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Growth rate, internal rates of return and turnpikes in an investment model*

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Summary. This paper describes the relationship between the model's growth rate, the set of vectors of equilibrium growth and the set of internal rates of return of the investment matrix. This matrix specifies the renewable and reproducible scale-neutral investment possibilities. An explicit description of quasioptimal strategies and turnpikes is given.

1. Introduction. Main results

The main aim of this paper is to look from a new angle at the relations between three important characteristics of the investment models: the *growth rate* of the system described by the model, the set of the *internal rates of return* of the projects defining the investment possibilities in the model, and the set of vectors ensuring balanced (stationary) development of the system – *turnpikes*.

Thus the content of this paper lies on the boundary of two powerful streams in theoretical economics: the theory of economic growth, with such key words as resources, von Neumann–Gale model, balanced growth equilibrium, the turnpikes theorems and the theory of optimal (sequential) investments with key words: investment programme (project), capital, profit, internal rates of return, the equilibrium rate of interest, and selection criteria. We consider a rather specific model with rigid assumptions about the possibility to reproduce every project at any scale and infinitely many times, and do not make any attempts to trace in depth the history of the problem or to embed derived relations into more general economic context. We hope that the economists with broader perspectives and better knowledge of economic literature can use the results of this paper for more far

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reaching conclusions. We also refer the reader to the important papers of Cantor and Lippman (1983) and Atsumi (1991), (see also the interesting comments of McKenzie following Atsumi's paper), where very similar models with detailed explication and justification of assumptions were considered.

The starting point for our analysis is the work of Cantor and Lippman (CL (1983)), where the following model of economic dynamics with a unique resource, money, is considered. Time is discrete and runs through $0, 1, \ldots, n$. Later n tends to infinity. The investor's initial money capital is given and his goal is to maximize the *terminal wealth*, i.e. total cash on hand, by date n. The investment possibilities are specified by the *project* $\mathbf{a} = (a_0, a_1, \ldots, a_r)$, where r is the *duration* of the project, a_0 is the initial investment, and a_i represents the cash received from the project in the ith period after investing. The main case is when $a_0 < 0$ and then, without loss of generality, $a_0 = -1$, where the other a_i may have any sign. A negative number is an input (a cash outlay from the investor) and a positive number is an output. Three basic assumptions are made in their model: (A) (*Stationarity*). The project $\mathbf{a}$ is available for investment in each period of the planning horizon. (B) (*Divisibility*). The project $\mathbf{a}$ can be implemented at any scale. (C) (*Imperfect capital market*). The investors decisions are constrained so that a nonnegative cash position must be maintained at all points in time.

In addition to the basic project $\mathbf{a}$, the auxiliary project $\mathbf{a}^0 = (-1, 1)$, the "keeping" of capital, is available. It has the same properties as $\mathbf{a}$.

Since the aim of the investor is to maximize total cash at terminal moment n and he must be free of obligations at n, all the undertaken projects must be completed by this moment and this implies that there is no investment in the project during the last r periods of the planning horizon. Denote by V_n the optimal n-period return (the *value function*). The well-known concept of the *investment polynomial* $J(\lambda)$ of the project $\mathbf{a}$, $J(\lambda) = \sum_{i=0}^{r} a_i \lambda^{-i} = \sum_i a_i/(1+\rho)^i$, plays a key role in the analysis of the asymptotic behaviour of V_n. The value of $\rho = \lambda - 1$ is called the *rate of return*, $1/\lambda$ is the *discount factor*, so $J(\lambda)$ is equal to the net present value of the project with discount $1/\lambda$. The project $\mathbf{a}$ is *productive* if $J(1) = \sum_i a_i > 0$. If $a_0 < 0$, then $J(\lambda) < 0$ for large λ and hence in this case the equation $J(\lambda) = 0$ has at least one real root greater than one. If $J(1) = 0$, the project $\mathbf{a}$ is called *cashtransfer*, if $J(1) < 0$, the project $\mathbf{a}$ is called *nonproductive*. The values of the roots of $J(\lambda)$ minus one are called *internal rates of return* of the project $\mathbf{a}$.

The main aim of Cantor and Lippman was to find the relationship between the *model's growth rate* $g = \lim_n V_n^{1/n}$ and the roots of $J(\lambda)$. First, by means of the so called *fundamental equation* (see Section 4a), they found that

$$V_n \leq C\lambda_*^n, \tag{1.1}$$

where λ_* is the *minimal real root of* $J(\lambda)$ *greater than* 1, C is the initial capital, in the sequel $C = 1$. Second, Cantor and Lippman proved that if $\lambda_* < \infty$ and has multiplicity $m+1$, then there exist positive constants b_1 and b_2 such that

$$b_1 \lambda_*^n / n^m \leq V_n \leq b_2 \lambda_*^n / n^m. \tag{1.2}$$

These statements seem to be inconsistent with the works of other authors, where in very similar conditions, the model growth rate is equal to *maximal* real root of investment polynomial. For instance in Dorfman (1981), an abstract states that "It has long been recognized that a firm will grow asymptotically at a rate equal to the largest real positive root of an individual project's rate of return equation if the net cash flows are continually reinvested in the projects of the same type." Atsumi (1991) comes to the same conclusion about the maximal root under the condition of non-positive discounted profit for an equilibrium interest factor. Cantor and Lippman noted this divergence, but don't give an explanation of it. Their proofs are based on nontrivial results from polynomial theory, and are rather complicated.

In this work we consider the case of a (finite) set of projects of both types, *investments* ($a_0 < 0$) and *loans* ($a_0 > 0$), explain the above "inconsistency", and give an explicit description of *quasioptimal strategies*, i.e. the strategies ensuring the maximal growth rate. Briefly our methods and results may be described as follows.

The basis for our analysis is the representation of the investment model as the von Neumann type model. As a state of the system we consider the vector $\mathbf{z} = (z_0, z_1, \ldots, z_{r-1}) \in R^r$, where z_0 is cash at hand, and z_i is the money received (paid) i-periods after the present moment if no additional investment will be made. The analogues of (A)–(C) and assumption of independence of the projects are assumed.

At state $\mathbf{z}$ with k basic projects and an auxiliary one, the investor may choose any distribution of "investments" (activity levels) $\mathbf{u} = (u^0, u^1, \ldots, u^k)$, $u^j \geq 0$, compatible with financial position $\mathbf{z}$. The goal of the investor is to maximize the zero coordinate of final state $\mathbf{z}_n$ under the condition that other coordinates are equal zero (nonnegative). The described model is very similar to the model of von Neumann. But in contrast to von Neumann's model the vectors $\mathbf{z}$ may have negative coordinates. Moreover, in the important case without loans there are admissible states $\mathbf{z}$ such that the subsequent motion is possible only for a finite number of steps. We refer to such states as *deadends*. Finally the considered terminal functional $h(\mathbf{z}) = z_0$ if $z_i \geq 0$, $i = 0, 1, \ldots, r-1$, $\mathbf{z} = (z_0, z_1, \ldots, z_{r-1})$, $h(\mathbf{z}) = 0$ otherwise, is linear only on a *subset* of state space and is neither linear nor continuous on the whole space. This property is specified by "free of obligation at final moment" constraint. So we failed to use any facts from the vast theory related to the Neumann–Gale models, though the key idea to consider the vectors of balanced growth is adopted from this theory.

We say that a vector $\mathbf{c}$ is a *turnpike vector* (the vector of balanced growth) or briefly a *t-vector* with rate $\lambda > 0$, if under an admissible investment the vector $\mathbf{c}$ can be reproduced at scale λ with may be some nonnegative cash. The *turnpike* $\{\mathbf{c}\}$ is the set of all vectors proportional to $\mathbf{c}$. Denote by $J^j(\lambda)$ the investment polynomial for the jth project, $J(\lambda) = \max_j^{J^j}(\lambda)$, $\nabla = \{\lambda \geq 1\colon J(\lambda) \geq 0\}$, $C(\lambda)$ is the set of all t-vectors with fixed rate of growth λ, and $\lambda_* = \min(\lambda \geq 1, J(\lambda) = 0)$. In the Cantor–Lippman case (one basic project) λ_* coincides with the minimal root of the investment polynomial.

It will be proved that 1) for a fixed λ the set $C(\lambda)$ is nonempty if and only if $\lambda \in \{J(\lambda) \geq 0\}$, 2) the transition from initial state $\mathbf{z}(0) = \mathbf{e} \equiv (1, 0, \ldots, 0)$ to every turnpike $\{\mathbf{c}(\lambda)\}$, $\lambda \in \nabla$ is possible in no more than $(r-1)$ steps. The statements 1), 2) and the definition of the turnpike vector immediately imply that it is possible to reach the turnpike corresponding to the *maximal* root of $J(\lambda) = 0$ and to move along it with this rate. At first glance, this fact, in accordance with usual notions of von

Neumann–Gale models, imply that V_n also will grow at this rate. This however contradicts (1.1) and (1.2). The explanation of this contradiction is the following. For all $\lambda > \lambda_*$ there is no way to leave the turnpike $\{\mathbf{c}(\lambda)\}$ and to reach a final set $\Phi = \{\mathbf{z} = (z_0, 0, \ldots, 0), z_0 \geq 0\}$. On such a turnpike *the investor is doomed to stay on it forever* (or to pass to the turnpike with bigger λ if it exists) because the financial obligations connected with previous investments can be met only on such turnpikes. In the case without loans, getting off the turnpike will lead to the deadends.

Of course in the similar model with other functionals the growth rate of the value function may be different, though, under reasonable assumptions, bounded from above by the maximal root of $J(\lambda)$. In any case the noted effect casts a shadow on any functional without "tail" term, for instance additive consumption, and stresses a necessity of the classification of the possible states.

