ITCS 6150
Intelligent Systems

Lecture 11
Logical Agents
Chapter 7
Reasoning w/ propositional logic

Remember what we’ve developed so far

- Logical sentences
- And, or, not, implies (entailment), iff (equivalence)
- Syntax vs. semantics
- Truth tables
- Satisfiability, proof by contradiction
Logical Equivalences

Know these equivalences

\[
\begin{align*}
(\alpha \land \beta) &\equiv (\beta \land \alpha) & \text{commutativity of } \land \\
(\alpha \lor \beta) &\equiv (\beta \lor \alpha) & \text{commutativity of } \lor \\
((\alpha \land \beta) \land \gamma) &\equiv (\alpha \land (\beta \land \gamma)) & \text{associativity of } \land \\
((\alpha \lor \beta) \lor \gamma) &\equiv (\alpha \lor (\beta \lor \gamma)) & \text{associativity of } \lor \\
\neg(\neg \alpha) &\equiv \alpha & \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg \beta \Rightarrow \neg \alpha) & \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg \alpha \lor \beta) & \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)) & \text{biconditional elimination} \\
\neg(\alpha \land \beta) &\equiv (\neg \alpha \lor \neg \beta) & \text{de Morgan} \\
\neg(\alpha \lor \beta) &\equiv (\neg \alpha \land \neg \beta) & \text{de Morgan} \\
(\alpha \land (\beta \lor \gamma)) &\equiv ((\alpha \land \beta) \lor (\alpha \land \gamma)) & \text{distributivity of } \land \text{ over } \lor \\
(\alpha \lor (\beta \land \gamma)) &\equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma)) & \text{distributivity of } \lor \text{ over } \land
\end{align*}
\]

Figure 7.11  Standard logical equivalences. The symbols $\alpha$, $\beta$, and $\gamma$ stand for arbitrary sentences of propositional logic.
Reasoning with propositional logic

**Inference Rules**

- Modus Ponens:
  - Whenever sentences of form \( \alpha \Rightarrow \beta \) and \( \alpha \) are given, the sentence \( \beta \) can be inferred.

- \( R_1: \) Green \( \Rightarrow \) Martian
- \( R_2: \) Green
- Inferred: Martian
Reasoning with propositional logic

**Inference Rules**

- And-Elimination
  - Any of conjuncts can be inferred
    - $R_1$: Martian $\land$ Green
    - Inferred: Martian
    - Inferred: Green

*Use truth tables if you want to confirm inference rules*
## Inference Rules

<table>
<thead>
<tr>
<th>Name</th>
<th>Rule of Inference</th>
<th>Tautology</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modus [ponendo] ponens = “the way that affirms by affirming”</td>
<td>$p$</td>
<td>$(p \land (p \rightarrow q)) \rightarrow q$</td>
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<tr>
<td></td>
<td>$p \rightarrow q$</td>
<td></td>
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<tr>
<td></td>
<td>$\therefore q$</td>
<td></td>
</tr>
<tr>
<td>Modus [tollendo] tollens = “the way that denies by denying”</td>
<td>$\neg q$</td>
<td>$(\neg q \land (p \rightarrow q)) \rightarrow \neg p$</td>
</tr>
<tr>
<td></td>
<td>$p \rightarrow q$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\therefore \neg p$</td>
<td></td>
</tr>
<tr>
<td>Transitivity (Hypothetical syllogism)</td>
<td>$p \rightarrow q$</td>
<td>$((p \rightarrow q) \land (q \rightarrow r) \rightarrow (p \rightarrow r)$</td>
</tr>
<tr>
<td></td>
<td>$q \rightarrow r$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\therefore p \rightarrow r$</td>
<td></td>
</tr>
<tr>
<td>Elimination (Disjunctive syllogism)</td>
<td>$p \lor q$</td>
<td>$((p \lor q) \land \neg p) \rightarrow q$</td>
</tr>
<tr>
<td></td>
<td>$\neg p$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\therefore q$</td>
<td></td>
</tr>
<tr>
<td>Generalization</td>
<td>$p$</td>
<td>$p \rightarrow (p \lor q)$</td>
</tr>
<tr>
<td></td>
<td>$\therefore p \lor q$</td>
<td></td>
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<tr>
<td>Simplification</td>
<td>$p \land q$</td>
<td>$(p \land q) \rightarrow p$</td>
</tr>
<tr>
<td></td>
<td>$\therefore p$</td>
<td></td>
</tr>
<tr>
<td>Conjunction</td>
<td>$p$</td>
<td>$((p \land (q)) \rightarrow (p \land q)$</td>
</tr>
<tr>
<td></td>
<td>$q$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\therefore p \land q$</td>
<td></td>
</tr>
<tr>
<td>Resolution</td>
<td>$p \lor q$</td>
<td>$((p \lor q) \land \neg (p \lor r)) \rightarrow (q \lor r)$</td>
</tr>
<tr>
<td></td>
<td>$\neg p \lor r$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\therefore q \lor r$</td>
<td></td>
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</tbody>
</table>
Example of a proof

