G}_r$ and afterwards coincides with the optimal control. From the optimality of $\alpha^*(\eta)$ in $G_r$ proved above, it follows that the exit time from $\overline{G}_r$ coincides with $t_r = \nu + (r - 1)\gamma^1$ and $\eta(t_r) = 0$.

To prove statement (b) it is sufficient to show that for the control under consideration

$$F_r^\alpha(t, \eta) = 1 - F_r(t)e^{\eta/2}/(1 + e^\eta), \tag{6.41}$$

where $F_r(t)$ is defined in (6.37).

Since $F_0(t, \eta) :\equiv 1$, (6.41) will hold for $r = 0$ if we set $F_0(t) :\equiv 0$.

From (6.33) and the equality $z(s|t, \eta) = z(t_r|t, \eta)z(s|t_r, \eta(t_r))$, which follows from (6.31), we have, in light of the optimality of $\alpha^*(\eta)$ for $s > t_r$ and the equality $\eta(t_r) = 0$ $(r \geq 1)$

$$\begin{aligned} F_r^\alpha(t, \eta) = &\int_t^{t_r} z(s) \sum_{j=1}^{2} \alpha^j(s)\tilde{p}^j(\eta(s))F_{r-1}(s, \eta(s) + \gamma^j)\, ds \\ &+ z(t_r|t, \eta)F_r(t_r, 0). \end{aligned} \tag{6.42}$$

By direct substitution in (6.31) using (6.30) it is easy to check that

$$\begin{aligned} &z(s|t, \eta) \\ &= [(e^{\eta(s)} + 1)/(e^\eta + 1)] \exp\{-\frac{1}{2}[\eta(s) - \eta + (\lambda^1 + \lambda^2)(s - t)]\}, \\ &\qquad \eta(s) := \eta(s|t, \eta). \end{aligned} \tag{6.43}$$

From $(\nu - s, \eta) \in \overline{G}_r$ it follows that $(\nu - s, \eta + \gamma^j) \in \overline{G}_{r-1}$ and this means that formula (6.36) can be used for $F_{r-1}(s, \eta(s) + \gamma^j)$, which gives

$$\begin{aligned} &F_{r-1}(s, \eta(s) + \gamma^j) \\ &= 1 - [\tilde{p}^j(\eta(s))]^{-1}\sqrt{\lambda^1\lambda^2}\, e^{\eta(s)/2}F_{r-1}(s)/(e^{\eta(s)} + 1). \end{aligned} \tag{6.44}$$

Substituting (6.44) and (6.43) into (6.42), we have

$$F_r^\alpha(t,\eta) = \int_t^{t_r} z(s|t,\eta)\sum_{j=1}^{2}\alpha^j(s)\tilde{p}^j(\eta(s|t,\eta))\,ds$$
$$-\frac{e^{\eta/2}}{e^\eta+1}\int_t^{t_r}\sqrt{\lambda^1\lambda^2}\exp\{-(\lambda^1+\lambda^2)(s-t)/2\}F_{r-1}(s)\,ds$$
$$+ z(t_r|t,\eta)F_r(t_r,0). \tag{6.45}$$

According to (6.31), the first integral in (6.45) equals $1 - z(t_r|t,\eta)$. Substituting this equality into (6.45) and then using (6.43) for $s = t_r$ and then the fact that $\eta(t_r) = 0$, we obtain formula (6.41), which completes the proof of statements (a) and (b).

Consider statement (c). The equality $F_r(t,\eta) = F_r(t,-\eta)$ follows from the proven symmetry of the optimal synthesis, the relation $\tilde{p}^1(\eta) = \tilde{p}^2(-\eta)$ and formula (6.33).

To prove the continuous differentiability of $F_r(t,\eta)$ it suffices to consider $\eta \geq 0$. We substitute an optimal control in (6.33) and use the fact that $\tilde{p}^1(0) = \tilde{p}^2(0)$, $F_{r-1}(s,\gamma^1) = F_{r-1}(s,\gamma^2)$ and $\eta(s|t,\eta) \geq 0$ if $\eta > 0$ to obtain that the function under the integral sign equals $z(s)\tilde{p}^2(\eta(s))F_{r-1}(s,\eta(s)+\gamma^2)$. Denote by $\tau(t,\eta)$ the time the line $\eta = 0$ is reached by a trajectory which exits from the point $(t,\eta)$. The functions $z(s|t,\eta)$ and $\eta(s|t,\eta)$ have continuous bounded one-sided derivatives with respect to $t$ and $\eta$ which coincide for all $s$ except at time $\tau(t,\eta)$ and the function $F_{r-1}(s,\eta)$ is continuously differentiable by the assumed truth of statement (c) for $k = r-1$. (If $r = 1$, then $F_0(t,\eta) \equiv 1$.) Therefore, we can differentiate under the integral sign and the function so obtained is continuous as a function of $t$ and $\eta$ for $\eta \geq 0$.