The states from which the gradual transition to the final set is possible are called *liquid* and the set of all such states is denoted by L. It will be proved that all the turnpikes $\{\mathbf{c}(\lambda)\}$ for $\lambda < \lambda_*$ belong to L. The turnpike $\{\mathbf{c}(\lambda_*)\}$ may belong or not belong to L depending on the form of the corresponding project $\mathbf{a}$. This fact specifies two different patterns of the final behaviour of the quasioptimal trajectory: sharp pass or smooth slipping off the turnpike. In the Appendix a sufficient condition for $\mathbf{c}(\lambda_*) \in L$ is presented. In particular it holds when the investment project $\mathbf{a}$ is *simple*, i.e. the sequence $a_0, a_1, \ldots, a_r$ has exactly one sign change. It will be shown also that $C(\lambda_*) \cap L = \varnothing$ if λ_* has multiplicity more than one, and that $C(\lambda) \cap L = \varnothing$ for $\lambda > \lambda_*$. Since the set of turnpikes $\bigcup_{\lambda > \lambda_*} C(\lambda) \neq \varnothing$ in the case of multiple internal rates of return, we get that besides liquid states and deadends there are states from which it is not possible to reach liquid and hence final states, but the infinite motion is possible. We call such states *flying by states*.

In Section 4b we describe the quasioptimal strategies. An interesting phenomenon is that it is possible and not exceptional that the *nonproductive* projects are implemented most of the time. Their use may be optimal. It is important only that together with these projects there are other projects that ensure transition from nonproductive turnpike to the final state. The using of productive projects may be necessary only in the last interval of time. This phenomenon without referring to the turnpikes and quasioptimal strategies was noted in the last section of CL (1983), where the combinations of nonproductive project with the auxiliary project $(-1, 1+\rho)$, $\rho > 0$, was considered and the effect of possible "cooperation" of projects was stressed. So in this relation we elaborate the idea of Cantor and Lippman.

Our final remark in this introductory Section is the following. If the presented model captures at least some features of investment policies then it suggests that the growth of an enterprise or economy is not necessarily the best indicator of good policy because high growth rates can be associated with non-liquid turnpikes. Why is this bad, particularly when high growth can sustain high consumption? Nothing, provided there are unlimited investment possibilities. But if there are unexpected demands for cash or events change investment prospects, bankruptcy is a possible outcome. We are reminded of the old saying about the 1929 crash, "All was started when someone on Wall Street requested one dollar in cash."

The terminology in this paper slightly deviates from the usual in two points. First we call a turnpike any ray which specifies the proportional development, not necessarily with maximal rate. Second, Cantor and Lippman and some other authors refer to $g = \lim_n V_n^{1/n}$ as the *project's* growth rate. It seems more appropriate to refer to g as *model's* or *system's* growth rate because g depends not only on the project's parameters but also on the assumptions and the form of the functional.

The content of the other Sections is as follows. Section 2a contains the rigorous presentation of the model. In Section 2b the classification of the states is presented. In Section 2c all turnpikes are described and the investment polynomials naturally emerge for the first time. In Section 3 the formulation of the main theorem and the scheme of its proof are presented. In Section 4a the fundamental equation is treated and the upper bounds for the growth rate are proved. Here again the investment polynomials appear independently of Section 2c. Section 4b contains the description of quasioptimal strategies. In Section 5 we make some conclusive remarks and present some open problems. Some proofs are presented in the Appendix. The numeration inside every Section and every part of the Appendix is independent. The formula (2.1) means the first formula of the Section 2 and so on. The vectors are presented by bold face except vector α from the k-dimensional simplex, $\Sigma = \{\alpha = (\alpha^0, \alpha^1, \ldots, \alpha^k), \alpha^j \geq 0, \sum^j \alpha^j = 1\}$, and all vectors are considered as column-vectors under matrix multiplication. The time is denoted as a rule with low indices and the numbers of projects with high indices. The notation $f_n \simeq g_n$ means that $\lim_n f_n/g_n = 1$, $|\mathbf{z}| = \sum_i |z_i|$. As a rule we will denote by the same letter b all positive constants. Let $\mathbf{e}$ denote the vector $(1, 0, \ldots, 0)$, where its dimension is clear from context.

2a. The decision model with many projects

We describe the model as a standard Markov decision model defined by the tuple $(Z, U(\mathbf{z}), T(\mathbf{u}), H_n(\ ))$, where Z is the *state space*, $U(\mathbf{z})$ is the *set of actions admissible at the state* $\mathbf{z}$, $T(\mathbf{u})$ are the *transitions operators* specifying the next state of the system if at present state $\mathbf{z}$ the admissible action $\mathbf{u}$ is chosen, and $H_n(\ )$ is the *functional* defined on the trajectories of the system, terminal in this model.

The state space is defined as $Z = \{\mathbf{z}:\mathbf{z} = (z_0, z_1, \ldots, z_{r-1}),\ z_i \in R\}$. The investment possibilities are specified by the *investment matrix* $A = \{a_i^j\}$, $i = 0, 1, \ldots, r$, $j = 0, 1, \ldots, k$, where r is the maximal duration of the projects, k is the number of basic projects. The jth project is described by the vector-column $\mathbf{a}^j = (a_0^j, a_1^j, \ldots, a_r^j)$, $a_0^j \neq 0, j = 0, 1, \ldots, k$. The projects with $a_0^j < 0$, $0 \leq j \leq m$, $0 < m \leq k$, are interpreted as *investment projects* and the projects with $a_0^j > 0$ as *loans*. Without loss of generality $a_0^j = -1$ for $0 \leq j \leq m$, $a_0^j = +1$ for $m+1 \leq j \leq k$. For the auxiliary project "keeping money" $\mathbf{a}^0$ we have $a_1^0 = 1$, $a_i^0 = 0$, $i \geq 2$. For other, basic projects a_i^j for $i \geq 1$ may have any sign. In particular the investment matrix may contain the projects of the form $(-1, 1 + \rho_0, 0, \ldots, 0)$, $\rho_0 > 0$, i.e. keeping money with percent ρ_0.

At state $\mathbf{z}$, the investor may choose any distribution of "investments" (activity levels) $\mathbf{u} = (u^0, u^1, \ldots, u^k)$, $u^j \geq 0$, compatible with financial position $\mathbf{z}$, i.e. $\sum_{0 \leq j \leq m} u^j = z_0 + \sum_{m+1 \leq j \leq k} u^j$ or the same $\sum_{0 \leq j \leq k} a_0^j u^j + z_0 = 0$. We write this condition as $\mathbf{u} \in U(\mathbf{z})$. (This is the analogue of assumption (C) for our model). In fact, u^j for

$0 \le j \le m$ is an investment into the jth project, and u^j for $m+1 \le j \le k$ is the intensity of the jth loan, but we will refer to u^j as to investments in both cases. Formally, the transitions of the system for the investments $\mathbf{u} = (u^0, u^1, \ldots, u^k)$, $\mathbf{u} \in U(\mathbf{z})$ are specified by the operators $T(\mathbf{u})$

$$T(\mathbf{u})\mathbf{z} = S(\mathbf{z} + A\mathbf{u}), \tag{2.1}$$

where a) S is shift operator: $R^{r+1} \to R^r$, $S\mathbf{v} = (v_1, \ldots, v_r)$ for $\mathbf{v} = (v_0, v_1, \ldots, v_r))$, b) A is the investment matrix and c) $\mathbf{z}$ is given by $\mathbf{z} = (z_0, z_1, \ldots, z_{r-1}, 0)$. Note that the operators $T(\mathbf{u})$ are defined for all $\mathbf{u} \in R^{k+1}$, not only for $\mathbf{u} \in U(\mathbf{z})$.

We will use also the notation T^α, where $\alpha = (\alpha^0, \alpha^1, \ldots, \alpha^k)$ is a point of k-dimensional simplex Σ, for the operators $T(\mathbf{u})$ for the normalized investments $\mathbf{u} = \alpha$ and T^j, $j = 0, 1, \ldots, k$, for the case when all normalized investments are made into project j, i.e. when α coincides with a vertex of simplex Σ,

$$T^j\mathbf{z} = S(\mathbf{z} + \mathbf{a}^j), \quad T^\alpha \mathbf{z} = S(\mathbf{z} + \mathrm{A}\alpha) \equiv \textstyle\sum_{j=0}^k \alpha^j T^j \mathbf{z}. \tag{2.2}$$

Let us present some useful formulae related to the operators $T(\mathbf{u})$. It is easy to verify that

$$T(\mathbf{u})\mathbf{z} \equiv dT^\alpha(\mathbf{z}/d) \quad \text{for } \mathbf{u} \in U(\mathbf{z}), \quad \mathbf{u} = d\alpha, \quad \alpha \in \Sigma, \quad d > 0, \tag{2.3}$$

$$T(\mathbf{u})(\mathbf{z}_1 + \mathbf{z}_2) = T(\mathbf{u})\mathbf{z}_1 + S\mathbf{z}_2, \quad T(b\mathbf{u})b\mathbf{z} = bT(\mathbf{u})\mathbf{z}, \quad b > 0. \tag{2.4}$$

Formula (2.1) implies also the independence of the projects

$$T(\mathbf{u}_1 + \mathbf{u}_2)(\mathbf{z}_1 + \mathbf{z}_2) = T(\mathbf{u}_1)\mathbf{z}_1 + T(\mathbf{u}_2)\mathbf{z}_2. \tag{2.5}$$

The tuple of actions $\mathbf{u}_1, \ldots, \mathbf{u}_s$ is denoted briefly as $\mathbf{u}_1^s$, the product of the operators $T(\mathbf{u}_s) \cdots T(\mathbf{u}_1)$ as $T(\mathbf{u}_1^s)$. The sequence (tuple) of actions $\mathbf{u}_1, \mathbf{u}_2, \ldots$ is called *admissible* or a *strategy* for the state $\mathbf{z}$ if $\mathbf{u}_1 \in U(\mathbf{z})$ and $\mathbf{u}_{i+1} \in U(T(\mathbf{u}_1^i)\mathbf{z})$ for every $i = 1, 2, \ldots$ Hereafter, speaking of infinite (finite) *trajectories* $\mathbf{z}_1, \mathbf{z}_2, \ldots$ we always mean that they are derived by means of some admissible sequence (tuple) of actions.