If today is Tuesday, I have a test in Mathematics or Economics. If my Economics professor is sick, I will not have a test in Economics. Today is Tuesday, and my Economics professor is sick. Therefore, I will have a test in Mathematics.

Converting to logical notation:

\[ t = \text{today is Tuesday} \]
\[ m = \text{I have a test in Mathematics} \]
\[ e = \text{I have a test in Economics} \]
\[ s = \text{My Economics Professor is sick.} \]

So the argument is:

\[ t \rightarrow (m \lor e) \]
\[ s \rightarrow \neg e \]
\[ t \land s \]
\[ \therefore m \]

1. \[ t \land s \] premise
2. \[ t \] from (1) by simplification Law
3. \[ t \rightarrow (m \lor e) \] premise
4. \[ m \lor e \] from (2) and (3) by modus ponens
5. \[ s \] from (1) by simplification Law
6. \[ s \rightarrow \neg e \] premise
7. \[ \neg e \] from (5) and (6) by modus ponens
8. \[ e \lor m \] from (4) by commutative law
9. \[ m \] from (7) and (8) by elimination Law
Constructing a proof

Proving *is like* searching

- Find sequence of logical inference rules that lead to desired result
- Note the explosion of propositions
  - Good proof methods ignore the countless irrelevant propositions
How many inferences?

*Previous example relied on application of inference rules to generate new sentences*

- When have you drawn enough inferences to prove something?
  - Too many make search process take longer
  - Too few halt logical progression and make proof process incomplete
Completeness Theorem

Definition:
Formula $\alpha$ is a **tautology** if for every valuation $v$ (in every model $v$), $v(\alpha) = 1$ (true)

Axioms
A1: $A \rightarrow (B \rightarrow A)$
A2: $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
A3: $(\neg B \rightarrow \neg A) \rightarrow [(\neg B \rightarrow A) \rightarrow B]$

Rule of Inference:
R: $[A, A \rightarrow B]/B$

Definition:
Formula $\alpha$ is a theorem if it can be proved from Axioms A1, A2, A3 using rule R.

**Proposition 1.** If formula $\alpha$ is a theorem, then $\alpha$ is a tautology (soundness)

**Proposition 2.** If formula $\alpha$ is a tautology, then $\alpha$ is a theorem (completeness)

**Proposition 3.** $\models \neg \alpha$ iff $\models \alpha$ ; $F \models \neg \alpha$ iff $F \models \alpha$, where $F$ – set of formulas.
What about “and” clauses?

Resolution only applies to “or” clauses

- Every sentence of propositional logic can be transformed to a logically equivalent conjunction of disjunctions of literals

Conjunctive Normal Form (CNF)

- A sentence expressed as conjunction of disjunction of literals
  - k-CNF: exactly k literals per clause

\[(\ell_{1,1} \lor \ldots \lor \ell_{1,k}) \land \ldots \land (\ell_{n,1} \lor \ldots \lor \ell_{n,k})\]
The steps are as follows:

1. Eliminate $\leftrightarrow$ replacing $\alpha \leftrightarrow \beta$ by $(\alpha \to \beta) \land (\beta \to \alpha)$
2. Eliminate $\to$ by replacing $\alpha \to \beta$ by $\lnot \alpha \lor \beta$
3. CNF requires $\lnot$ to appear only in literals so we move $\lnot$ inwards by repeated application of the following equivalence forms:
   - $\lnot (\lnot \alpha) = \alpha$ (double negation elimination)
   - $\lnot (\alpha \land \beta) = (\lnot \alpha \lor \lnot \beta)$ (de Morgan)
   - $\lnot (\alpha \lor \beta) = (\lnot \alpha \land \lnot \beta)$ (de Morgan)
4. Now, we apply the distributivity law
   
   $(\alpha \land \beta) \lor \gamma = (\alpha \lor \gamma) \land (\beta \lor \gamma)$

   by distributing $\lor$ over $\land$ wherever possible.