Using the equalities $(\partial/\partial\eta)\eta(s|t,\eta) = 0$, valid for $s > \tau(t,\eta)$, and $(\partial^+/\partial\eta)z(s|t,\eta) = 0$, we obtain that $(\partial^+/\partial\eta)F_r(t,\eta) = 0$. Thus, the continuous differentiability of $F_r(t,\eta)$ has been proved, and this completes the proof of statement c) for $k = r$.

Consider statement (d). The coincidence of $\phi(t,\eta)$ and $F_r(t,\eta)$ was proved in Lemma 6.1, therefore to check statement (d) for $k = r$ it remains to prove the coincidence of $\psi_r(t,\eta)$ and $(\partial F_r/\partial\eta)(t,\eta)$.

Notice that for the part of the optimal trajectory on the line $\eta = 0$, we have $(\partial/\partial\eta)F_r(t,\eta) = 0$, $\psi_r(t,\eta) = 0$ (the last equality is obtained immediately from the equation for $\psi(t)$ and the boundary condition $\psi(\nu) = 0$). Therefore, it is sufficient to show that for $\eta > 0$ the function $(\partial/\partial\eta)F_r(t,\eta)$ satisfies the same equation as $\psi(t)$ on optimal trajectories. The optimal trajectories here have the form $\eta(t) = \eta_0 - \delta^2 t$ for $\eta(t) > 0$.

From the equality $\phi(t,\eta) = F_r(t,\eta)$ and equation (6.11) for $\phi$ we find that

$$\left(\frac{\partial}{\partial t} - \delta^2\frac{\partial}{\partial\eta}\right)F_r(t,\eta) = \tilde{p}^2(\eta)F_{r-1}(t,\eta) - \tilde{p}^2(\eta)F_{r-1}(t,\eta-\gamma^2) \tag{6.46}$$

holds for $\eta > 0$ (see also (6.17)). The right-hand side of this relation is continuously differentiable with respect to $\eta$. If we consider $\partial/\partial l^2 = \partial/\partial t - \delta^2(\partial/\partial\eta)$ as the derivative in the direction $\eta = -\delta^2 t$ in the $(t,\eta)$ plane, then from existence and continuity of the repeated derivative $(\partial/\partial\eta)(\partial/\partial l^2)F_r(t,\eta)$ it follows that the other repeated derivatives exist and the equality $(\partial/\partial l^2)(\partial/\partial\eta)F_r(t,\eta) = (\partial/\partial\eta)(\partial/\partial l^2)F_r(t,\eta)$ holds. Using this equality and differentiating (6.46) with respect to $\eta$, we obtain that for $r \geq 1$ the function $(\partial/\partial\eta)F_r(t,\eta)$ satisfies the same differential equation as $\psi(t)$ on optimal trajectories. Therefore, statement (d) is proved.

Consider statement (e). The boundary conditions for $L_r(s,\eta)$ at $s = \nu$, $r \geq 1$ are obtained from (6.12), (6.19) and the nature of the function $q(s,\eta)$. Statement (e), for $k = r$ and $s < \nu$, follows immediately from the proven relations (6.40), equality $\tilde{L}_r(s) = z(s)L_r(s,\eta(s))$ and the above boundary condition for $L_r(s,\eta)$. Here the equality $L_r(s,\eta) = -L_r(s,-\eta)$ follows from the proven symmetry with respect to the line $\eta = 0$.

Considering the proof of statement (f), we assume without loss of generality that $\eta > 0$. As before, we denote $\tau(t,\eta)$ as the time the line $\eta = 0$ is hit by a trajectory starting at the point $(t,\eta)$. Then by statement (e) for $\eta = 0$ and formulae (6.38) we have

$$-\tilde{L}_r(s|t,\eta)$$
$$= \int_s^{\tau(t,\eta)} \dot{\tilde{L}}_r(v|t,\eta)\,dv = \int_s^{\tau(t,\eta)} [D_1(v|t,\eta) - D_2(v|t,\eta)]\,dv, \tag{6.47}$$

where

$$D_1(v|t,\eta) := -z(v|t,\eta)\tilde{p}^1(\eta(v|t,\eta))L_{r-1}(v,\eta(v|t,\eta)+\gamma^1) \times \alpha^*(\eta(v|t,\eta)+\gamma^1), \quad (6.48)$$

$$D_2(v|t,\eta) := z(v|t,\eta)\tilde{p}^2(\eta(v|t,\eta))L_{r-1}(v,\eta(v|t,\eta)+\gamma^2) \times [1-\alpha^*(\eta(v|t,\eta)+\gamma^2)]. \quad (6.49)$$

By (6.40), we have that $D_1(v|t,\eta) - D_2(v|t,\eta) > 0$ and, moreover, it is obvious that $\tau(t,\eta)$ is increasing with increasing $\eta$. Therefore it follows from (6.47) that to prove statement (f) it suffices to show that $D_1(v|t,\eta)$ is monotonically increasing and $D_2(v|t,\eta)$ is monotonically decreasing with $\eta$ for fixed $t$ and $v$.