From the definition of the operators $T(\mathbf{u})$ and (2.4), (2.5) it follows obviously

Proposition 2.1. *Let* $\mathbf{u}_1, \ldots, \mathbf{u}_s$ *and* $\mathbf{v}_1, \ldots, \mathbf{v}_s$ *be two tuples of actions admissible for* $\mathbf{z}$ *and* $\mathbf{z}'$. *Then a)* $\mathbf{u}_1 + \mathbf{v}_1, \ldots, \mathbf{u}_s + \mathbf{v}_s$ *is a tuple of actions admissible for* $\mathbf{z} + \mathbf{z}'$ *and* $T(\mathbf{u} + \mathbf{v})_1^s(\mathbf{z} + \mathbf{z}') = T(\mathbf{u}_1^s)\mathbf{z} + T(\mathbf{v}_1^s)\mathbf{z}'$, *b) for any* $b > 0$ *the tuple actions* $b\mathbf{u}_1, \ldots, b\mathbf{u}_s$ *is admissible for* $b\mathbf{z}$ *and* $T(b\mathbf{u}_1^s)(b\mathbf{z}) = bT(\mathbf{u}_1^s)\mathbf{z}$.

To complete the description of the model it is necessary to define the functional H_n on all trajectories. In accordance with Section 1 this functional has the form

$$H_n(\mathbf{z}_1, \ldots, \mathbf{z}_n) \equiv h_n(\mathbf{z}_n) = \begin{cases} z_{n,0} & \text{if} \quad z_{n,i} = (\ge) 0 \quad \text{for} \quad i = 0, 1, \ldots, r-1 \\ 0 & \text{otherwise.} \end{cases}$$

The value of a functional for a given strategy (the *payoff*) is denoted as W_n, so the value function $V_n = \sup W_n$, where sup is taken over all strategies for a given initial point.

2b. The classification of states in the model

It is easy to see that in the case when $m < k$, i.e. when there are loans, $U(\mathbf{z}) \ne \varnothing$ for any $\mathbf{z} \in Z$. In the case when there is no loans ($m = k$) it is possible that $U(\mathbf{z}) = \varnothing$ (if

$z_0 < 0$) or $U(\mathbf{z}) \neq \varnothing$ but $U(T(\mathbf{u})\mathbf{z}) = \varnothing$ for any $\mathbf{u} \in U(\mathbf{z})$. The existence of such states and the considered functional motivates the following classification. We define the set of *final* states $\Phi = \{\mathbf{z} = (z_0, 0, \ldots, 0), z_0 \geq 0\}$; the set of *deadend states* $D = \{\mathbf{z} \in Z$, there does not exists infinite trajectory starting in $\mathbf{z}\}$; the set of *liquid states* $L = \{\mathbf{z} \in Z$, there exist a finite trajectory starting in $\mathbf{z}$ and reaching $\Phi\}$; the set of *selfliquid* states $L_0 = \{\mathbf{z} \in L$, where it is possible to stop investment into the basic projects and using only auxiliary project $(-1, 1)$ come through admissible states to the final set $\Phi\}$; the set of *flying by* states $F = Z \backslash (D \cup L)$, i.e. states which are not deadend but from which it is not possible to reach liquid and hence final states.

So every trajectory starting in a *deadend* after a finite number of steps comes to a state such that the subsequent motion is impossible. The deadends exist only in the case without loans. Its easy to see that $L_0 = \{\mathbf{z}: \sum_{i=0}^{s} z_i \geq 0,\ s = 0, 1, \ldots, r-1\}$ and no more than $(r-1)$ steps is needed to reach Φ from L_0.

In Section 4 we prove that in the case of multiple internal rates of return the t-vectors $\mathbf{c}(\lambda)$ for $\lambda > \lambda_*$, are flying by states and the t-vectors $\mathbf{c}(\lambda)$ for $\lambda < \lambda_*$, are liquid states.

Let us define the ordering of R^r by L_0, i.e. $\mathbf{z}' \succ \mathbf{z}$ iff $\mathbf{z}' = \mathbf{z} + \mathbf{y}, \mathbf{y} \in L_0$. The definition of $|\mathbf{z}| = \sum_i |z_i|$ and the ordering $\succ$ immediately imply that if the vectors satisfy $|\mathbf{a} - \mathbf{b}| < \varepsilon$ then $\mathbf{a} + \varepsilon\mathbf{e} \succ \mathbf{b}$.

2c. The description of turnpikes. The investment polynomials

In this section we describe all solutions $\mathbf{c}$ of the equation

$$T^\alpha \mathbf{c} = \lambda \mathbf{c} + \delta \mathbf{e}, \quad \alpha \in \Sigma, \quad \alpha \in U(\mathbf{c}), \quad \lambda > 0, \delta \geq 0. \tag{2.6}$$

(The solutions of similar equation for $T(\mathbf{u})$ are proportional to the solutions of (2.6) by (2.3)). These solutions exist only for some special values of λ and α, and, as well as proportional ones, are called turnpike vectors or t-vectors with given rate λ. In the von Neumann–Gale theory the vectors $\mathbf{c}$ from (2.6) with $\delta = 0$ are referred to often as the vectors of equilibrium growth.

At first we find the solution of (2.6) for the particular case $T^\alpha \equiv T^j$ and for δ of any sign,

$$T^j \mathbf{c} = \lambda \mathbf{c} + \delta \mathbf{e}, \quad \mathbf{c} \in Z, \quad \lambda > 0, \tag{2.7}$$

where the operators $T^j, j = 0, 1, \ldots, k$ have the form $T^j \mathbf{z} = S(\mathbf{z} + \mathbf{a}^j)$ (see (2.2)).

From the definition of $U(\mathbf{z})$ it follows that $c_0^j = -a_0^j$ and hence $c_0^j = +1$ for $0 \leq j \leq m$, $c_0^j = -1$ for $m < j \leq k$. Rewriting the relation (2.7) in coordinates for fixed λ, we get

$$c_1^j + a_1^j = \lambda c_0^j + \delta, \quad c_2^j + a_2^j = \lambda c_1^j, \ldots, c_{r-1}^j + a_{r-1}^j = \lambda c_{r-2}^j, \quad a_r^j = \lambda c_{r-1}^j. \tag{2.8}$$

Substituting successively the equality for c_i^j, $i = r-1, \ldots, 2$ into the foregoing equality we get the relation for δ

$$\delta \equiv \delta^j(\lambda) = \lambda \sum_{i=0}^{r} a_i^j \lambda^{-i} \equiv \lambda J^j(\lambda) \tag{2.9}$$

and the explicit form for the vector $\mathbf{c}^j(\lambda)$

$$\mathbf{c}^j(\lambda) = P(\lambda)\mathbf{a}^j - J^j(\lambda)\mathbf{e}, \tag{2.10}$$

where ith row of the matrix $P(\lambda)$, $i=0,1,\ldots,r-1$, has the form $(0,\ldots,0,\lambda^{-1},\ldots,\lambda^{-r+i})$ with $(i+1)$ zeros.

For the solutions of (2.6) for δ of *any sign* and $\lambda>0$, using (2.5) and (2.4), we get that at least one solution exists for any λ and any $\alpha\in\Sigma$ and has the form

$$\mathbf{c}^{\alpha}(\lambda)=\sum_{j=0}^{k}\alpha^{j}\mathbf{c}^{j}(\lambda),\quad \delta=\lambda J^{\alpha}(\lambda)\equiv\lambda\sum_{j=0}^{k}\alpha^{j}J^{j}(\lambda). \tag{2.11}$$

If $\mathbf{b}$ is another solution of (2.6), i.e. $T^{\alpha}\mathbf{b}=\lambda\mathbf{b}+\delta'\mathbf{e}$, then using the first of formulae (2.4), we have, writing $\mathbf{c}^{\alpha}(\lambda)$ simply as $\mathbf{c}$, $T^{\alpha}\mathbf{b}=T^{\alpha}(\mathbf{c}+\mathbf{b}-\mathbf{c})=\lambda\mathbf{c}+\delta\mathbf{e}+S(\mathbf{b}-\mathbf{c})=\lambda\mathbf{c}+\lambda(\mathbf{b}-\mathbf{c})+\delta'\mathbf{e}$ and hence $S(\mathbf{b}-\mathbf{c})=\lambda(\mathbf{b}-\mathbf{c})+(\delta-\delta')\mathbf{e}$. The definition of S implies that $(\mathbf{b}-\mathbf{c})=0$.

The function $J^{j}(\lambda)=\sum_{i=0}^{r}\alpha_{i}^{j}\lambda^{-i}$ is called the *investment polynomial* of the jth project, $j=0,1,\ldots,k$. Denote by $J(\lambda)=\max_{j}J^{j}(\lambda)$, $\lambda_{*}=\min(\lambda\geq 1\colon J(\lambda)=0)$, $\nabla=\{\lambda\geq 1\colon J(\lambda)\geq 0\}$, $R(\lambda)=\{\alpha\in\Sigma\colon J^{\alpha}(\lambda)\geq 0\}$, $C(\lambda)=\{\mathbf{c}^{\alpha}(\lambda)\colon\alpha\in R(\lambda),\lambda\geq 1\}$ – the set of solutions of (2.6) (t-vectors) with given rate λ. The multiplicity of the root λ_{*} is defined now as the minimal value of multiplicities for those polynomials $J^{j}(\lambda)$ for which λ_{*} is a root and $J^{j}(\lambda-\varepsilon)>0$ for all positive ε.

It is easy to see that a) $C(\lambda)\neq\varnothing$ if and only if $\lambda\in\nabla$, b) $J^{j}(\lambda)>0$ for some j if $\lambda<\lambda_{*}$, and c) $\lambda_{*}<\infty$ if all projects are investment projects, i.e. if $m=k$.