Example: Convert the formula $A \leftrightarrow (B \lor C)$ to CNF
We wish to prove $KB$ entails $\alpha$

- Must show $(KB \land \neg \alpha)$ is unsatisfiable
  - No possible way for $KB$ to entail (not $\alpha$)
  - Proof by contradiction
**Proof by Contradiction - Example**

**Proof attributed to Euclid (300 BC)**

**Theorem.** There are infinitely many prime numbers.

**Proof.** Assume to the contrary that there are only finitely many prime numbers, and all of them are listed as follows: $p_1, p_2, \ldots, p_n$. Consider the number $q = p_1 p_2 \cdots p_n + 1$. □ The number $q$ is either prime or composite. If we divided any of the listed primes $p_i$ into $q$, there would result a remainder of 1 for each $i = 1, 2, \ldots, n$. Thus, $q$ cannot be composite. We conclude that $q$ is a prime number, not among the primes listed above, contradicting our assumption that all primes are in the list $p_1, p_2, \ldots, p_n$. 
Example of resolution in KB

\[
\neg P_{2,1} \lor B_{1,1} \quad \neg B_{1,1} \lor P_{1,2} \lor P_{2,1} \\
\neg B_{1,1} \lor P_{1,2} \lor \neg P_{1,2} \quad \neg B_{1,1} \lor P_{2,1} \lor B_{1,1} \\
P_{1,2} \lor P_{2,1} \lor \neg P_{2,1} \\
\neg P_{2,1} \quad \neg P_{2,1} \\
\]

Partial application of PL-RESOLUTION to a simple inference.
.. \neg P_{1,2} is shown to follow from the first four clauses in the top row.

**Proof that there is not a pit in P_{1,2}: \neg P_{1,2}**

- KB ^ P_{1,2} leads to empty clause (resolution refutation)
- Therefore \neg P_{1,2} is true
Resolution Refutation

**Problem 1.** Show by resolution refutation that each of the following formulas is a tautology:

(a) \((P \rightarrow Q) \rightarrow [(R \lor P) \rightarrow (R \lor Q)]\)
(b) \([(P \rightarrow Q) \rightarrow P] \rightarrow P\)
(c) \((\neg P \rightarrow P) \rightarrow P\)
(d) \((P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)\)

**Problem 2.** /language alphabet – \{P, Q, R\}/

Let’s consider interpretation \(v\) where \(v(P) = F, v(Q) = T, v(R) = T\). Does \(v\) satisfy the following propositional formulas?

\((P \rightarrow \neg Q) \lor \neg (R \land Q)\)
\((\neg P \lor \neg Q) \rightarrow (P \lor \neg R)\)
\((\neg (\neg P \rightarrow \neg Q) \land R)\)
\((\neg (\neg P \rightarrow Q \land \neg R)\)
Horn Clauses

**Horn Clause**

- Disjunction of literals with at most one is positive
  - \((\neg a \lor \neg b \lor \neg c \lor d)\)
  - \((\neg a \lor b \lor c \lor \neg d)\) **Not a Horn Clause**
Horn Clauses

Can be written as a special implication

- $(\neg a \lor \neg b \lor c) \iff (a \land b) \Rightarrow c$
  - $(\neg a \lor \neg b \lor c) \equiv (\neg(a \land b) \lor c) \quad \ldots \text{de Morgan}$
  - $(\neg(a \land b) \lor c) \equiv ((a \land b) \Rightarrow c) \quad \ldots \text{implication elimination}$
Horn Clauses

*Permit straightforward inference determination*

- **Forward chaining** (repeated application of modus ponens)
- **Backward chaining** (an inference method described as working backward from the goal).

Example of backward chaining: RS-strategy (Project 1)
Forward Chaining

**Properties**

- Sound
- Complete
  - All entailed atomic sentence will be derived

**Data Driven**

- Start with what we know
- Derive new info until we discover what we want
Backward Chaining

Start with what you want to know, a query \( (q) \)

Look for implications that conclude \( q \)

Goal-Directed Reasoning