Consider $D_1(v|t,\eta)$. From the definition $\alpha^*(\eta)$ it follows that either $D_1(v|t,\eta)$ is equal to 0 or all items in (6.48) are positive. But for $t < v < \tau(t,\eta)$

$$\eta(v|t,\eta) = \eta - \delta^2(v-t), \quad (6.50)$$

holds, and therefore by statement (f) for $k = r-1$ the function $L_{r-1}(v,\eta(v|t,\eta)+\gamma^1)$ is decreasing with $\eta$. From (6.50) and (6.32) we obtain that $\tilde{p}^1(\eta(v|t,\eta))$ is also monotonically decreasing with $\eta$. The same statement is obtained from (6.50) and (6.43) with respect to $z(v|t,\eta)$. Thus, the proof of monotonically increasing of $D_1(v|t,\eta)$ with respect to $\eta$ is complete.

Consider $D_2(v|t,\eta)$. From (6.50), (6.43) and (6.32) it is not difficult to show that

$$z(v|t,\eta)\tilde{p}^2(\eta(v|t,\eta)) = \tilde{p}^2(\eta)z(v|t,\eta+\gamma^2). \quad (6.51)$$

From (6.50) we have $\eta(v|t,\eta)+\gamma^1 = \eta(v|t,\eta+\gamma^2)$. From this and from (6.51) and (6.49) it follows that

$$D_2(v|t,\eta) = \tilde{p}^2(\eta)z(v|t,\eta+\gamma^2)L_{r-1}(v,\eta(v|t,\eta+\gamma^2)). \quad (6.52)$$

But $\tilde{p}^2(\eta)$ is increasing with $\eta$, and the product of the last two terms in the right-hand side of (6.52) is negative and is decreasing with $\eta$ by statements (e) and (f) for $k = r-1$. Thus, $D_2(v|t,\eta)$ is decreasing with $\eta$ for fixed $t$ and $v$, which completes the proof of statement (f) and Theorem 6.1. ■



**Remark 6.2** Using a formula similar to (6.51) a recurrence relation can be given for $F_k(t,\eta)$ not only in the region $(\nu - t,\eta) \in \overline{G}_k$, but in the whole half plane. ■

## 6.6 Problems with an infinite number of jumps

In this section we briefly present a heuristic method for considering the problem with a criterion functional of type (6.9), when the value function enters on both sides of the corresponding equation. In the previous section we overcame the difficulties by considering the sequence of problems with recursive functional equation of the type

$$F_k(t,x) = \sup_{\alpha} \int_t^{\nu} z(s|\alpha) \sum_{j=1}^{m} \alpha^j(s)p^j(x(s|a))[\lambda q(s),\Gamma^j x(s|a)) + F_{k-1}\,(s,\Gamma^j x(s|a))]\,ds,$$

where $a := (t,x,\alpha(\cdot))$ and $F_k(t,x)$ converge as $k \to \infty$ to the value function $F(t,x)$ sought. However, this method was successful only because the optimal synthesis has the same form in all intermediate problems and therefore in the initial problem. In the general case such a solution method, although in principle possible, can be practically unrealizable because of the complexity of the construction of the syntheses for the intermediate problems.

Another method of solution consists of the following. Suppose that the value function is smooth. Then we can obtain a formula similar to that obtained in §6.4 except that $L_k^j$ and $L_{k-1}^j$ are both the function $L^j$, which is the coefficient of $\alpha^j$ in the Hamiltonian $\mathcal{H}$. This formula is a necessary condition which can be used to construct the optimal synthesis. The second step in the solution consists in the proof that the functional $F^\alpha(t,x)$ corresponding to this synthesis is a smooth function. Since the synthesis was constructed with the assumption that the value function is smooth, it does not follow automatically that $F^\alpha(t,x) = F(t,x)$. Finally, based on the smoothness of $F^\alpha(t,x)$ and the necessary optimality conditions, it can be shown that $F^\alpha(t,x)$ satisfies the Bellman equation. From this it follows that $F^\alpha(t,x)$ coincides with $F(t,x)$. By this method we can, for example, prove the optimality of the strategy given in §6.5 for the problem of maximization of the number of successes.