For $\mathbf{c}\in C(\lambda)$ we denote by $\alpha(\mathbf{c})$ and $\delta(\mathbf{c})$ the corresponding values of α and δ from (2.6). In summary we have

Proposition 2.2. a) *The solution of* (2.7) *exists if and only if* $\delta=\lambda J^{j}(\lambda)$ *and* $\mathbf{c}\equiv\mathbf{c}^{j}(\lambda)$ *has the form* (2.10),
b) *the solutions of* (2.6) *do exist if and only if* $\lambda\in\nabla$, $\alpha\in R(\lambda)$, $\delta=\lambda J^{\alpha}(\lambda)$ *and* $\mathbf{c}\equiv\mathbf{c}^{\alpha}(\lambda)$ *have the form* (2.11).
c) *for any* $\lambda<\lambda_{*}$ *there exists a vector* $\mathbf{c}\in C(\lambda)$ *with* $\delta(\mathbf{c})>0$.

Now we formulate four auxiliary Propositions which will be used in Section 4b in description of the "slipping off" the turnpikes. The relation (2.6) and the formulae (2.3), (2.4) immediately imply

Proposition 2.3. *Let* $(\mathbf{c},\alpha,\lambda,\delta)$ *be a solution of* (2.6). *Then the tuple of actions* $\mathbf{u}_1,\mathbf{u}_2,\ldots$ *of the form* $\mathbf{u}_1=\alpha$, $\mathbf{u}_k=\lambda\mathbf{u}_{k-1}+\delta\mathbf{e}\equiv\lambda^{k-1}\mathbf{u}_1+\delta_k\mathbf{e}$, $\delta_k=\delta(1+\lambda^1+\cdots+\lambda^{k-2})$, $k\geq 2$, *is admissible for* $\mathbf{c}$, *and the corresponding trajectory has the form* $\mathbf{c}_1=\mathbf{c}$, $\mathbf{c}_k=\lambda\mathbf{c}_{k-1}+\delta\mathbf{e}\equiv\lambda^{k-1}\mathbf{c}+\delta_k\mathbf{e}$, $k\geq 2$.

Hereafter we refer to this trajectory as to the *motion along the turnpike* $\{\mathbf{c}\}$ with rate λ, though in the case $\delta>0$ only nonzero coordinates of the vectors $\mathbf{c}_k$ are growing with the rate λ, and zero coordinate is growing with the rate $(\lambda^k+\delta_{k+1})/(\lambda^{k-1}+\delta_k)$ which tends to λ from above.

In the Appendix we present a sufficient condition when the t-vector for the root of an investment polynomial $J^{j}(\lambda)$ is a selfliquid state. It is possible obviously only when $j\leq m$, i.e. when jth project is of investment type. In particular it is true when the sequence $a_0,a_1,\ldots,a_r$ has only one sign change, the so-called *simple project* case.

Proposition 2.4. *There exists a positive constant* b *such that for* $j=0,1,\ldots,k$

$$|\mathbf{c}^{j}(\lambda)-\mathbf{c}^{j}(\lambda')|\leq b|1/\lambda-1/\lambda'|,\quad 1\leq\lambda,\quad \lambda'<\lambda_{*}\leq\infty.$$

The proofs follows immediately from the formula (2.10).

Proposition 2.5. a) *for any* $\lambda < \lambda_*$ *and* $\mathbf{c} \in C(\lambda), \delta(\mathbf{c}) > 0$ *there exists continuous function* $\alpha(s) \in R(s)$ *and constants* $A = A(\lambda, \mathbf{c}) > 0, B = B(\lambda, \mathbf{c}) > 0$ *such that for* $1 \leqslant s \leqslant \lambda$

$$J^{\alpha(s)} \geq AJ(s) > 0, \tag{2.12}$$

the functions $\alpha(s)$ *and* $\mathbf{c}(s) \equiv \mathbf{c}^{\alpha(s)}(s)$ *satisfy the condition*

$$|\alpha(s) - \alpha(s')| \leq B|1/s - 1/s'|, |\mathbf{c}(s) - \mathbf{c}(s')| \leq B|1/s - 1/s'|, \tag{2.13}$$

and the initial condition $\mathbf{c}(\lambda) = \mathbf{c}$,
b) *there exist continuous function* $\alpha(s) \in R(s)$, *and constants* $A > 0, B > 0$ *such that the functions* $\alpha(s)$ *and* $\mathbf{c}(s) \equiv \mathbf{c}^{\alpha(s)}(s)$ *satisfy the inequalities (2.12), (2.13) for* $1 \leq s < \lambda_* \leq \infty$.

The proof is in the Appendix. Note also that for the case $k = 1$ (one basic project) the statement is trivial since $\alpha(\lambda) \equiv 1$ and (2.13) follows from (2.10).

Proposition 2.6. *Let* $\mathbf{c} \in C(1)$ *and* $\delta(\mathbf{c}) > 0$. *Then* $\mathbf{c} \in L$.

Proof: Denote $\alpha(\mathbf{c}) = \alpha, \delta(\mathbf{c}) = \delta$. The equality $T^{\alpha}\mathbf{c} = \mathbf{c} + \delta\mathbf{e}$ implies that the sequence of actions $\mathbf{u}_k, \mathbf{u}_k = \alpha + \delta(k-1)\mathbf{e}, k = 1, 2, \ldots$ is admissible for $\mathbf{c}$ and the corresponding trajectory has a form $\mathbf{c}_k = \mathbf{c} + \delta(k-1)\mathbf{e}$, $k = 1, 2, \ldots$, $\delta > 0$. Obviously, $\mathbf{c}_k \in L_0$ for sufficiently large k. From here and from the definition of the set L we get the statement.

3. Formulation of the main theorem and the scheme of the proof

First let us introduce the class of strategies the quasioptimal strategies may be chosen from. Let $\mathbf{c}$ be a t-vector i.e. $\mathbf{c} \in C(\lambda)$ for some $\lambda \in \Delta$, let s_n be some moment of time, $0 < s_n \leq n$. The strategy $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n$, defined on the time interval $(0, n)$, is called the $(\mathbf{c}, s_n)$ *(turnpike) strategy* if it is specified by the three following stages: at the first stage, the actions $\mathbf{u}_0, \ldots, \mathbf{u}_k$, which ensure to get on the turnpike $\{\mathbf{c}\}$, are used; at the second stage, after reaching the turnpike, the actions which ensure the motion along the turnpike are used and at the third stage, at the moment s_n, the system slips off the turnpike and transits to the final set Φ. The description of possible situations is given by

Theorem 3.1. 1) *If the root* λ_* *has multiplicity one then there exist positive constants* b_1 *and* b_2 *such that*

$$b_1 \lambda_*^n \leq V_n \leq b_2 \lambda_*^n, \tag{3.1}$$

and a) *if* $\mathbf{c} \in C(\lambda_*) \cap L \neq \emptyset$, *then the* $(\mathbf{c}, s_n)$ *turnpike strategy may be taken as the quasioptimal strategy on the interval* $(0, n)$, *where* $s_n = n - l$ *and* l *depends only on the* t*-vector* $\mathbf{c}$ *and does not depend on* n,

b) *if* $C(\lambda_*) \cap L = \emptyset$, *then the* $(\mathbf{c}, s_n)$ *turnpike strategy may be taken as the quasioptimal strategy on the interval* $(0, n)$ *with* $\mathbf{c} \in C(\lambda^{(n)}) \cap L$, $\lambda_* - \lambda^{(n)} \simeq 1/n$, $n - s_n \simeq b \ln n$.

2) *If the root* λ_* *has multiplicity* $m + 1 \geq 2$, *then* $C(\lambda_*) \cap L = \emptyset$ *and*

$$b_1 \lambda_*^{n(1 - g_m(n))} \leq V_n \leq b_2 \lambda_*^n / n^m, \tag{3.2}$$

where $g_m(n) \simeq bn^{-1/m}$ *and the strategies ensuring the lower bound in* (*3.2*) *have the same form as in* b) *with* $\lambda_* - \lambda^{(n)} \simeq n^{-1/m}$, $n - s_n \simeq b_0 n$, $b_0 < 1$.
3) *If* $\lambda_* = \infty$ *then there exists a finite* n *for which* $V_n = \infty$.

The general scheme of the proof of theorem 3.1 follows the classical pattern of the proof of the final-state turnpike theorems (see Radner (1961), McKenzie (1971)): to obtain an upper bound (evaluation ceiling) for the value function, using the general properties of the model and a lower bound (comparison floor) constructing corresponding strategies. Our upper bound is the direct generalization of the Cantor–Lippman estimate (1.2). The proof is based on an explicit formula relating the discounted sum of investment, assets, and the investment polynomials. This formula for the case of one basic project was called in CL (1983) the *fundamental equation*. To get the lower bound we use the $(\mathbf{c}, s_n)$ turnpike strategies and estimates for the growth rate.

The Proposition 2.3 implies that unlimited motion along the turnpike with growth rate λ of state vector is possible and the second stage is always realizable. As it will be shown below in Proposition 4.2, *any* turnpike is reachable from the initial state $\mathbf{e} = (1, 0, \ldots, 0)$ in no more than $(r-1)$ steps and thus the first stage is also always realizable. The growth rate of state vector on this stage does not influence the order of growth in the whole since the number of steps is limited for all n. To realize the conclusive stage, it is necessary that the system be in the *liquid* state at moment s_n and the transition from this state to final set Φ be possible in the remaining time $n - s_n$. To estimate the growth rate of the state vector on the third stage, it is convenient to represent the payoff W_n for the $(\mathbf{c}, s_n)$ strategies ($\mathbf{c} \in C(\lambda)$) in the following form. Let $x_s = |\mathbf{z}_{s+1}|/|\mathbf{z}_s|$. Since $\mathbf{z}_n \in \Phi$, then $W_n = z_{n0} = |\mathbf{z}_n|$ and taking into account that $|z_0| = 1$ we have

$$W_n = \textstyle\prod_{s=1}^{n-1} x_s = \prod_{s=1}^{s_n-1} x_s \prod_{s=s_n}^{n-1} x_s. \tag{3.3}$$

For the turnpike strategy $x_s = \lambda + \delta(\mathbf{c})/|\mathbf{z}_s| \simeq \lambda$ for all $r \le s \le s_n$. Now let us represent x_s for $s_n \le s \le n-1$ as $\lambda \exp(\ln(x_s/\lambda))$. Then

$$W_n \simeq b\lambda^n \exp L_n, \quad b > 0, \tag{3.4}$$

where

$$L_n = \textstyle\sum_{s=s_n}^{n-1} \ln(x_s/\lambda) \tag{3.5}$$

is the *logarithmic loss* for given strategy on the third stage.

The simplest case is 1a) where t-vector $\mathbf{c} \equiv \mathbf{c}(\lambda_*)$ is liquid and hence the corresponding turnpike strategy can be used. So in (3.4) λ coinsides with λ_* and L_n has no more than l terms for all n. In cases 1b) and 2) the situation is more complicated since the t-vectors with rate λ_* are not liquid and hence can't be used. So most of the time on interval $(0, n)$, the system (investor) must stay on a liquid turnpike with a rate $\lambda^{(n)}$ near λ_*, $\lambda^{(n)} \to \lambda_*$ as $n \to \infty$. The proof that all $\mathbf{c}(\lambda)$ for $\lambda < \lambda_*$ are liquid, the process of smooth "slipping off" such turnpike and the estimates for L_n are presented in Section 4b (Proposition 4.6). In fact, if at any time the system is on a turnpike $\{\mathbf{c}(\lambda)\}$, $\lambda < \lambda_*$, then in a bounded number of steps it may transit to the turnpike $\{\mathbf{c}(\lambda')\}$ with $\lambda' < \lambda$, $|\lambda - \lambda'| \simeq bJ(\lambda)$, $b > 0$. So the transition from a turnpike $\{\mathbf{c}(\lambda)\}$ to the selfliquid turnpike $\{\mathbf{c}(1)\}$ is possible if $J(s) > 0$ on $1 \leq s \leq \lambda$ and in the process of such slipping it is necessary to use subsequently all the projects from a tuple of projects B such that $\max_{j \in B} J^j(s) > 0$ for all $1 \leq s \leq \lambda$. At the initial period of slipping, such transition is slow because $J(\lambda^{(n)}) \simeq J(\lambda_*) = 0$ and the speed of such transition is specified by the multiplicity of the root λ_*. This explains the relation between multiplicity of λ_* and the model's growth rate.

In the case of multiplicity more than one, our strategies do not ensure the lower bound $b_1 \lambda_*^n / n^m$, which can be expected by an analogy with one project model, so the more careful investigation of the behaviour of the trajectories near turnpikes is necessary to cover this situation. The statement of point 3) is similar to Cantor and Lippman but our proof is different. We prove that in a fixed number of steps, the transition from any turnpike $\{\mathbf{c}(\lambda)\}$, $1 \leq \lambda < \infty$, to final set is possible.

4. Proof of theorem 3.1.

4a. The fundamental equation. The upper bound for growth rate

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}$ be any admissible strategy for initial state $\mathbf{z}_0$, $\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_n$ corresponding trajectory, $Z_s(\lambda) = \sum_{i=0}^{r-1} z_{si} \lambda^{-i}$ – the discounted assets at the moment s, $U_{n-1}^j(\lambda) = \sum_{s=1}^{n-1} u_s^j \lambda^{-s+1}$ – the discounted sum of investments into the jth project up to moment $n-1$, $U_{n-1}(\lambda) = \sum_{j=0}^{k} U_{n-1}^j(\lambda)$ – the discounted sum of investments into all projects up to moment $n-1$. In other words, $Z_s(\lambda)$, $U_{n-1}^j(\lambda)$, $U_{n-1}(\lambda)$ are the investment polynomials for the vectors $\mathbf{z}_s = (z_{s0}, \ldots, z_{s,r-1})$, $(u_1^j, \ldots, u_{n-1}^j)$, $(\sum_j u_1^j, \ldots, \sum_j u_{n-1}^j)$.

First we get the relation between investment polynomials $Y(\lambda)$ and $Z(\lambda)$ for two vectors $\mathbf{y}$, $\mathbf{z}$ such that $T(\mathbf{u})\mathbf{z} = \mathbf{y}$, $\mathbf{u} \in U(\mathbf{z})$. The definitions of admissible action and operator $T(\mathbf{u})\mathbf{z}$ (2.1) imply $z_0 + \sum_{j=0}^{k} a_0^j u^j = 0$, $z_{i+1} + \sum_{j=0}^{k} a_{i+1}^j u^j = y_i$, $i = 0, 1, \ldots, r-1$. Multiplying these equalities by λ^{-i}, $i = 0, 1, \ldots, r-1$ and summing up, we have

$$Z(\lambda) + \sum_{j=0}^{k} J^j(\lambda) u^j = \lambda^{-1} Y(\lambda). \tag{4.1}$$

Then, writing this relation subsequently for the equalities

$T(\mathbf{u}_s)\mathbf{z}_{s-1} = \mathbf{z}_s$, $s = 1, \ldots, n-1$, we get the *fundamental equation*

$$Z_0(\lambda) + \sum_{j=0}^{k} J^j(\lambda) U_{n-1}^j(\lambda) = \lambda^{-n} Z_n(\lambda), \tag{4.2}$$

having the natural interpretation in terms of discounted assets, investments and net values.

If $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1})$ is a final strategy i.e. $\mathbf{u}_{n-r+1} = \cdots \mathbf{u}_{n-1} = 0$, then $\mathbf{z}_n = (z_{n0}, 0, \ldots, 0)$ and $Z_n(\lambda) = z_{n0} \lambda^{-n}$ and hence for the optimal strategy and $\mathbf{z}_0 = (1, 0, \ldots, 0)$,

$$1 + \sum_{j=0}^{k} J^j(\lambda) U_{n-1}^j(\lambda) = \lambda^{-n} V_n. \tag{4.3}$$

Since $(\mathbf{u}_1, \mathbf{u}_2, \ldots)$ is an admissible strategy then $u_s^j \geq 0$, $s = 0, 1, \ldots, j = 0, 1, \ldots, k$ and hence $U_{n-1}^j(\lambda) \geq 0$. For λ_* we have $J^j(\lambda_*) \leq 0$ for all j and we get from (4.3) the upper bound

$$V_n \leq \lambda_*^n. \tag{4.4}$$

To get the more precise estimate

$$V_n \le b\lambda_*^n/n^m, \tag{4.5}$$

it is necessary to consider the decomposition of the left side of (4.3) into Taylor series to $(m+1)$th term at the point λ_*. The proof of (4.5) is in the Appendix.

4b. The lower bound for the growth rate. The description of quasioptimal strategies.

At first we prove a useful auxiliary

Proposition 4.1. *Let $\mathbf{z}_1 \in Z$ and $\mathbf{v}_1, \ldots, \mathbf{v}_s$ be a tuple of actions admissible for $\mathbf{z}_1$. Let $\mathbf{y}_1 \succ \mathbf{z}_1$ and $\delta_i = \sum_{k=0}^{i-1}(y_{1k} - z_{1k})$, $i = 1, 2, \ldots$. Then the tuple of actions $\mathbf{u}_1, \ldots, \mathbf{u}_s$, $\mathbf{u}_i = \mathbf{v}_i + \delta_i \mathbf{e}$, $i = 1, \ldots, s$, is admissible for $\mathbf{y}_1$ and for $\mathbf{y}_{i+1} \equiv T(\mathbf{u}_1^i)\mathbf{y}_1$, $\mathbf{z}_{i+1} \equiv T(\mathbf{v}_1^i)\mathbf{z}_1$, $1 \le i \le s$, we have*

$$\begin{aligned} &\mathbf{y}_{i+1} = \mathbf{z}_{i+1} + \delta_i \mathbf{e} + S^i(\mathbf{y}_1 - \mathbf{z}_1), \quad 1 \le i \le s, \\ &\mathbf{y}_{i+1} \succ \mathbf{z}_{i+1}, \\ &\mathbf{y}_r = \mathbf{z}_r + \delta_r \mathbf{e}, \quad \delta_r \ge 0, \quad (\textit{if} \quad s \ge r-1). \end{aligned} \tag{4.6}$$

Proof: The relation $\mathbf{y}_1 \succ \mathbf{z}_1$ is equivalent to $\delta_i \ge 0$, $i = 1, \ldots, r$. It is easy to check that the tuple of actions $\delta_1 \mathbf{e}, \ldots, \delta_s \mathbf{e}$ is admissible for $(\mathbf{y}_1 - \mathbf{z}_1)$ and $T(\delta_1^i \mathbf{e})(\mathbf{y}_1 - \mathbf{z}_1) = \delta_i \mathbf{e} + S^i(\mathbf{y}_1 - \mathbf{z}_1) \succ 0$, $i = 1, \ldots, s$. Since $S^r \mathbf{z} = 0$ for all $\mathbf{z}$, $\delta_{r-1}\mathbf{e} + S^{r-1}(\mathbf{y}_1 - \mathbf{z}_1) = \delta_r \mathbf{e}$ and $\delta_i = \delta_r$ for $i \ge r$, we have also that if $s \ge r-1$ then $\delta_i \mathbf{e} + S^i(\mathbf{y}_1 - \mathbf{z}_1) \equiv \delta_r \mathbf{e}$ for $r-1 \le i \le s$. Hence by Proposition 2.1 $\mathbf{y}_{i+1} = T(\mathbf{u}_1^i)\mathbf{y}_1 = T(\mathbf{v}_1^i)\mathbf{z}_1 + T(\delta_1^i \mathbf{e})(\mathbf{y}_1 - \mathbf{z}_1) = \mathbf{z}_{i+1} + \delta_i \mathbf{e} + S^i(\mathbf{y}_1 - \mathbf{z}_1) \succ \mathbf{z}_{i+1}$, $i = 1, \ldots, s-1$ and we get all three formulae (4.6).

Proposition 4.2. *Let $\mathbf{c} \in C(\lambda)$, $\lambda \in \Delta$. Then there exists an admissible tuple of actions $\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}$ such that*

$$T(\mathbf{u}_1^{r-1})\mathbf{e} = b\mathbf{c} + \varepsilon' \mathbf{e}, \quad b > 0, \quad \varepsilon' \ge 0.$$

Proof: The definition of vector $\mathbf{c}$ and Proposition 2.3 implies the existence of the tuple of actions $\mathbf{q}_1, \ldots, \mathbf{q}_{r-1}$ such that $T(\mathbf{q}_1^{r-1})\mathbf{c} = \lambda^{r-1}\mathbf{c} + \delta \mathbf{e}$, $\delta \ge 0$. Since $\mathbf{e} \succ \varepsilon \mathbf{c}$ for any vector $\mathbf{c}$ and $\varepsilon = 1/\max_{0 \le s \le r-1} |\sum_{i=0}^s c_i|$, we may apply Proposition 4.1 for $\mathbf{y}_1 = \mathbf{e}$, $\mathbf{z}_1 = \varepsilon \mathbf{c}$ and tuple of actions $\varepsilon \mathbf{q}_1, \ldots, \varepsilon \mathbf{q}_{r-1}$. The third formula (4.6) implies that $T(\mathbf{u}_1^{r-1})\mathbf{e} = \varepsilon \lambda^{r-1}\mathbf{c} + \delta' \mathbf{e}$, $\delta' \ge 0$, where $\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}$ are given in Proposition 4.1.

Thus the first stage of any $(\mathbf{c}, s_n)$ turnpike strategy is always realizable and does not influence the order of growth. Now we consider the simplest situation, when number of terms in (3.5) is bounded for all n.

Proposition 4.3. *Let $C(\lambda) \cap L \ne \varnothing$, $\mathbf{c} \in C(\lambda) \cap L$ and $l(\mathbf{c})$ be the number of steps necessary to reach final set Φ from $\mathbf{c}$. Then there exists a $(\mathbf{c}, n - l(\mathbf{c}))$ strategy ensuring the payoff*

$$W_n \simeq b\lambda^n. \tag{4.7}$$

Proof: Let $\mathbf{c} \in C(\lambda) \cap L$. By the definition of the set of liquid states L and $l(\mathbf{c})$ there exists an admissible tuple of actions $\mathbf{u}_1, \ldots, \mathbf{u}_l$ such that $T(\mathbf{u}_1^l)\mathbf{c} \in \Phi$. Then it is easy to see that for all n, the $(\mathbf{c}(\lambda), n - l(\mathbf{c}))$ strategy, using at the third stage the actions proportional to $\mathbf{u}_1, \ldots, \mathbf{u}_l$ is the admissible strategy with the same tuple of values x_s for $n-l \leq s \leq n$ and hence the function $L_n \equiv b(\mathbf{c}) > -\infty$. From here and (3.4), (3.5) we get (4.7).

Below up to the proof of point 3) of Theorem 3.1 we consider the case $\lambda_* < \infty$. The Proposition 4.3 and the bound (4.4) imply also that if $C(\lambda_*) \cap L \neq \emptyset$ and $\mathbf{c} \in C(\lambda_*) \cap L$ then for any n there exists quasioptimal $(\mathbf{c}, n - l(\mathbf{c}))$ strategy, i.e.

$$W_n \simeq b_1 \lambda_*^n \leq V_n \leq \lambda_*^n. \tag{4.8}$$

Formula (4.8) and the bound (4.5) imply immediately also

Proposition 4.4. *Let $C(\lambda_*) \cap L \neq \emptyset$. Then the root λ_* has multiplicity one.*

The Propositions 4.3, 4.4 and (4.8) imply the point 1a) of Theorem 3.1. It is an open problem whether inverse to Proposition 4.4 is true, i.e. if the root λ_* has multiplicity one then $C(\lambda_*) \cap L \neq \emptyset$.

Proposition 4.3 and the bound (4.4) immediately simply also

Proposition 4.5. *For all $\lambda > \lambda_*$, $\lambda \in \Delta$ we have $C(\lambda) \cap L = \emptyset$ and hence the t-vectors $\mathbf{c} \in C(\lambda)$ for these λ are flying by states.*

To prove that the turnpikes for $\lambda < \lambda_*$ are liquid and to deal with the case $C(\lambda_*) \cap L = \emptyset$, we prove the important Proposition 4.6. Its proof describes explicitly how the transition from a trunpike $\mathbf{c} \in C(\lambda)$, $\lambda < \lambda_*$ to the final set may be realized.

Proposition 4.6. *Let $1 \leqslant \lambda < \lambda_*$, $\mathbf{c} \in C(\lambda)$, $\delta(\mathbf{c}) > 0$. Then $\mathbf{c} \in L$ and there exists the $(\mathbf{c}, s_n)$ turnpike strategy such that $s_n = n - N(r-1) - b$,*

$$L_n(\lambda) \simeq b \textstyle\sum_{i=1}^{N} \ln(\lambda_i/\lambda), \tag{4.9}$$

where all constants b depend only on λ and $\mathbf{c}$, and λ_i and $N = N(\lambda, \mathbf{c})$ are defined as

$$\lambda_0 = \lambda, \quad \lambda_{i+1} = \lambda_i - bJ(\lambda_i), \quad i = 1, \ldots, N-1, \quad N = \min(i: \lambda_i \leq 1). \tag{4.10}$$

Proof: Let $\mathbf{c} \in C(\lambda)$, $\delta(\mathbf{c}) > 0$, $\alpha(s)$ and $\mathbf{c}(s)$ are the functions, A and B are the constants from the point a) Proposition 2.5. Then $\mathbf{c}(\lambda) = \mathbf{c}$ and

$$T^{\alpha(s)}\mathbf{c}(s) = s\mathbf{c}(s) + \delta(s)\mathbf{e}, \quad \delta(s) \equiv \delta(\mathbf{c}(s)) \equiv sJ^{\alpha(s)}(s) \geq AsJ(s) > 0, \quad 1 \leq s \leq \lambda \tag{4.11}$$

and in particular

$$T(\mathbf{u})\mathbf{c}(\lambda) = \lambda\mathbf{c}(\lambda) + \delta\mathbf{e}, \quad \delta \geq A\lambda J(\lambda), \quad \mathbf{u} = \alpha(\lambda). \tag{4.12}$$

Denote $\mathbf{z}_0 = \lambda\mathbf{c}(\lambda) + \delta\mathbf{e}$. To prove Proposition we will construct the sequence of states $\mathbf{z}_i = k_i(\mathbf{c}(\lambda_i) + d_i\mathbf{e})$, and the tuples of actions $\mathbf{u}_1^{r-1}(i) \equiv (\mathbf{u}_1(i), \ldots, \mathbf{u}_{r-1}(i))$ admissible for $\mathbf{z}_i$, such that $\mathbf{z}_{i+1} \equiv T(\mathbf{u}_1^{r-1}(i))\mathbf{z}_i$, $i = 0, 1, \ldots, N-1$ and $\mathbf{z}_N = k\mathbf{c}(1) + \varepsilon\mathbf{e}$, $k > 0$, $\varepsilon \geq 0$, $\mathbf{c}(1) \in C(1)$, $\delta(\mathbf{c}(1)) > 0$. By Proposition 2.6 the state $\mathbf{c}(1)$ is liquid. Hence $\mathbf{z}_N$ and $\mathbf{z}_0$ are also liquid and by (4.12) $\mathbf{c}(\lambda) = \mathbf{c}$ is also liquid. To construct $\mathbf{z}_i$ we need the following lemma.

Lemma. *For any $s \leq \lambda$, any vector $\mathbf{z} = \mathbf{c}(s) + d\mathbf{e}$, $d \geq AJ(s)$, and any $s' \leq \lambda$, such that $|\mathbf{c}(s) - \mathbf{c}(s')| \leq AJ(s)$, there exists a tuple of actions $\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}$ such that*

$$\mathbf{z}' \equiv T(\mathbf{u}_1^{r-1})\mathbf{z} = s'^{r-1}(\mathbf{c}(s') + d'\mathbf{e}), \quad d' \geq AJ(s'). \tag{4.13}$$

Proof: Let $|\mathbf{c}(s')-\mathbf{c}(s)|\leq AJ(s)$. Then by remark in the end of the Section 2b we have $\mathbf{c}(s)+AJ(s)\mathbf{e}\succ\mathbf{c}(s')$ and hence $\mathbf{z}\succ\mathbf{c}(s')$. Formula (4.11) for s' and Proposition 2.3 imply the existence of actions $\mathbf{v}_1,\ldots,\mathbf{v}_{r-1}$ such that $T(\mathbf{v}_1^{r-1})\mathbf{c}(s')=\lambda'^{r-1}(\mathbf{c}(s')+\varepsilon\mathbf{e})$, $\varepsilon\geq AJ(s')$. By Proposition 4.1 applied to vectors $\mathbf{y}_1\equiv\mathbf{z}$, $\mathbf{z}_1=\mathbf{c}(s')$ and the tuple of actions $\mathbf{v}_1,\ldots,\mathbf{v}_{r-1}$ there exists a tuple of actions $\mathbf{u}_1,\ldots,\mathbf{u}_{r-1}$ such that the third formula (4.6)) is valid and hence (4.13) holds.

Consider now the sequence λ_i, $i=0,1,2,\ldots,\lambda_0=\lambda$, $\lambda_{i+1}=\lambda_i-bJ(\lambda_i)$, $b=A/B$. By (2.13) $\mathbf{c}(s)$ satisfies Lipschitz condition with coefficient B and hence we have $|\mathbf{c}(\lambda_{i+1})-\mathbf{c}(\lambda_i)|\leq AJ(\lambda_i)$. Now we may apply repeatedly the lemma to $\mathbf{z}_0,\mathbf{z}_1,\ldots$. Taking into account point b) of Proposition 2.1 we get the sequences $\mathbf{z}_i$ and tuples of actions $\mathbf{u}_1'^{r-1}(i)$ such that $\mathbf{z}_{i+1}\equiv T(\mathbf{u}_1'^{r-1}(i))\mathbf{z}_i$, $\mathbf{z}_i=k_i(\mathbf{c}(\lambda_i)+d_i\mathbf{e})$, $i=0,1,\ldots,N-1$, where $N\equiv N(\lambda,\mathbf{c})=\min(i:\lambda_i\leq 1)$, $(\lambda_N=1)$, $d_i\geq AJ(\lambda_i)$ and

$$k_i=\textstyle\prod_{s=1}^{i}(\lambda_s)^{r-1}. \tag{4.14}$$

Since $J(s)\geq p\geq 0$ for all $1\leqslant s\leqslant\lambda$, the size of every step $\lambda_i-\lambda_{i+1}\geq q>0$ and hence in a finite number of steps $N(\lambda,\mathbf{c})$ the transition to the turnpike $\{\mathbf{c}(1)\}$ (the state $d(\mathbf{c}(1)+\varepsilon\mathbf{e})$, $\varepsilon\geq AJ(1)>0$) is possible. The real trajectory of transition from $\{\mathbf{c}(\lambda)\}$ to $\{\mathbf{c}(1)\}$ is described by the states of the form $\mathbf{z}_{s+1,i}=T(\mathbf{u}_1^s(i))\mathbf{z}_i$, $s=1,\ldots,r-1$, $\mathbf{z}_{i+1}\equiv\mathbf{z}_{r,i}$, $i=0,1,\ldots,N-1$.

The number of moments of time necessary for the transition from $\mathbf{c}(\lambda)\equiv\mathbf{c}$ to the final set Φ is obviously equal to $N(\lambda,\mathbf{c})(r-1)+N'$, where N' does not depend on λ and depend only on $\mathbf{c}(1)$. Since N' does not influence the asymptotics when $n\to\infty$ we omit it below. The initial point λ will further depend on n, $\lambda=\lambda_0^{(n)}$ and we will study the assymptotics $N(\lambda_0^{(n)})$.

We will refer further to the $(\mathbf{c},s_n)$ strategy with transition from turnpike $\mathbf{c}\in C(\lambda)$, $\lambda<\lambda_*$ to the final set of the form described above as to $(\mathbf{c},s_n)$ *slipping strategy.*

To complete the proof of Theorem 3.1 let us assume now that $\alpha(s)$ and $\mathbf{c}(s)\in C(s)$, $s<\lambda_*$, be the functions from point b) of Proposition 2.5. The proof of Proposition 4.6 applied to the vector $\mathbf{c}(s)$ as to the initial vector $\mathbf{c}$ immediately imply that the formula (4.9), (4.10) are valid, but now the constants b are the same for all $\lambda<\lambda_*$. Let $\lambda_0^{(n)}$ be a sequence, $\lambda_0^{(n)}\to\lambda_*$ as $n\to\infty$, λ_*, L_n, s_n are defined in (4.9), (4.10) for the sequence of constructed above $(\mathbf{c}(\lambda_0^{(n)},s_n)$ slipping strategies (constants b do not depend on n). It is easy to see that the asymptotics of L_n and s_n depend only on behaviour of the function $J(\lambda)$ near λ_*. If $m=0$ then $J(\lambda)=a(\lambda-\lambda_*)+O(\lambda-\lambda_*)^2$, $a<0$, and for fixed n the sequence λ_i in Proposition 4.6 may be taken to have the form $\lambda_{i+1}^{(n)}=\lambda_i^{(n)}-b(\lambda_*-\lambda_i^{(n)})$, $i=0,1,\ldots,N(\lambda_0^{(n)})-1$. Let us take $\lambda_0^{(n)}=\lambda_*-1/n$. Then $\lambda_i^{(n)}=\lambda_*-(1+b)^i/n$ and hence $N(\lambda_0^{(n)})\approx b\ln n$, i.e. $s_n\approx n-b\ln n$. Putting the values of $\lambda_0^{(n)}$ and $N(\lambda_0^{(n)})$ into (4.9), it is easy to check that $L_n\geq -b$ for all n. Now putting $\lambda=\lambda_*-1/n$ into (3.4) and taking into account that $\lim(\lambda_*-1/n)^n/\lambda_*^n=b>0$, we get that for the described $(\mathbf{c}(\lambda_0^{(n)},s_n)$ slipping strategies $W_n\approx b\lambda_*^n$ and hence the point 1b) of Theorem 3.1 is proved. For $m\geq 1$ we have $J(\lambda)=a(\lambda-\lambda_*)^{m+1}+O(\lambda-\lambda_*)^{m+2}$ and we may consider the sequence $\lambda_i^{(n)}$ of the form $\lambda_{i+1}^{(n)}=\lambda_i^{(n)}-b(\lambda_*-\lambda_i^{(n)})^{m+1}$, $i=0,1,\ldots,N(\lambda_0^{(n)})-1$. To get the asymptotics of s_n and L_n we can use the approximation of this difference scheme by the differential equation $d\lambda(t)/dt=-b(\lambda_*-\lambda)^{m+1}$, $\lambda(0)=\lambda_0^{(n)}$. By standard analytical technique, it can be shown that $N(\lambda_0^{(n)})\approx b\tau(\lambda_0^{(n)})$, where $\tau(\lambda_0^{(n)})=\min(s:\lambda(s)=1)$ and $L_n(\lambda_0^{(n)})\approx b\int_0^\tau\ln(\lambda(t)/$

$\lambda_0^{(n)})dt$. Direct solution of differential equation implies that $\tau(\lambda_0^{(n)}) \approx b(\lambda_* - \lambda_0^{(n)})^{-m}$ for all $m \geq 1$ and $L_n = -bln(\lambda_* - \lambda_0^{(n)})$ for $m = 1$, $L_n \approx -b(\lambda_* - \lambda_0^{(n)})^{-m+1}$ for $m > 1$. Let us take $(\lambda_0^{(n)}) = \lambda_* - (b_0 n)^{-1/m}$. Then $N(\lambda_0^{(n)}) \approx b_0 bn$ and hence $s_n = n - b_0 bn > 0$ for sufficiently small b_0. Since $(\lambda_0^{(n)})^n = (\lambda_* - (b_0 n)^{-1/m})^n \simeq \lambda_*^{n(1-bn^{-1/m})}$ and $L_n \simeq -b(ln\, n)$ for $m = 1$ and $L_n \simeq -bn^{(m-1)/m}$ for $m \geq 2$ we get the statement of point 3) of Theorem 3.1.

Point 3). The case $\lambda_* = \infty$ is possible only when there are loans and hence $\lim J(\lambda) = +1$ as $\lambda \to \infty$ and $J(\lambda) \geq d > 0$ for all λ. Let $\alpha(\lambda)$, $\mathbf{c}(\lambda)$ be the functions specified by point b) of Proposition 2.5 for this case. We will prove that there exists n_0 such that for any λ_0, $1 \leq \lambda_0 < \infty$ the transition from the turnpike $\{\mathbf{c}(\lambda_0)\}$ to the final set is possible in no more than n_0 steps. Then the $(\mathbf{c}(\lambda_0), s_n)$ strategy on the time interval $(0, r + n_0)$, with the second stage consisting of one step will ensure payoff at least $b\lambda_0$, where b does not depend on λ_0, and hence $V_n = \infty$ for $n \geq r + n_0 + 1$. In Proposition 4.6 we constructed the sequence $\mathbf{z}_i$ ensuring the transition from the turnpike $\{\mathbf{c}(\lambda)\}$ to the final set. In this construction we used the sequence λ_i, $i = 0, 1, \ldots, N(\lambda_0, \mathbf{c}(\lambda_0))$ with the properties

$$|\mathbf{c}(\lambda_{i+1}) - \mathbf{c}(\lambda_i)| \leq bJ(\lambda_i), \quad i = 0, 1, \ldots, N-1, \quad \lambda_N = 1. \tag{4.15}$$

Now, aiming to minimize $N(\lambda, \mathbf{c})$, its more appropriate to use for big values of λ, the inequality $|\mathbf{c}(s') - \mathbf{c}(s)| \leq B|1/s - 1/s'|$ rather than Lipschitz condition. By (2.13) we have $|\mathbf{c}(\lambda') - \mathbf{c}(\lambda)| \leq B|1/\lambda' - 1/\lambda|$. Since $J(\lambda) \geq d > 0$ for all λ, there exists $n_0 \equiv n_0(b, B, d)$ such that for any $\lambda_0, 1 < \lambda_0, < \infty$, there exists a tuple of numbers $\lambda_0, \lambda_1, \ldots, \lambda_{n_0} = 1$ satisfying (4.15) ($\lambda_1 = 1/c$, $\lambda_{i+1} = \lambda_i/(c\lambda_i + 1)$, $c = bd/B$, $i \geq 1$), and the whole construction of Proposition 4.6 may be repeated. Thus $N(\mathbf{c}(\lambda), \lambda_0) \leq n_0$ for all $\lambda_0 \geq 1$.

5. Conclusion

Many interesting problems remain uninvestigated. One of the most intriguing, perhaps, is the description of liquid, flying by and deadend states in terms of the investment matrix. One of the related questions is the following. Denote by $\omega(\mathbf{z}) = \{\mathbf{y}: \mathbf{y} = T(\mathbf{u})\mathbf{z},\ \mathbf{u} \in U(\mathbf{z})\}$, $Z = Z^0$, $Z^k = \{\mathbf{z} \in Z^{k-1}, \omega(\mathbf{z}) \cap Z^{k-1} \neq \varnothing\}$ – the set of all states from which at least k admissible steps are possible, $k = 1, 2, \ldots$. Obviously $Z \supseteq Z^1 \supseteq \ldots$ and $\cap Z^k = L \cup F$, where L is the set of liquid and F is the set of flying by states. How can we describe Z^k and is it true that $Z^i = Z^{i+1}$ for all i greater than some s? If $\mathbf{z}$ is a liquid state and $l(\mathbf{z})$ is a number of steps to reach final set, how do we get the estimates for this function? For the deadends there is another question: how long does the agony last? That is, what is the maximal number of steps before reaching a point where we can not move?

We didn't consider separately the case with and without loans, though the presence of loans implies essential difference, for instance the possibility of $\lambda_* = \infty$.

We don't pay attention to the meaning of complex roots of investment polynomials. Some interesting remarks may be found in Atsumi (1991). We didn't consider the relation between the flying turnpikes with the rate $\lambda > \lambda_*$. It may be shown that from any such turnpike, the turnpikes with bigger rates may be reached but not vice versa. Note that consideration of the turnpikes with the rate $\lambda < 1$ makes sense if we are concerned with the optimization of slowing down a system.

There are many possible generalizations of the model considered. It is interesting to study the related model with consumption. Probably, in this case not only the structure of the set of roots but also the values of the investment polynomials will play an important role. Still more interesting would be the study of the stochastic analog of this model. Many significant effects can be anticipated, for instance the instability of some turnpikes. Note also that in this case it makes sense to consider the projects $\mathbf{a}^j$ with $a_0^j = 0$.

6. Appendix

6a. Some sufficient conditions for the liquidity of turnpikes. Let $\mathbf{a}^j = (a_0^j, a_1^j, \ldots, a_r^j)$ be any productive or cash-transfer project, i.e. $\sum_{k=0}^r a_k^j \geq 0$. Denote by $A_i^j = \sum_{k=0}^i a_k^j$, $i = 0, 1, \ldots, r$.

Proposition 6.1. *If the sequence $A_0^j, A_1^j, \ldots, A_r^j$ for the investment project $\mathbf{a}^j$ has exactly one sign change then $J^j(\lambda) = 0$ implies $\mathbf{c}^j(\lambda) \in L_0$.*

Proof: By the definition of the set L_0, $\mathbf{c} = (c_0, c_1, \ldots, c_{r-1}) \in L_0$ iff the operator T^0 (keeping money) is applicable to the states $S^i \mathbf{c}$ for $i = 0, 1, \ldots, r-1$, where S^i is ith power of the operator S or equivalently $\sum_{k=0}^i c_k \geq 0$, $i = 0, 1, \ldots, r-1$. Rewrite this condition for $\mathbf{c} = \mathbf{c}^j(\lambda)$, using the equality (2.10) ((2.8)) in coordinates. The index j is omitted below. Denoting by $C_i = \sum_{k=0}^i c_k$, we get

$$C_0 = c_0 = 1, \quad C_{s+1} = \lambda C_s - A_{s+1}, \quad s = 0, 1, \ldots, r-2, \quad C_{r-1} = \lambda C_{r-1} - A_r. \quad (6.1)$$

We have $A_0 = a_0 = -1$. Let $A_k < 0$, $A_i \geq 0$, $i \geq k+1$. By (6.1) and $\lambda \geq 1$, we have $1 = C_0 < C_1 < \cdots < C_k$. Let us suppose that $C_i < 0$ for some $i > k$. Then (6.1) implies that $C_i > C_{i+1} > \cdots > C_{r-1}$. Since $\mathbf{a}^j$ is a productive or cash-transfer project then $A_r \geq 0$ and $C_{r-1} = A_r/(\lambda - 1) \geq 0$, and we get the contradiction.

The project $\mathbf{a}^j = (a_0, a_1, \ldots, a_r)$ is called *simple* if the sequence $a_0, a_1, \ldots, a_r$ has only one sign change. Obviously for the simple projects the sequence A_i, $i = 0, 1, \ldots, r-1$ also has only one sign change and hence the proposition 6.1 implies immediately

Proposition 6.2. *Let $\mathbf{a}^j$ be a simple investment project, $J^j(\lambda) = 0$. Then $\mathbf{c}^j(\lambda) \in L_0$.*

6b. Proof of Proposition 2.5. We consider only the proof of the point a). The point b) may be proven similarly. Consider at first the case when $\mathbf{c} = \mathbf{c}^j(\lambda)$, $J^j(\lambda) > 0$ for some j. Let $U(d)$ be the neighbourhood of the point λ with radius d, such that $J^j(s) > 0$ for $s \in U(d)$. Let $h(s)$ be the smooth function defined on the segment $[1, \lambda_*]$ and $0 \leq h(s) \leq 1$, $h(s) = 0$ for $s \in U(d/2)$, $h(s) \equiv 1$ for $s \notin U(d)$. Now consider the function $\alpha(s) = (\alpha^0, \alpha^1, \ldots, \alpha^k)$ of the form $\alpha^i(s) = b(s)h(s)J_+^i(s)$, $i \neq j$, $\alpha^j(s) = b(s)J_+^j(s)$, where $f_+ = \max(f, 0)$, $b(s)$ is positive and specified by the condition $\sum_j \alpha^j(s) = 1$. Taking into account Proposition 2.4 it is easy to show that $\alpha(s)$ and $\mathbf{c}(s) \equiv \mathbf{c}^{\alpha(s)}(s)$ satisfy all the conditions of the Proposition. Consider now the case of any $\mathbf{c} \in C(\lambda)$. Let $\alpha_0 \equiv \alpha(\mathbf{c}) = (\alpha_0^0, \alpha_0^1, \ldots, \alpha_0^k)$ and let j be one of the indexes such that $J^j(\lambda) > 0$. It exists because of $\delta(\mathbf{c}) > 0$ and (2.11). Let $U(d)$ be the same as defined above and let $f(s)$ be the linear function, $f(\lambda)=0$, $df/ds=1/d$. Now define the function $\alpha(s)=(\alpha^0,\alpha^1,\ldots,\alpha^k)$ on $U(d)$ by the relations $\alpha^i(s)=\alpha_0^i(1+f(s))b(s)$, $i\neq j$, $\alpha^j(s)=\alpha_0^j(1-f(s))b(s)$, where $b(s)$ is again a norming function. It is easy to check that $\alpha(\lambda)=\alpha_0$, that for d sufficiently small $\alpha(s)\in R(s)$ for $s\in[\lambda-d,\lambda]$ and $\mathbf{c}(s)=\mathbf{c}^{\alpha(s)}(s)$ satisfy the condition of Proposition on this segment. Since $\alpha^j(\lambda-d)=1$, on the segment $[1,\lambda-d]$ we have the case considered above.

6c. Proof of the Estimates (4.5). Let us put to the right side of (4.3) all terms of the sum in (4.3) with $J^j(\lambda_*)<0$ and with $J^j(\lambda_*)=0$, $J^j(\lambda-\varepsilon)<0$ for small positive ε. Hence by the definition of λ_* we leave in the left side only the terms with $J^j(\lambda_*)=0$ and multiplicities no less than $m+1$. To simplify differentiation in the sequel, we make the change of variables $z=1/\lambda$, leaving the same notations for functions. We have

$$1+\textstyle\sum_j^0 J^j(z)U_{n-1}^j(z)=z^nV_n-\sum_j^- J^j(z)U_{n-1}^j(z). \tag{6.2}$$

Since in the left side of (6.2) all terms of the sum have the multiplicity of the root $z_*=1/\lambda_*$ equal at least to $m+1$ then this sum and its first m derivatives vanish when $z=z_*$. Denote by p_i, q_i^j, v_i^j the value of ith derivatives of the functions z^n, $J^j(z)$ and $U_{n-1}^j(z)$ at the point z_*. Then at that point the k-th derivative of the right hand side of (6.2) for $k=0,1,\ldots,m$ is equal to

$$D_k=p_kV_n-\textstyle\sum_j^-\sum_{i=0}^k C_k^i v_i^j g_{k-i}^j, \tag{6.3}$$

where $C_k^i=k!/i!(k-i)!$, $D_k=0$ for $k=1,\ldots,j$; $D_k=1$ for $k=0$, $(v_0^j\equiv U_n^j(\lambda_*))$, $g_0^j\equiv J^j(\lambda_*))$. Note that nonnegativity of investments imply

$$v_i^j\geq 0 \quad \text{for all} \quad i,j, \tag{6.4}$$

and that $g_0^j\leq 0$ for all j in the sum (6.3). For sake of brevity we consider the case $g_0^j<0$ for all j. Now multiply the equalities (6.3) for $k=0,1,\ldots,m$ by positive constants $d_0,d_1,\ldots,d_m$, which will be specified later and sum to obtain

$$d_0=V_n\textstyle\sum_{k=0}^m p_kd_k+\sum_{k=0}^m d_k\left(-\sum_j^-\sum_{i=0}^k C_k^i v_i^j g_{k-i}^j\right). \tag{6.5}$$

Changing the order of summation we may rewrite the second sum as

$$\textstyle\sum_{i=0}^m\sum_{k=i}^m d_k\left(-\sum_j^- C_k^i v_i^j g_{k-i}^j\right)\equiv\sum_{i=0}^m\sum_{k=i}^m d_kM_{ki}\equiv\sum_{i=0}^m N_i. \tag{6.6}$$

Let us show by induction on i, $i=m, m-1,\ldots,0$, that positive constants $d_m,d_{m-1},\ldots,d_i$ may be chosen to ensure $N_m,\ldots,N_i\geq 0$. Note that by (6.4) and remark after this formula $M_{ii}\equiv-\sum_j^- v_i^jg_0^j\geq 0$ and if $M_{ii}=0$ then $v_i^j=0$ for all j in the sum on j and hence $M_{ki}=0$ for all $k=i,\ldots,m$ and $N_i=0$ for all $d_m,\ldots,d_i$. Since $M_{mm}\geq 0$, then $N_m\equiv d_mM_{mm}\geq 0$ for any $d_m>0$. Let positive constants $d_m,\ldots,d_{i+1}$ be chosen so that $N_m,N_{m-1},\ldots,N_{i+1}\geq 0$. If $M_{ii}=0$, then $N_i=0$ for any $d_i>0$. If $M_{ii}>0$, we may take $d_i>0$ such that $N_i=d_iM_{ii}+\sum_{k=i+1}^m d_kM_{ki}\geq 0$. Let positive $d_0,d_1,\ldots,d_m$ be such that $N_i\geq 0$, $i=0,1,\ldots,m$. The formulae (6.5), (6.6) and positiveness of $p_0,p_1,\ldots,p_m$ imply

$$V_np_md_m\leq V_n\textstyle\sum_{k=0}^m p_kd_k\leq V_n\sum_{k=0}^m p_kd_k+\sum_{i=0}^m N_i=d_0. \tag{6.7}$$

Since $p_m=z_*^{n-m}n(n-1)\cdots(n-m+1)\geq bz_*^nn^m$, $b>0$, then (6.7) implies (4.5)

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